Fourth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning

# Remarkable delta:03

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# Preface

The Remarkable Delta03 conference is co-hosted by the Department of Mathematics and Statistics of the University of Otago and the Department of Mathematics and the Department of Statistics of the University of Auckland. Delta03 is backed by an International Steering Committee made up of representatives from the Southern Hemisphere countries of Argentina, Australia, New Zealand, South Africa, and Uruguay. This steering committee currently consists of: Derek Holton (New Zealand, Chair), Nestor Aguilera (Argentina), Pat Cretchley (Australia), Johann Engelbrecht (South Africa), Victor Martinez Luaces (Uruguay), Jan Persens (South Africa), Ivan Reilly (New Zealand) and Christina Varavsky (Australia).

This Conference is the fourth in a series of conferences on the undergraduate teaching and learning of Mathematics and Statistics, as part of a collaboration between Southern Hemisphere countries. The first conference, Delta97 (delta implying change) was held in Brisbane, Australia, in November 1997. The second conference, Delta99, took place in November 1999 at Laguna Quays, Queensland, Australia. The third conference, Warthog Delta01, was held in Kruger National Park, South Africa, in July 2001. The Delta conferences take place biennially, and the next one, Delta05, will probably be in Australia.

The theme for Delta97 was "What Can We Do to Improve Learning?", for Delta99 it was "The Challenge of Diversity", and for Delta01 it was "Gearing for Flexibility". For Delta03 the theme is "From all Angles".

The 120 delegates, attending from 12 different countries worldwide, are responsible for about 80 contributed presentations, panel discussions and round table discussions. The conference has two publications: the Proceedings, consisting of peer reviewed research papers, and the Communications (subject to editorial scrutiny) largely comprising reports on teaching experiences and research in progress.

We believe that the deliberations at this meeting will influence the course of future tertiary mathematics and statistics education worldwide. It is broadly accepted that skills and training in the quantitative sciences will be crucial to success in the future. Thus our belief is that this meeting will be one of the most important of 2003.

Our thanks to the group who worked on editing, compiling and producing this issue of the Communications, especially to Margaret Woolgrove, Mike Thomas, Greg Oates and Bill Barton.

While all papers in these Communications have been subject to peer review (and some submissions were rejected), no substantive editing was undertaken. All opinions expressed herein are those of the authors themselves, and not necessarily of the Delta '03 organising committee.

Derek Holton and Ivan Reilly
Co-Convenors
Remarkable Delta '03

# Acknowledgements

It is a pleasure to acknowledge the work of the team that has put together these Communications.

Editorial work was conducted by Bill Barton, Greg Oates, Ivan Reilly and Mike Thomas of the Mathematics Education Unit, Department of Mathematics, The University of Auckland.

Production management was expertly handled by Margaret Woolgrove, of the New Zealand Institute of Mathematics and its Applications (NZIMA). Min-Young Lee, of the Department of Mathematics, University of Auckland ably assisted with the formatting of TEX files.

Special thanks also to the sponsors of this conference:

- The University of Auckland, Department of Mathematics
- The University of Auckland, Department of Statistics
- Australian Mathematical Society
- New Zealand Mathematical Society
- New Zealand Statistical Association
- The University of Otago, Department of Mathematics and Statistics
- The Royal Society of New Zealand
- Texas Instruments, Inc.

# Table of Contents

# A NEW APPROACH TO DISTANCE LEARNING

### Level 100 Mathematics Distance Learning Course for Secondary School Students

Liz Ackerly

## Background: Why this programme was developed

Many secondary schools have set up accelerated programmes which allow exceptional students to take mathematics at a faster rate, often resulting in them sitting the Bursary Calculus paper in year 12. A problem then arises – what mathematics should these students be doing in their final year? This is where universities can help.

These students also face another problem. They are often offered direct entry into second year courses at university. Even for exceptional students, the difficulties they face in missing out the vital work in first year mathematics is very real, and may cause lasting problems.

The Department of Mathematics and Statistics at the University of Canterbury has been successfully running a first year mathematics course for the local Christchurch secondary school students for the past seven years. To date, 26 of the 27 local schools have participated in this programme. In a typical week, students attend three hours of lectures and a one-hour small group tutorial or laboratory, at University, and after their school day. The course has given students an opportunity to study stimulating and interesting work while at school, and has prepared them well for second year mathematics.

Students who successfully complete this course are given a Certificate of Completion. Once they have enrolled at the University of Canterbury, they are given 12 points credit towards a Canterbury degree, and are able to enter our second year courses.

With the help of a Teaching and Learning Committee grant and further funding support from Recruitment, the Department has been able to offer this course to schools outside the local Christchurch area.

## NetTutor: Why our distance course is different

There are major disadvantages associated with distance learning. The key to the success of our course is the web-based programme **NetTutor** which allows for an innovative and new approach to distance teaching.

NetTutor is designed principally for *communicating* mathematics. It enables students to work directly with the lecturer, on an individual or group basis, via a "whiteboard". This provides an area of the screen where the lecturer/students can write messages and equations, draw figures, refer to notes etc. The work on the whiteboard can be seen and accessed by all participants simultaneously. Although other programmes have whiteboards, this one is different because it contains the specialised symbols that mathematics requires. These symbols are easy to use and students very quickly become adept at handling the technical aspects of NetTutor.

Students need access to an internet browser such as Internet Explorer or Netscape but no further software is required, so apart from the course fees, the cost to the students is minimal.

At this stage all course materials (notes, exercises and assignments) are posted to students at appropriate times, as well as being available through NetTutor.

## Key Features of NetTutor

NetTutor is well-organised into six main centres, each with a clearly defined purpose. The main ones are listed below.

- **Live Classroom**

  This is the centre where the weekly tutorials are conducted. The "live chat" sessions which each student has with the course lecturer are the key to the success of this course. Although labour intensive – the tutorials are held in groups of two/three students – it is in these sessions that the real learning takes place.

  The language students use on the web is very informal, and they very quickly adapt to this way of holding tutorials. *In fact many of the students commented at the mid-year workshop on campus that without NetTutor and the weekly tutorials they would have had difficulty coping with the demands of this course.*

  There is also a "live chat board" feature which is suitable for students to chat with each other without the lecturer present.

- **Q & A Centre**

  Students can use this centre outside their regular tutorial time to ask questions about any aspect of the course work. They can also submit their assignments on a whiteboard in this centre. It has the advantage that the assignments are marked within 24 hours, so the students get feedback almost straight away.

- **Message Centre**

  The formal course material is available here through web links. There is also an email feature connecting the class members.

- **Archive Centre**

  All "Live Chat" sessions are archived, along with other entries selected by the lecturer. This is proving to be a very valuable aspect of NetTutor. Boards can be very quickly and informally set up on problems that are of common interest to the whole class. Students can browse through these archives in their own time and at their own pace.

## Problems and Possible Solutions

- **Unreliable Connections**

  In the pilot year 2001, the most frustrating aspect of working with NetTutor was the bad connections the students and I regularly encountered during the live tutorial sessions. Many dial-up ISPs do not provide a quality, continuous connection, and having a dedicated or permanent line, such as jetstream, is very much an advantage. This year disconnections and slow connections are still a problem at times, but overall the quality of the connections has greatly improved.

- **This is a time-consuming course to run properly**

  The amount of time that it takes to *effectively* run the distance course is huge and out of all proportion to the money it generates in fees. It does, however, create a great deal of goodwill and provides a valuable service to our most able students. (With the decreasing numbers of honours students in mathematics, departments might well feel the need to put extra resources in at this level towards these top students. It is a matter of priorities.)

I have chosen to use the "Live Chat" option in tutorials because this *allows the students to do the mathematics.* Two or three students can comfortably work through set problems without the boards getting too messy.

NetTutor does have a "Live Tutorial" option which would enable a tutor to teach larger groups at any one time. In such a session the tutor can develop work on the whiteboard but the students cannot participate except to ask questions via a "Chat" board.

- **This is an expensive course to run**

  A single classroom NetTutor licence for a year costs just over $4,000 (NZ). A more robust version of NetTutor has been released but at $35,000 per year for the cheapest locally based option (though multi classroom), it is much more that we are prepared to pay out for just one rather small class. (There is a cheaper option where NetTutor is hosted through Links-Systems International in Florida rather than through a local server.)

  However, the University sees the distance course as an effective way of promoting Canterbury and attracting top students, and is now underwriting some of the costs.

Before the start of the course, workshops are given in the students' home areas to teach them how to use NetTutor, and to give them an opportunity to meet and work with the course tutor. A mid-year workshop is held on campus as well. Students particularly enjoyed the opportunity to meet the other students in the course.

It is important that I make personal contact with the students at the *beginning* of the course. The students are more at ease during the weekly sessions if they can put a friendly face to their tutor and if they have had a trial run on NetTutor with me present.

What the Department is doing has been received extremely well by the schools, and will cement links between schools and Canterbury University. Schools, however, do need a couple of years to develop programmes that have this course as a natural goal.

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# STUDENTS' DERIVATIONS OF MODELING FUNCTIONS BASED ON NUMERICAL PATTERNS OF RATES OF CHANGES[1]

KAROLINE AFAMASAGA-FUATA'I

This paper discusses a research study undertaken with Samoan university mathematics students to examine their strategies for deriving modeling functions based on numerical patterns with rates of changes in contrast to the equation-graph matching approach prevalent in schools. Students involved were final year students some of whom were practicing teachers of mathematics or were intending to teach. Students had already examined the cases of linear, quadratic, cubic and some exponential functions and were requested to extend their projects to quartics, other exponential functions and a trigonometric (sine or cosine) or logarithmic function. This paper discusses similarities and differences between students' strategies for quartic and exponential modeling functions only. Data collected showed that students' general formulas were in terms of $n^{th}$ differences and ratios obtained by generalizing numerical patterns. The paper also offers some implications for enhancing the teaching and learning of these functions in schools.

## Introduction

The traditional teaching of functions in Samoan schools, in general, is mostly algebraic and graphical (Afamasaga-Fuata'i, 2003). For example, students mostly learn to recognize function types by inspecting the structure of equations and shapes of graphs. In contrast, the identification, characterization and derivation of modeling functions based on numerical values, beyond the first degree are rarely practiced. In Samoa, it is not until after first year courses that students then have a formal method of deriving polynomials (degree $n \geq 2$) by solving a system of up to n-equations. This approach is different from the author's rates-of-changes research which explores how students use numerical patterns of rates of changes to classify and derive modeling functions. Findings from such research can contribute significantly to the development of enrichment curriculum materials for the teaching of functions in schools. Students involved in the study reported here had already taken first and second year courses in calculus and linear algebra. In this paper, a *modeling function* is defined as a function that collocates at all given points or one that best models a given set of data values, *rates of changes* refer to rates of changes of differences of consecutive y-values for constant changes in x-values whilst *strategies* generally refer to different methods, constructions and schemes students use to solve a problem.

From past experiences in teaching first year calculus, the researcher found that students often struggle to understand the conceptual development of limits particularly in how a simple ratio of change evolves to that of a limit of a ratio of change. However, this difficulty is usually resolved to some extent when numerical values are used to illustrate the variation of the ratio ($\Delta y/\Delta x$) as $\Delta x$ approaches zero. But in spite of this understanding, proving limits and continuities of a function formally in terms of $\varepsilon$-$\delta$ definitions is still problematic. It seems that formal notations compound student difficulties rather than enhance them. In this case, the concepts of $\varepsilon$-$\delta$ are relatively easier to understand when actual numerical values are used. Tall (2000) promotes the use of technology as a valuable tool "in encouraging experimentation which helps, before any formal theory is developed, to give a sense of the given phenomenon and suggest what kind of properties are involved" (p.212). Thus, in order to enrich Samoan students' conceptualizations of functions beyond the prevalent algebraic and geometric views, the researcher argues that the use of numerical values (and patterns) should be promoted as springboards for characterizing and deriving functions. The practice of seeking out numerical patterns can highlight the *co-variation* nature of functional relationships and reinforce the necessity for *variables* as labels to represent co-varying sets. Using characteristic numerical patterns would provide students with an efficient means of categorizing functions and deriving their equations. Furthermore, the *practice* of generalizing

[1] This research was funded by a grant from the Institute of Samoan Studies at the National University of Samoa, Samoa.

numerical patterns can promote the development of critical skills that facilitate statistical and scientific reasoning (e.g., conjecturing about modeling functions for data values).

## The Research Problem

The researcher was interested in examining students' use of numerical patterns and computed rates of changes to derive modeling functions, and subsequently identifying how this numerical approach impacts on students' conceptualisation of functional relationships. The researcher was also interested in the types of classroom practices that support and facilitate students' articulations and presentations of their own strategies. Subsequently, the research study was guided by the three focus questions namely: (a) What are *students' characterizations of patterns of rates of changes* of polynomial and transcendental functions? (b) In what ways could the rates of changes be used to *derive modeling functions*? (c) In what ways are students' general formulas different from or similar to *Newton's formula* for collocation polynomials for equally spaced x-values? For the purpose of this paper, only the quartic and exponential (Type: $y = A(1+r)^x + B$) sub-projects data will be presented and discussed. The rest of the data from the trigonometric and logarithmic sub-projects will form the focus of a subsequent paper. These two sub-projects required students to extend their investigations to trigonometric and logarithmic functions in order to identify their respective characteristic patterns for rates of changes and then to determine how these can be used to derive equations for the appropriate modeling functions. The researcher was specifically interested in how students coped with these two contexts given their findings with polynomials and exponential functions.

## Background to the Research Study & the Study

The research reported here for the National University of Samoa (NUS) students was built upon one conducted at Cornell University with four secondary and university students in 1991. The original rates-of-changes study examined students' actions and strategies in solving six contextual problems that could be modeled by quadratic functions (Afamasaga-Fuata'i, 1998a, 1998b, 1998c, 1994). The same set of contextual problems was given to a group of undergraduate mathematics students with an additional request that they also undertake a *project* at the end to examine cubics and exponential functions (Type: $y = AR^x + B$). Students were not given contextual problems for the project. Instead, they were asked to identify characteristic patterns for each function type using their own self-generated data and to suggest their own contextual problems that could be modeled by cubics and exponentials to verify their findings. *Contextual problems are defined as application problems with realistic settings.*

Using a teaching experiment approach in a class setting, students were asked to solve each of the six initial problems by generating their own values based on their interpretations of the given problem context. Their task was to eventually derive an equation that would predict one quantity in terms of another. The first contextual problem was the following. *"Farmer Joe found that over the years the average yield from his plantation of 25 avocado trees was 500 avocadoes per tree. Later on he planted more trees. He then found that the average yield decreased by 10 every time a new tree is planted. If he is to maximize the total yield from his plantation, how many additional trees should be plant?"* Students solved the problem by starting of with a few values, and then iteratively worked out columns of numbers until they observed a maximum value. Because of the stronger mathematics background of the NUS group of students (in contrast to the secondary students of the 1991 study), the NUS' students readily obtained general formulas for predicting total yield (y) in terms of additional trees (x) from which they determined maximum value thereby confirming their iteratively generated table value. However, the challenges were for students to identify characteristic patterns for the rates of changes of first ($\Delta y$) and second ($\Delta^2 y$) differences from the data, and then determine how these could be utilized to derive equations of functions. Subsequently, through their explorations of differences and observations of the impacts of varying given initial conditions on base line patterns, students concluded that unlike linear functions with constant $\Delta y$, the second type (identified to be quadratic based on the total yield equation), had constant $\Delta^2 y$ but linear $\Delta y$. They also proposed actual formulas to predict coefficients A and B of the modeling function ($y = Ax^2 + Bx + C$) in terms of $\Delta y$ and $\Delta^2 y$ such as $A = \Delta^2 y/2$ (where $\Delta x = 1$) and $B = E - A$ where E is the constant term in the equation $\Delta y = (\Delta^2 y)x + E$ which describes the linear $\Delta y$ values in terms of x, by successively trying out different numerical combinations. They also suggested a second formula for A; that it was equal to the coefficient from $\iint \Delta^2 y dx dx$ . These general formulas were subsequently tested for each contextual problem. To further challenge students, initial conditions were deliberately varied, and students had to conjecture and then confirm how the variations impact on general equations for $\Delta y$ and coefficients A and B.

One of the classroom practices introduced during the study was the requirement that students present their solutions to the rest of the group first before meeting with the researcher on a one-on-one basis. During these presentations, the researcher was particularly interested in multiple, workable methods that were general and applicable across similar situations, and were appropriate, original and elegant in their general form. Cycles of negotiations and re-negotiations became a normal part of the routine during class presentations. In practice, the researcher and rest of students critiqued and challenged the presented information whilst the presenter sought to justify and verify the correctness of his/her work. Given that the teaching experiment approach used was fundamentally different from that normally encountered in most traditional mathematics courses, it was expected that students would initially struggle to communicate their mathematical ideas both orally and in written form. However, with time and practice over the semester, students quickly adapted to the *newly* established classroom practice of justifying, clarifying, and verifying mathematical solutions publicly to others.

At the end of their quadratic investigations, students undertook a project to explore cubics and exponential functions of the form $y = AR^x + B$ (Afamasaga-Fuata'i, 2001). This time, students were not given contextual problems. Instead they generated their own values from which they suggested general formulas for deriving the coefficients in $y = Ax^3 + Bx^2 + Cx + D$ and $y = AR^x + B$. In addition, they were required to eventually design their own contextual problems to confirm the correctness of their findings. It was found that students extended their quadratic findings to conjecture about cubics. For example, they confirmed that for cubics, their third differences $(\Delta^3 y)$ are always constant, $\Delta^2 y$ are linear whilst $\Delta y$ vary quadratically with x.

For all students, the cubic project provided an opportunity to confirm their emerging conceptualization of polynomial functional relationships in terms of characteristic patterns for n$^{th}$-differences $(\Delta^n y)$. In particular, linear functions have constant $\Delta y$, quadratics have constant $\Delta^2 y$ compared to constant $\Delta^3 y$ for cubics. Furthermore, their quadratic problem solving experience suggested that it must be possible to convert these characteristic patterns into general equations to derive some of the coefficients. Subsequently, after trialling different combinations students eventually identified and confirmed that the possibilities include $A = \Delta^3 y/6$ and A is the coefficient from $\iiint \Delta^3 y\,dxdxdx$. They were also becoming aware that the coefficient of the $x^n$ term was directly related to the difference that is constant but were as yet unsure of the exact relationship. For coefficient B, some students suggested that $B = (\Delta^2 y_0 - \Delta^3 y)/2$ while the rest resorted to solving simultaneously for B and C after obtaining A, B and D. One student proposed a rather cumbersome general equation for C in terms of differences but it failed to predict accurately. Thus in spite of their success to derive all of the two coefficients for quadratics in terms of differences, students were unable to do the same for all three coefficients of cubics.

For exponential functions $(y = AR^x + B)$, students found that their explorations of rates of changes failed to find any constant values. This was the first anomaly to their tentative conceptualizations of functional relationships up to this point and students were unsure of how to cope with it. During discussions, it was suggested that perhaps they should attempt ratios of consecutive y values this time instead of differences. Subsequently, they found that the ratios of y-values $(y_{n+1}/y)$ were not exactly constant but the ratios $(\Delta y_{n+1}/\Delta y_n)$ were. They proposed and confirmed that $R = \Delta y_{n+1}/\Delta y_n$, $B = y_x - (\Delta y_0/R - 1)R^x$ whilst $A = \Delta y_0/R - 1$. However, their investigations showed that when B is zero, the constant ratio was $(y_{n+1}/y)$ not $(\Delta y_{n+1}/\Delta y_n)$. Thus in the case of $y = AR^x$, $A = y_0$ and $R = y_{n+1}/y_n$.

By the end of the cubic and $y = AR^x + B$ projects, students had demonstrated that their conceptualizations of functional relationships included constant $\Delta^n y$ values for polynomials of degree n, the coefficient of the $x^n$ term may be derived two different ways, one by dividing the constant $\Delta^n y$ value by an appropriate constant where $\Delta x = 1$ and the other by choosing the coefficient of the $x^n$ term from the integration of the constant $\Delta^n y$ value n-times with respect to x. In contrast, exponential functions $(y = AR^x + B)$ do not have constant differences; instead they have constant ratios of consecutive first differences. Thus far, it appeared that, students were not challenged sufficiently to pursue how coefficients C and D of cubics could be derived completely from differences to complement the usual algebraic simultaneous approach. Furthermore, they did not have a unifying general method for exponential functions. Therefore, from the researcher's view, it would be interesting to challenge the students with a third variation (Type: $y = A(1+r)^x + B$) and the next higher polynomial $(y = Ax^4 + Bx^3 + Cx^2 + Dx + E)$. The progression to a higher polynomial and a more complex exponential functions was a deliberate attempt to challenge students' established schemes and to encourage further explorations on the usefulness of differences in deriving coefficients of modeling functions. Subsequently, eight undergraduate students undertook the quartic and exponential (Type : $y = A(1 + r)^x + B$ ) sub-projects (reported in this paper) already equipped with viable schemes to derive linear, quadratic, cubic and exponential (Type $y = AR^x + B$ ) modelling functions. Some of the specific questions guiding the sub-projects were: (1) What are the characteristic patterns of differences for quartic (or exponential Type : $y = A(1 + r)^x + B$) functions? (2) How do you determine each of the coefficients in terms of differences or other means?

## Theoretical Perspectives

Since the examination of students' strategies necessitate analyses of both individual student's constructions and the group as "the classroom community appropriating a different practice of learning mathematics," it was fitting that the study's guiding theoretical framework should be a combination of the constructivist and socio-cultural perspectives as suggested by Cobb (1994), Bereiter (1994), and Cobb & Bowers (1999). For example, Cobb (1994) argues that "*mathematical learning should be viewed as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society*" (p. 13). For example, it was vital that each student explored and formulated his/her strategies for deriving modeling functions individually before their constructions are presented for public scrutiny and critique. The requirement for presentations demanded that students explain, argue, justify and verify their work. The critiquing students on the other hand, seek to challenge the workability of presented strategies and determine if possible the existence of counterexamples. This interaction is reflexive and cyclic until presented strategies are validated and declared workable or otherwise, by the entire group. Thus, the pragmatic stance by Cobb & Bowers (1999) provides a theoretical framework guiding the study particularly as it investigated ways "of proactively supporting and organizing students' mathematical (strategies)" (p. 9) and reasoning within a classroom setting. Through the ensuing interactions and discussions after the presentation, each participant tries to understand the work that was presented whilst simultaneously critically assessing it for inconsistencies and contradictions. Where appropriate, the presenter agrees to revise/modify general methods if counterexamples are found. When a general method is declared acceptable and viable by the group, it is added to one's pool of knowledge for deriving modeling functions. Knowledge in this study is hence viewed as *"an entity that is acquired in one task setting and conveyed to other task settings"* by an individual who is participating in and contributing to the development of the mathematical practices established by the classroom community (Cobb & Bowers 1999, p. 5).

The study focused on individual student's reasoning with numerical values and patterns and how these are used to construct general methods applicable to exemplars of a particular family type. In this case, reasoning is viewed as *"an act of participating in communal mathematical practices"* whilst learning is a "*process in which students actively reorganize their ways of participating in classroom practices. ... individual student's reasoning and communal practices is viewed as reflexive in that students contribute to the evolution of the classroom practices that constitute the immediate social situation of their mathematical developments as they learn*" (Cobb & Bowers 1999, p. 9). In this study, mathematical classroom practices supportive of a teaching experiment approach were established earlier in a previous semester, and through those experiences, students' participation would vary and subject to successive renegotiations of meanings in a social setting during presentations. It is assumed that *"students reorganize the mathematical beliefs and values that constitute their mathematical disposition as they participate in the renegotiation of socio-mathematical norms"* (Yackel & Cobb, 1996). Hence, there would be qualitative differences in student' strategies over time and as a result of cycles of renegotiations and critique from others but eventually converging to more or less finalized forms.

## Methodology

A teaching experiment methodology was adopted for the study so that the researcher could learn about, understand and explore students' strategies (Steffe & D'Ambrosio, 1996) particularly as the researcher was interested in the extent to which students' strategies are generalized, transferred, and applied to similar/new situations.

**Procedures of the Study** - The study period spanned two semesters. Since this study was a continuation of previous work that originated from given quadratic contextual problems and then progressed to cubics and exponentials, and then extended further to quartics and more exponentials, students were expected to generate their own data values to answer the focus questions. They chose to select known functions and then worked backwards from the rates to derive the original known functions. Students were also required to design their own contextual problems to confirm the viability of their proposed strategies. During the study period, students submitted completed, written reports to the researcher. The marked scripts including researchers' comments were returned to students for further revisions. Normally, comments challenged students' proposed solutions and requested more clarifications and justifications if there were anomalies and inconsistencies. Students' seminar presentations and one-on-one student interviews were scheduled later on in the study period. The one-on-one interviews with the researcher provided another opportunity for students to further explain and justify strategies for all sub-projects. The study was completed with the eight students submitting their final research report for grading.

**Criteria for Assessing Students' Strategies** - Two sets of criteria were used to classify students' solutions throughout the study when commenting on written work, during group and one-on-one presentations. The first one distinguishes between types of solutions (i.e., different, efficient, sophisticated and acceptable) as suggested by Cobb & Bowers (1999, p. 9) and is useful only as a means of describing the qualitatively different solutions proposed by students throughout the study. The second set is that proposed by Hashimoto & Becker (1999) for judging students' solutions in an open approach to teaching mathematics. For example, the researcher was interested in the number of distinct solutions or approaches used by students in their solutions (*multiplicity*), the number of different mathematical ideas used (*flexibility*), evidence of insightful observations (*originality*), and the extent to which the student expressed thinking in mathematical notation (*elegance*). These sets of criteria were considered appropriate for analysing students' project work particularly as students had already been initiated into the type of individual engagement and level of investigation expected of them within a classroom teaching experiment setting.

## Data Collected & Discussions

The data collected included students' final research reports, at least two versions of each sub-project including researcher's comments, video-tapes of students' seminar presentations and transcripts of end of semester one-on-one interviews.

**Quartics - Characteristic patterns of differences**. On the basis of their previous work with lower polynomials, all eight students conjectured that $\Delta y$ of quartics should behave like cubics, $\Delta^2 y$ like quadratics, $\Delta^3 y$ as linear functions whilst fourth differences ($\Delta^4 y$) should be constant. To confirm these conjectures, students systematically examined the impact of selectively varying the coefficients of known quartics upon their respective computed differences. By inspecting trends of differences graphically, identifying numerical patterns, and comparing patterns to previous findings, they confirmed the correctness of their initial conjectures. Sipa, a secondary mathematics teacher, was the only student that derived general equations to predict $\Delta^n y$ of quartics for any x-value in terms of lower differences, see Column 1 of Table 1. This was achieved by systematically pairing up each of the x-values from 0 to 5 with corresponding $\Delta^n y$ values to form two columns (e.g., x vs $\Delta^3 y$, x vs $\Delta^2 y$, and x vs $\Delta y$ ). With the new pairing, Sipa re-applied his cubic, quadratic and linear findings to derive the general equations shown in Column 1 (Table 1).

**Quartics – Derivation of modeling functions.** By inspecting various numerical patterns between known quartic coefficients and respective computed $\Delta^n y$ values, students proposed a variety of general equations for each of the 4 coefficients A, B, C and D whilst constant E was predicted to be the same as the initial y-value, $y_0$. All eight students, as they did in their cubic project, proposed two methods for coefficient A. The first one was: $A = \Delta^4 y/4!$ whilst the other equalled the coefficient of the $x^4$ term in $\int\int\int\int \Delta^4 y dx dx dx dx$. Students also noted the correspondence that seemed to exist between difference equations and derivative functions of each function. General equations, on the other hand, for coefficients B, C, and D were fundamentally generalized forms of numerical patterns that exist between computed differences ($\Delta y$ up to $\Delta^4 y$) and actual coefficients of quartics, see Column 2 of Table 1 for the range of general equations proposed by the students. Each of these equations were duly illustrated, verified and confirmed with appropriate examples. Students' written scripts and interview comments indicated that they were successful in transferring and effectively applying relevant findings from cubic functions to the new context of quartics.

Students were fairly confident at this stage that, for polynomials (degree n), the generic methods for the coefficient of the highest power of x would be given by $A = \Delta^n y/n!$ or the coefficient of the $x^n$ term in $\int\int\int ... \int \Delta^n y dx dx ... .dx$. Column 2 (Table 1) clearly shows that students' general equations for B, C and D were *similar* in that they are all expressed in terms of initial differences ($\Delta^p y_0, 1 \leq p \leq 3$) and constant $\Delta^4 y$, and that they represent acceptable formulas. However, their *differences* are reflected in their algebraic structure, level of sophistication and elegance in terms of mathematical notation used, which collectively impact on the equations' efficiency (Cobb & Bowers, 1999; Hashimoto & Becker, 1999) to compute the respective coefficient. In spite of challenging students to come up with more simplified and elegant versions for their general formulas, they chose to submit their findings as tabulated in Table 1 (Column 2). The last equation for B, C, and D in each box represent the most reduced form. As a group, students were motivated enough to propose general equations for all four coefficients (unlike the cubic project) in terms of differences. Those who resorted to simultaneous methods for coefficients C and D were unable to come up with general equations that work for all exemplars of quartics. The underlying *similarity* in problem solving activities of students was in the successive cycles of "identifying numerical patterns, generalizing numerical patterns into equations, testing equations, revising equations if necessary, and verifying its accuracy in predicting correct values" until a workable combination was obtained. Their individual engagement in this process helped prepare students for the peer and researcher critiques that were part of their public presentations and written work respectively.

***Table 1***

*Summary of: (1) Sipa's $\Delta^n y$ General Equations, & (2) Students' General Equations for Coefficients B, C & D of Quartics:* $y = Ax^4 + Bx^3 + Cx^2 + Dx + E$

| Sipa's $\Delta^n y$ General Equations | Students' General Equations for Coefficients |
|---|---|
| $\Delta^3 y = (\Delta^4 y)x + \Delta^3 y_0$ | $B = \frac{1}{6}\left[\Delta^3 y_0 - \frac{\Delta^4 y}{2} - \Delta^4 y\right]$, $B = \frac{\Delta^3 y_0 - (3!)^2 A}{3!}$,<br>$B = \frac{\Delta^3 y_0}{3!} - (3!)A$, $B = \frac{2\Delta^3 y_0 - 3\Delta^4 y}{12}$ |
| $\Delta^2 y = \frac{\Delta^4 y}{2}x^2 + \left[\Delta^3 y_0 - \frac{\Delta^4 y}{2}\right]x + \Delta^2 y_0$ | $C = \frac{1}{2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y\right] - \frac{\Delta^4 y}{24}$<br>$C = \frac{\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y - 2!\,A}{2!}$,<br>$C = \frac{12\Delta^2 y_0 - 12\Delta^3 y_0 + 11\Delta^4 y}{24}$ |
| $\Delta y = \frac{\Delta^4 y}{6}x^3 + \left[\frac{\Delta^3 y_0 - \Delta^4 y}{2}\right]x^2 + cx + \Delta y_0$ | $D = \Delta y_0 + \frac{1}{2}\left[-\Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y\right] - \frac{\Delta^3 y_0}{6} + \frac{\Delta^4 y}{4}$<br>$D = \frac{2\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y - 2!B}{2!}$<br>$D = \frac{12\Delta y_0 - 6\Delta^2 y_0 + 4\Delta^3 y_0 - 3\Delta^4 y}{12}$ |

**Quartics - Newton's formula for collocation polynomials** – Students' general equations for coefficients A, B, C and D correctly derived modeling functions just like Newton's formula. Applying Newton's formula to the values students used (i.e., $x_k = \{0,1,2,3,4,\ldots\}$ and $k = \{0,1,2,3,4,\ldots\}$ where $h = 1$), students' general equations for A match exactly that from Newton's formula. Other exact matches are the most simplified general equations for B, C and D in Table 1 (Column 2, last entry). In comparison, differences that do exist between the other versions of students' general equations and corresponding Newtonian coefficients are to do with the level of sophistication and elegance of expression in the mathematical equations. All of the eight students had not encountered Newton's formula for collocation polynomials before. Therefore, for them to produce the kind of results reported here, entirely based on generalizing characteristic numerical patterns in spite of their mathematics background, indicate that with the right supportive practices in a classroom setting students can be motivated to explore mathematical concepts beyond our expectations (Tall, 2000).

**Exponentials - Characteristic patterns & General Equations**. Students predicted the characteristic pattern of $\Delta^n y$ values to be describable by exponential functions, based on their previous project with Type : $y = AR^x + B$, confirmed by "x vs $\Delta^n y$ "graphs and justified by pointing out the correspondence between derivatives of functions and differences of functions as previously observed in the case of polynomials. Unlike polynomials, $\Delta^n y$ values of exponential functions never achieve constant values. However, ratios of consecutive $\Delta y$ values, like Type: $y = AR^X + B$, are always constant; that is, "$\Delta y_{n+1}/\Delta y_n$ behave like constant functions." Students also noted that ratios of consecutive y values; i.e. $y_{n+1}/y_n$ are not always exactly constant thus they prefer to use the former ratio in the derivation of coefficients, see students' range of general formulas in Table 2. These findings were also justified and verified with actual examples and contextual problems.

Students' general formulas below indicate quite a variety of combinations which use not only initial y and $\Delta y$ differences but also utilized $\Delta y_x$ (or $\Delta y_n$) values at any value of x (or n). This variety would not have emerged if it were not for the supportive classroom practices established earlier particularly in terms of encouraging students' interpretations and representations. Students participated willingly and freely suggested their variations of general equations for quartics and exponentials. The most significant criteria for all suggestions being general equations should correctly predict displayed values and work equally well with all exemplars of a function type. General formulas for deriving coefficient A and ratio R below appear original, sophisticated and elegant which suggest that students are beginning to gain confidence and competence in utilizing differences effectively to derive complete equations of modeling functions.

***Table 2***

*Summary of Students' General Equations for Coefficients A & B and Ratios R & r of Exponential Functions:* $y = A(1 + r)^x + B$ *and* $y = AR^x + B$

| General Equations | | | | |
|---|---|---|---|---|
| | $A = \frac{y_1 - y_0}{r}$, | $A = \frac{\Delta y_0}{r}$, | $A = \frac{\Delta y_0(\Delta y_x)}{\Delta^2 y_x}$ | |
| $R = \frac{\Delta y_{n+1}}{\Delta y_n}$, | $r = \frac{\Delta y_0}{A}$, | $r = \frac{\Delta^2 y_{n+1}}{\Delta y_n}$, | $r = \frac{\Delta^2 y_x}{\Delta y_x}$, | $r = R - 1$ |
| $B = y_0 - \frac{\Delta y_0}{r}$, | $B = \frac{(r+1)y_n - y_{n+1}}{r}$, | $B = \frac{y_0(1+r) - y_1}{r}$, | $B = y_0 - \Delta y_0\left(\frac{\Delta y_x}{\Delta^2 y_x}\right)$ | |

## Conclusions & Implications

In both sub-projects, students' initial conjectures were extensions of their cubics and Type : $y = AR^x + B$ findings, confirmed by graphing and explicitly expressed by general equations as a result of successive cycles of pattern recognition, conjecturing, justifying and verifying using computed differences of exemplars of each function type. In accordance with established classroom practice, they illustrated and verified with examples the viability of their general equations. In the absence of counterexamples, their strategies were considered acceptable solutions. The types of socio-mathematical norms suggested by Cobb & Bowers (1999) provided a useful means of describing students' solutions. For the researcher, it was important during presenting/teaching episodes that students understood what constitutes a solution and an acceptable justification. In both cases, it was vital that students demonstrated to others that "their solution" works for all function examples and consistent with the rest of their problem interpretations. During both group and one-on-one presentations, students were encouraged to provide multiple methods and appropriate justifications and verifications both verbally and in written form. Students also challenged their peers' solutions particularly if they thought that their own work might be more elegant or different.

Some students proposed strategies that satisfied the criteria of multiplicity, flexibility, originality and elegance. For example, most students showed two distinct methods (multiplicity) for determining A the coefficient of the $x^4$-term of quartics and both methods were elegantly and precisely expressed as formulas in terms of differences (elegance). Students also indicated flexibility in their thinking when they used the descriptive, graphic and algebraic representations to confirm and verify their initial conjectures about the general trends of $\Delta^n y$ values (flexibility). Students' various general equations in Tables 1 and 2 were the results of a number of insightful observations based on identified numerical patterns between computed differences and known quartics/exponentials (originality). Although some of students' general equations in Column 2 (Table 1) and Table 2 could be re-arranged in more elegant and simplified equivalent forms, they, nonetheless, represent schemes that are effective and viable for the correct prediction of values.

Through the sub-projects, students had effectively used, and transferred schemes, knowledge and skills that they had appropriated and found successful from previous situations. As expected, they provided appropriate justifications and verifications of their general findings; they derived their own general equations in terms of rates of changes by reasoning from data values, numerical patterns and computed differences. Through their explorations of numerical patterns, students systematically unpacked inherent characteristics of each function type most relevant to derivations of equations. These practices provided an added dimension to students' conceptual understanding of functional relationships to complement the usual algebraic-graphical approaches prevalent in schools. Although students' general equations were not always exact replicas of Newton's formula for collocation polynomials, their strategies are their own and they had demonstrated that for the purpose of the research project, their general equations were just as successful.

Students' strategies evolved and became general equations that work, partially because of the supportive teaching experiment approach of the course and mainly because having understood the researcher's expectations of them as students and a member of the classroom community, they were motivated to produce their own methods. Finally, the inclusion of this numerical approach to the teaching and learning of functions can promote an enriched and integrated conceptualisation of functions. The findings of the study also imply that the individual constructivist approach combined with the socio-cultural aspects of criticism, renegotiation and researcher's expectations motivated students to produce their own methods. An area for future research that arose out of this study is to explore further the relationship between difference equations and derivative functions for each function type.

## References


Afamasaga-Fuata'i, K. (1994). Reconceptualising quadratics through contextual problems. In A. Jones, A. Begg, B. Bell, F. Biddulph, M. Carr, J. McChesney, E. McKinley, & J. Loveridge (Eds). *SAMEPapers 1994*. Science and Mathematics Education Papers-1994, Centre for Science and Mathematics Education Research, University of Waikato, Hamilton, New Zealand, 56-95.

Afamasaga-Fuata'i, K. (1998a). Mapping and multiple-representations of contextual problems as alternative ways in learning about functions. *Directions* 20(1): 74-87. Journal of Educational Studies. Institute of Education, University of the South Pacific, Fiji.

Afamasaga-Fuata'i, K. (1998b). Abstract for: Realistic contexts as critical sites for problem solving. In Teaching in new times. *Proceedings of 21st Conference of the Mathematics Education Research Group of Australasia MERGA-21*, Griffith University, Gold Coast, Australia. Pp. 723.

Afamasaga-Fuata'i, K. (1998c). The use of difference equations to predict quadratic functions. In *PRISMCS: Problems, Research & Issues in Science, Mathematics, Computing & Statistics*. Journal of the Faculty of Science, National University of Samoa. 39-59.

Afamasaga-Fuata'i, K. (2001, October). *Rates as identifiers of functions*. Paper presented at the Institute of Samoan Studies Seminar Series, National University of Samoa, Samoa.

Afamasaga-Fuata'i, K. (2003, January). Numeracy in Samoa: From Trends & Concerns to Strategies. A paper presented at the Samoa Principal Conference, Department of Education, Samoa.

Bereiter, C. (1994). Constructivism, socioculturalism, and Popper's World 3. *Educational Researcher,* 23(7):21-23.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher,* 23(7); 13-20.

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspective in theory and practice. *Educational Researcher,* 28(2):4-15.

Hashimoto, Y., & Becker, J. (1999). The open approach to teaching mathematics – Creating a culture of mathematics in the classroom: Japan. In L. Sheffield (Ed). *Developing mathematically promising students* (pp. 101-119). Reston, V.A: National Council of Teachers of Mathematics.

Steffe, L. P., & D'Ambrosio, B. S. (1996). Using teaching experiments to enhance understanding of students' mathematics. In D. F. Treagust, R. Duit, & B. F. Fraser (Eds), *Improving teaching and learning in science and mathematics* (pp. 65-76a). Teachers College Press, Columbia University, New York.

Tall, D. (2000). Cognitive development in advanced mathematics using technology. *Mathematics Education Research Journal,* 27(3):196-218.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education,* 27:458-477.



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# EXPERIENCE WITH TEXAS STYLE TOPOLOGY

BILL BARITOMPA

Students often don't get the chance to do mathematics for themselves. This talk reports on the experience of a "Texas Style" course in Topology for honours students at the University of Canterbury. They worked as a group through material (Definitions, Examples and Propositions similar to basic textbook material) and practiced finding their own proofs and explanations. They discovered the proofs themselves. Textbooks were forbidden. Students took turns presenting and writing up the material

## Introduction

Our honours students at Canterbury get good exposure to a wide range of mathematical topics. However most of the courses are traditional lectures. A course run as a Texas Style seminar, provides a useful complement to their education. The Texas style of teaching topology comes from the teaching style of R. L. Moore who taught at the University of Texas from 1920 till he was 86. My supervisor R. D. Anderson was one of his students, and used this method.

I have used this method on and off over the last 30 years, when there was interest in a topology course. This year there is strong interest. Math 428 meets two hours per week for the second half of the year and has 10 students. I will report on the specifics in my talk, but in this paper, I will set the background, outline the realities of how it is adapted to our situation, and report briefly on past experiences.

## Background

The following extract from [1] explains the pure "Texas Style."

These methods are described by F Burton Jones, who himself was a student of Moore, and himself taught very successfully with a modified version, in [3]:-

> Moore would begin his graduate course in topology by carefully selecting the members of the class. If a student had already studied topology elsewhere or had read too much, he would exclude him (in some cases, he would run a separate class for such students). The idea was to have a class as homogeneously ignorant (topologically) as possible. He would usually caution the group not to read topology but simply to use their own ability. Plainly he wanted the competition to be as fair as possible, for competition was one of the driving forces...
>
> Having selected the class he would tell them briefly his view of the axiomatic method: there were certain undefined terms (e.g., "point" and "region") which had meaning restricted (or controlled) by the axioms (e.g., a region is a point set). He would then state the axioms that the class was to start with...
>
> After stating the axioms and giving motivating examples to illustrate their meaning he would then state some definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened, ' omitted, or partially omitted.
>
> When the class returned for the next meeting he would call on some student to prove Theorem 1. After he became familiar with the abilities of the class members, he would call on them in reverse order and in this way give the more unsuccessful students first chance when they did get a proof. He was not inflexible in this procedure but it was clear that he preferred it.

> When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present his proof. Then the other students, especially those who had not been able to discover a proof, would make sure that the proof presented was correct and convincing. Moore sternly prevented heckling. This was seldom necessary because the whole atmosphere was one of a serious community effort to understand the argument.
>
> When a flaw appeared in a "proof" everyone would patiently wait for the student at the board to "patch it up." If he could not, he would sit down. Moore would then ask the next student to try or if he thought the difficulty encountered was sufficiently interesting, he would save that theorem until next time and go on to the next unproved theorem (starting again at the bottom of the class).

Mary Ellen Rudin, who was also a student of Moore's presents a similar picture [2]:-

> His way of teaching was to present you with things that had not yet been proved, and with all kinds of things which might turn out to have a counterexample, and sometimes unsolved problems - that is, unsolved by anyone, not only unsolved by you. So you had some idea of what it meant to be a mathematician - more than the average undergraduate does today.

## Adaptation to Canterbury

As described above, the method is quite competitive and requires students to be Topologically innocent. This does not apply to our Honours students, who by their third year are quite mathematically sophisticated. Also we have a mix of students interested in the course, ranging from physics and engineering students to those interested in pure mathematics. The course is only a 3 point course that meets for 2 hrs per week for 12 weeks.

However the students do not have a lot of experience working with their peers in a seminar situation. For these reasons the focus is more on developing mathematical communication skills and confidence in finding their own way of looking at things.

The main differences between the pure Texas ideal and what we do in this course are:

The class agrees to not consult other reference material, and tries to look with a fresh view.

Some scheduling of who does what and when is done.

Working together is encouraged.

I sometimes work with presenters prior to the seminar and ask "helpful" questions!

There is an oral exam at the end.

## A First Session

This year's course started on 10 July. Notes stating the basic definition of a topology were passed out. Volunteers were asked to read it, and the class discussed what it meant. Of course for this first session, they had no time to digest or think about the material before hand. Much of the discussion centred on figuring out what the formal definition means intuitively, and also learning how to take it a face value and not be influenced by their prior conceptions with regards to "open" sets say.

## A Typical Session

From past experience, 3 or 4 topics per session might be covered, with groups of two or three students having made OHP's or handouts to explain their proof to the others.

## Problems and Benefits

The main difficulty is that not much material is covered. I would expect that the basic tools of point set topology – bases, sub-bases, products, and quotients would be covered. Some of the 'standard' properties would be met. We might meet Tychonoff's theorem, or have excursions into a bit of set theory.

However in the context of their complete honours program, this is not a great disadvantage, as in the more traditional course they have learned a wide variety of topics. In the past, students do end up more confident in presenting mathematics, as often find that they have their own unique ways of finding proofs.

## References


1. J. J. O'Connor and E F Robertson, http://www-gap.dcs.st-and.ac.uk/~history/mathematicians/moore_robert.html
2. M. A. M. Murray, *Women becoming mathematicians* (Cambridge, Massachusetts, 2000).
3. F. Burton Jones, *The Moore method, Amer. Math. Monthly* **84** (4) (1977), 273-277.



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# THE MISMATCH BETWEEN THE THEORY AND PRACTICE OF OUTCOMES BASED EDUCATION AT TERTIARY LEVEL: TRAINING PRE-SERVICE MATHEMATICS TEACHERS

HAYLEY BARNES

Two fundamental components in the training of pre-service mathematics teachers are; the subject content of mathematics and the methodology of how to teach mathematics. Is it possible that the presentation of these two components could contradict each other resulting in students making little attempt to adopt any new or alternative approaches to teaching mathematics? For example, if the methodology that is being used to present the subject content to the students is not modelling the methodology being taught to the students in training them to teach mathematics, does this affect their ability or willingness to try new approaches? Is it viable to expect pre-service teachers to adopt a learner-centred, outcomes–based approach to teaching mathematics if their own experience of learning mathematics is limited to a more traditional "chalk and talk" approach, even at a tertiary level? This paper advocates that while the theory of outcomes-based education is being proposed in the educational reform taking place in South Africa currently and preached in the mathematics method courses, it also needs to be modelled within the undergraduate mathematic content courses for pre-services mathematics students at tertiary institutions.

## Introduction

In a number of countries worldwide, educational reform has been taking place in various phases over the last few decades as new educational theories are being proposed, the increased focus and utility of technology is being propagated and traditional "chalk and talk" approaches to teaching and rote learning are being critically questioned. In the area of mathematics education, there has also been a surge of interest for some years now in the constructivist pedagogy and in cultivating an approach to teaching mathematics that empowers its students to be independent thinkers with good reasoning skills and a sound conceptual grasp of the subject.

The focus of this paper is on the impact (or lack thereof) of this reform on mathematics education in South Africa and more specifically the undergraduate mathematics training of pre-service mathematics teachers in response to the intended changes taking place. The author is suggesting that although pre-service mathematics teachers are being taught the theory behind outcomes-based education (OBE) and are being "trained" to work with the outcomes set out by the government, use continuous assessment and use more learner-centered and activity-based approaches in their teaching, anecdotal evidence suggests that the philosophy of OBE is not being embraced or modelled within the undergraduate mathematics content courses that these pre-service teachers are required to complete, even though the methods courses may be modelling or preaching this theory to students. Are the conceptions and beliefs about mathematics teaching and learning that students bring with them from their own schooling likely to be challenged if this remains the case? And are students likely to adopt the new approach to teaching mathematics being presented in the methods course if their own experience of learning mathematics remains similar to that which they were exposed to at school for between twelve and fourteen years?

Additional questions then also need to be asked such as:

1. Is it viable to expect pre-service teachers to adopt a student-centred, outcomes–based approach to teaching mathematics if their own experience of learning mathematics is limited to a more traditional "chalk and talk" approach, even at a tertiary level?
2. If pre-service teachers end up teaching mathematics in a school where little change has occurred at a pedagogical level and they themselves have not had much experience in a change at this level in their learning of mathematics, will they in fact reform their approach to teaching to be in line with that of the intended curriculum?
3. Are pre-service teachers likely to change their attitudes and beliefs to the teaching of mathematics based purely on theory and research presented to them during their training?

While it is not within the scope of this paper to provide well-researched answers to all of these questions, it is hoped that through integrating the reflections and experiences of the author with some existing literature in the light of these questions, that further interest and research can be generated to take a critical look at existing undergraduate content courses for training mathematics teachers in South Africa. The university where the author is employed serves as a basis for the authors' experiences and although the intention is not to generalise or evaluate the courses at the particular university, the courses directly involving mathematics will be alluded to in order to serve as a catalyst for further investigation into the potential value of modelling a desired methodology of teaching mathematics within the teaching of the subject content course.

## Outcomes Based Education (OBE)

According to van Niekerk and Killen (2000)[2], OBE can be viewed as a classroom practice, as a theory of education or as a systemic structure for education. As it embodies and expresses a certain set of beliefs and assumptions, it can also be thought of as philosophy of education. One of the people whose ideas on OBE have had a considerable influence on the approach to OBE that the South African government has adopted is William Spady (van Niekerk & Killen, 2000)[2]. Spady defines OBE as:

> ...clearly focusing and organizing everything in an educational system around what is essential for all students to be able to do successfully at the end of their learning experiences. This means starting with a clear picture of what is important for students to be able to do, then organizing the curriculum, instruction, and assessment to make sure this learning ultimately happens. (Spady, 1994, p. 1)[3]

Outcomes based education therefore strives to enable all students to achieve to their maximum ability. It aims to do this by setting the outcomes to be achieved at the end of the process. The outcomes encourage a student-centred and activity-based approach to education and have consequently theoretically redefined the roles of both learner[1] and educator[2] in the process in South Africa. These roles include:

> involving them as participants in the curriculum and learning
>
> ensuring that they accept responsibilities for assessment
>
> that they become lifelong learners who are confident and independent, literate, numerate, multi-skilled, compassionate with a respect for the environment and the ability to participate in society as a critical and active citizen (DoE, 2002)4.

According to the policy, student-centred education also goes beyond only ensuring that all students achieve the set outcomes and accept their new roles. It also responds to the learning styles and cultures of students and builds on their life experiences and needs. Continuous formative assessment is commended and enables the assessment of competence and complex performances. This means moving beyond the use of simply of making use of written tests for assessment purposes so that the assessment of critical outcomes such as teamwork, communication and problem-solving can be done in the context of "real performances" (DoE, 2001)[1].

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[1] The term "learner" is used in South Africa in policies and curriculum documents to indicate students who are still at school. The term "student" usually refers to those studying at tertiary institutions. The terms are used interchangeably in this paper.

[2] The term "educator" has replaced the term teacher in South African policies and curriculum documents and is used interchangeably with the term teacher in this paper

In summary, the theory of OBE has challenged the traditional authoritarian role of the teacher and passive role of the learner in South Africa, is encouraging the development of defined critical skills that strive to create active and effective citizens who are also lifelong learners and is an attempt to shift the focus in teaching and learning from a more rote and recall orientated discipline to one which involves acquisition of knowledge, skills and values.

## Setting the scene

The two main components in the training of pre-service mathematics teachers that are being taken into account in this paper are the specific content course of academic mathematics and the mathematics methodology courses that are compulsory during the training of the pre-service mathematics teachers. While it is acknowledged that the concept of pedagogical content knowledge is an important factor in pre-service teachers' training, further elaboration on this remains beyond the scope of this paper.

For students who are doing a four-year Bachelors of Education degree and who select mathematics or general mathematics[3] as an elective to specialise in, their mathematics related courses consist of: four 50 minute periods of academic mathematics (one of these is a practical period in the computer lab) per week for the first three years and one 50 minute methodology period per week during their second and third years. This undergraduate programme has recently been revised and the fourth year is currently being developed so it is unclear as to whether or not students will be required to complete any additional mathematics related components to the ones already mentioned in the future during their fourth year. The intention though is that the fourth year is mainly made up of practice teaching and a small research project. The total credits therefore that students presently receive over the four years is 64 credits (20 in the first and second years respectively and 24 in the third year) for the academic mathematics content courses and 12 credits[4] for the methodology courses (6 credits per year).

In the academic content courses the main topics dealt with are pre-calculus, calculus (limits, continuity, differentiation and integration), financial mathematics (arithmetic and geometric sequences and series, annuities, mathematical induction, the binomial theorem, permutations and combinations, probability, applications of differentiation and integration to economics), geometry and 3-D vectors. In the methodology courses topics such as assessment, misconceptions, teaching and learning of mathematics and the new revised school curriculum are covered.

Teaching practice is a separate core module that students complete in actual schools for three weeks both in their second and third years and then again for a longer period in their fourth year (as mentioned this is still under construction). Students will mostly have more than one elective though so their teaching practice will not only entail teaching mathematics. During their teaching practice students are assessed approximately twice in their third year and fourth years by lecturers and in their second years by teachers from within the school where they undertake their practice teaching.

## Pre-service teachers' beliefs and conceptions of mathematics

It is obvious from the information above that a far greater emphasis of the course is on the subject content knowledge of students in mathematics as opposed to the methodology course. This part of their training can therefore play a crucial part in shaping not only the knowledge and understanding of the content but also the experiences and hence the beliefs and conceptions of what mathematics is to the students. With the ratio of mathematics content periods to methodology periods being 6:1 over the four years, it must be acknowledged that the content course has far more opportunity to challenge and influence the students' beliefs and conceptions of mathematics.

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[3] Students have the option of choosing mathematics or general mathematics. General mathematics is an easier course which covers less content and is understood to qualify students to teach up to Grade 10 (there are 12 grades in the South African school system) while taking mathematics qualifies students to teach up to Grade 12.

[4] In the South African system, one credit is the equivalent of ten notional hours of study.

As the Cognitively Guided Instruction (CGI) research model developed by Thomas Carpenter and Elizabeth Fennema (1988) below in Figure 1 suggests (found in Romberg, 1992)[7], teachers' beliefs and knowledge of a subject may have a direct impact on their decisions, which in turn could affect the classroom instruction they embark on. To quote John Dussey (1992)[8]:

> The conception of mathematics held by the teacher may have a great deal to do with the way in which mathematics is characterized in classroom teaching. (p. 42)

Hersh (1986)[9] makes the same point:

> One's conceptions of what mathematics *is* affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it... (p. 13)

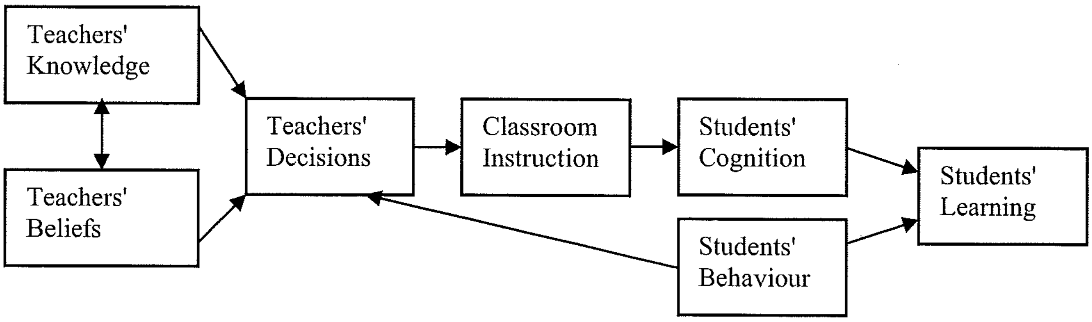


*FIGURE 1.* Cognitively Guided Instruction (CGI) research model

Students' beliefs and conceptions about mathematics are therefore an important element and determinant of how they will teach mathematics as various research has shown (Thompson, 1992[10]; Mapolelo, 2003[11]). But surely one's conceptions and beliefs about mathematics are largely determined through ones' experiences of mathematics (Thompson, 1992[10]; Boaler, 2002[12])? For the majority of mathematics teachers, these experiences will have been mostly gained during their own schooling and their training at a tertiary institution. Once these pre-service teachers start their teaching career, these experiences will be expanded into their own teaching and interaction with students and other teachers as well (which Brown and Borko, 1992[5], refer to as "teacher socialization"). Vast opportunities do not exist within the majority of schools, where pre-service teachers start teaching at, that will cause them to question or change their beliefs and it is notoriously difficult for young teachers to implement new methodologies (Brown & Borko, 1995[5]). This is not to say that schools in South Africa are not implementing the new curriculum but that most changes in the mathematics classes are due to curriculum changes rather than belief or conception changes by the teachers. And although the teaching and learning of mathematics at school level is dealt with in the methodology course, if the students' beliefs and conceptions are not in line with the methodology being presented within this course, and the proposed methodology is not being supported and modelled in the content course, can the methodology course hope to achieve more than merely creating an awareness within students of alternate approaches to the teaching and learning of mathematics?

If one looks at a model of the formal stages that contribute to the development of mathematics experiences that teachers have (Figure 2), the proportion of time that the student has at the tertiary institution is a rather small portion of the "bigger picture" of their experiences. Is it not therefore vital that we should strive to ensure that the experiences we afford students during their training will not simply re-enforce a traditional authoritarian view of teaching mathematics but rather challenge their existing beliefs and conceptions of mathematics teaching whilst also modelling the approach currently being suggested by the curriculum reform?

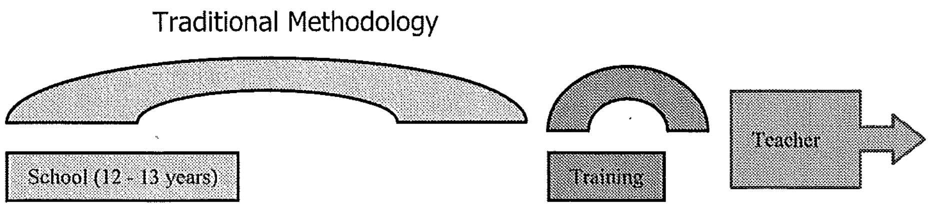


*FIGURE 2. Model depicting formal stages of mathematics experiences of teachers.*

Results of research conducted by amongst others, Wright and Tuska in 1968 (as cited in Brown and Borko, 1992[5]), indicate that early experiences of students (while they are at school) exercise a powerful influence on the images these students have of teachers and teaching that continue to influence pre-service teachers even once they have entered the profession as teachers themselves. Researchers who have studied the source of pre-service teachers' beliefs about mathematics teaching and learning have noted those beliefs are mostly formed during the teachers' schooling years and that they have been wrought by their own experience as students of mathematics (Ball, 1988; Bush, 1983; Owens, 1987, as cited in Thompson, 1992[10]). This research therefore suggests that unless formal teacher education at tertiary level can change these traditional images, pre-service teachers will utilise methods of teaching mathematics similar to the methods their own teachers used (Brown & Borko, 1992[5]).

## Mathematics Education and OBE

The fact that the philosophy of OBE is not being subscribed to by the undergraduate mathematics content courses can be firstly argued from the lack of congruency between Spady's definition of OBE and the approach to the undergraduate mathematics programmes being maintained in the tertiary institution where the author is employed. It is not clear what exactly the undergraduate content courses want the students that take these courses to all "...be able to do at the end of their learning experiences" in relation to the teaching of mathematics.

Secondly, in addition to this the assessment of many of these courses remains mainly summative and does not extend much beyond the usual form of semester tests and examinations, which in turn limits the students' experiences of forms of assessment within mathematics. The content of the assessments and the nature of the items also appear to imply that correct answers regarding the content (such as calculus problems etc) are the focus of these courses. Research however has indicated that while students may be able to find the correct answer, they may not be able to explain why their answers are correct or be able to demonstrate any connectedness of topics in mathematics (Ball, 1990a as cited in Brown and Borko,1992)[5]. From the analysis of her research, Ball argued that:

> ...prospective teachers do not have an understanding of the principles underlying mathematical procedures adequate for teaching. Also, their knowledge of mathematics is not sufficiently connected to enable them to break away from the common approach to teaching and learning math by compartmentalizing topics. (Brown & Borko, 1992, p. 215)

Boaler also deduced from her research that when good test results are the ultimate objective, mathematics students form the belief that their goal is to memorise assorted, unrelated procedures for the purpose of reproducing them on different test questions (Brown, 1981; Boaler, 1997, 2002a as cited in Boaler, 2002). The concern remains therefore that without clear course outcomes that endeavour to offer students a real experience of OBE, strive for conceptual understanding and challenge students' perceptions of mathematics as a collection of "disconnected standard procedures" (Boaler, 2002a; Schoenfeld, 1992 as cited in Boaler, 2002), will our students seek to set such outcomes for their learners one day? As Thompson (1992)[10] states:

> It appears that much about the nature of the discipline is effectively conveyed by the very manner in which instruction in the content of mathematics is conducted. (p. 141)

## Some anecdotal evidence

The source of this paper is rooted in the authors' experiences of observing second and third year students during their practice teaching as well as questioning them and analysing their written responses in the form of journal entries within the methods course. Two incidents from these experiences are reported below as anecdotal evidence of the cause for concern.

One of the students who is presently completing his third year of a general mathematics content class was asked by the author, during the reflection session following an observation of his practice teaching, to give a mathematical explanation on how to determine whether or not the gradient of a straight line is positive or negative. This question was prompted by his explanation to the Grade 10 class that he was observed to be teaching where he had explained to them that a positive gradient could be recognised by the fact that if you were walking along the line, it would be like walking up a mountain so you would feel really positive. On the other hand the negative gradient or slope is like coming down a mountain and one usually feels negative coming down a mountain! The student admitted to not knowing any mathematical reason for differentiating between a negative and positive gradient of a straight line but that he had always relied on his memorisation technique of the mountain to be able to tell the difference. Granted this is an anecdote, but the fact that such a student can be passing an undergraduate mathematics content course with a strong emphasis on calculus while obviously possessing virtually no conceptual understanding or mathematical meaning of the concept of gradient is a cause for concern.

Students in a third year methods class were asked if the calculation in Figure 3 could be performed by dividing the numerators and then dividing the denominators.

$$\frac{21}{35} \div \frac{3}{7} = \frac{^{-}7}{5}$$

*FIGURE 3.* Division of fractions calculation

The immediate response of most of the class was a resounding "no." After doing the calculation their own way though (by multiplying the dividend by the reciprocal of the divisor which for some was a lengthy process), most of the students then noted that the answer was in fact correct. At least half the class were still adamant however that the calculation could not be done using the technique first mentioned even though the answer was correct. When asked to write down why they thought it could not be done that way, the general response was that "we were not taught to do it that way." Students were then further requested to indicate how they would approach teaching the topic to a class. All the students focussed their approach on teaching learners to multiply by the reciprocal when confronted with the division of fractions and without exception, not one of them could produce a mathematically correct reason for why the method they were proposing to teach learners is acceptable and why it works. The most common reason they gave was that division and multiplication are inverse operations and that the second fraction should therefore be inverted. When confronted with the counter example of applying their conjecture to the addition and subtraction of fractions, although aware of the incorrectness in their thinking, students were unable to provide any improvement to their explanations.

These are only two of many examples where students demonstrate their lack of conceptual understanding and their limited, instrumentalist view of mathematics, which Ernest (1988)[6] proposes:

> ...is the view that mathematics, like a bag of tools, is made up of an accumulation of facts, rules and skills to be used by the trained artisan skilfully in the pursuance of some external end. Thus mathematics is a set of unrelated by utilitarian rules and facts. (p. 10)

However, students may continue to hold this view of mathematics as they themselves enter the teaching profession unless we intervene with a view to providing them with a range of experiences in the learning of mathematics that differ from those they were exposed to at school. This in turn will affect the way they approach the teaching of mathematics within their own classrooms.

## Conclusion

Many pre-service teachers in South Africa currently entering tertiary institutions are coming from traditional, teacher dominated mathematics classrooms where they often experienced mathematics as a discipline governed by rules and algorithms. Most of them will have had modest, if any, experience themselves of OBE at school. Guided by the educational reform currently taking place, mathematics method courses are training students to use a curriculum and approach to teaching mathematics that is underpinned by a theory of outcomes-based education. This paper has suggested though that this could have an insignificant effect on the way these pre-service teachers will eventually teach mathematics. It has argued instead that in order to increase the impact of teacher education on the beliefs and conceptions of pre-service teachers, regarding the teaching and learning of mathematics, which will in turn have an effect on the classroom instruction of these students when they become teachers, modelling the proposed methodology in the subject content courses is essential. The pre-service mathematics teachers are currently being expected to employ the underlying principles of OBE once they graduate and start their careers, with little actual experience of being learners themselves within this theory or philosophy of education. Unless teacher education offers students the experience (not only in theory) of an alternate approach to learning (and teaching) mathematics, are we being realistic in expecting their methodology to be in line with envisioned philosophy of OBE once they become teachers?

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## References

1. Department of Education (DoE), *Education in South Africa: Achievements since 1994*, 2001, 6.
2. L J. van Niekerk & R. Killen, *Recontextualising outcomes-based education for teacher education,* South African Journal of Higher Education, **14(3)** (2000), 90-100.
3. W. Spady, *Outcomes-based education: critical issues and answers*, Arlington, VA: American Association of School Administrators, 1994.
4. Department of Education (DoE), *Revised national curriculum statements Grades R- 9 (Schools)*, 2002.
5. C.A. Brown & H. Borko, Becoming a mathematics teacher, In A.D. Gouws (Ed.), *Handbook of research on mathematics teaching and learning* (209-239), New York: Macmillan, 1992.
6. P. Ernest, *The impact of beliefs on the teaching of mathematics,* Paper prepared for ICME VI, Budapest, Hungary, 1988.
7. T.A. Romberg, Perspectives on scholarship and research methods, In A.D. Gouws (Ed.), *Handbook of research on mathematics teaching and learning* (49-64), New York: Macmillan, 1992.
8. J.A. Dossey, The nature of mathematics: its role and its influence, In A.D. Gouws (Ed.), *Handbook of research on mathematics teaching and learning* (39-48), New York: Macmillan, 1992.
9. R. Hersh, Some proposals for revising the philosophy of mathematics, In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (9-28), Boston: Birkhauser, 1986.
10. A.G. Thompson, Teachers' beliefs and conceptions: A synthesis of the research, In A.D. Gouws (Ed.), *Handbook of research on mathematics teaching and learning* (127-143), New York: Macmillan, 1992.
11. D.C. Mapolelo, *Case studies of changes of beliefs of two in-service primary school teachers*, South African Journal of Education, **23(1),** (2003), 71-77.
12. J. Boaler, *Exploring the nature of mathematical activity: using theory, research and 'working hypotheses' to broaden conceptions of mathematics knowing*, Educational Studies in Mathematic, **51(1,2)**, (2002), 3-21.

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# THE LTSN MATHSTEAM STUDY 2003: DIAGNOSTIC TESTING AND STUDENT CENTRED SUPPORT

MIKE BARRY, DOUGLAS QUINNEY, BJOERN HASSLER, RICHARD ATKINSON AND CHRISTINE HIRST

In April 2003 the LTSN MathsTEAM in the United Kingdom published a 3-volume set of booklets: Diagnostic Testing for Mathematics, Maths Support for Students, and Maths for Engineering and Science. These booklets provide a comprehensive collection of case studies, which describe the execution of learning activities, the support needed, the implementation and related difficulties and the evidence of success. For those academics considering the implementation of any of the programmes, each case study is easily readable and provides an opportunity to review the learning processes and the tools involved. This paper considers the diagnostic testing of fresher students, as described in the first of the booklets. It is an adapted and updated version of a similar paper given by the same authors to the 4th IMA Conference, Mathematical Education of Engineers at Loughborough University in April 2003.

The mathematical preparedness of students to undertake degree study programmes, in which mathematics plays an important part, has been a concern for many years now. In addition, there is evidence of the degradation in the quality of highschool educational qualification benchmarks such as United Kingdom A Level grades with the real mathematical knowledge of students becoming increasingly porous. In response to this many universities are coming to rely increasingly upon the diagnostic testing of new entrants as a corroborative assessment of mathematical fitness. In some cases such measures are used, in part, to decide whether a student embarks upon a programme at standard first year level or at "zero-year" foundation level.

## Background to the Survey

Lawson, Halpin and Croft (2001) have already listed strategies that institutions should undertake to address the question of what action to take following the delivery of a mathematics diagnostic test to fresher students. Further details are in Quinney (2001) and Hawkes & Savage (2000). The LTSN MathsTEAM began coordinating the work of the Learning and Teaching Support Network (LTSN) centres for Mathematics, Statistics and Operational Research (MSOR), Physics, and Engineering, and in October 2001 conducted a survey of 91 university departments with regards to diagnostic testing and support; 54 departments responded. Following the survey the authors followed up a sample of these including some that provide both diagnostic testing and follow-up support, and others that provide testing without follow-up support. The sample was chosen to include both paper and computer based tests with visits made to 10 institutions. Each visit included the completion of separate questionnaires by randomly selected students, and members of staff responsible for the diagnostic testing. The findings of the full survey will be published later in 2003 but the early results, depicted in Tables 1 and 2, were included in the recently published LTSN MathsTEAM booklets, (see LTSN MathsTEAM, [A], [B], [C] 2003).

**TABLE 1**
***Universities visited, Type of Diagnostic Test and Follow-up Support Performance***

| University | Type | Support | Personnel |
|---|---|---|---|
| Anglia | Paper | None | BH/DQ |
| Cardiff | Paper | Paper | MB/RA |
| Brunel | Computer | Paper | MB/BH |
| Bristol | Computer | None | RA/DQ |
| Keele | Computer | Computer Assisted | MB/RA |
| Man Met | Paper | None | DQ/RA |
| QMW | Paper | Paper | BH/MB |
| Strathclyde | Paper | None | DQ/BH |
| Sussex | Paper | Paper | BH |
| UMIST | Paper | Computer Assisted | DQ/RA |

Copied from another paper

## The Survey of Staff Responses prepared for the LTSN MathsTEAM

The institutions visited were chosen due to their target intake and the need to cover a variety of testing and follow-up procedures. The types of degree vary from those, which are largely mathematical to those in which mathematics is a subsidiary subject, such as engineering. The survey has illustrated a wide variety of practice, even between institutions that appear in other respects to be relatively similar. The visits were carried out in September/October 2002 as institutional diagnostic testing was taking place. The testing procedures were evaluated by on-site visits and separate questionnaires administered to both staff and students. A large amount of data was collected, and the results of that analysis will be published in the full survey report. Early results are however interesting, see below in Table 2. Note the number of institutions that have invested in writing their own diagnostic test and those who would use a national test if it were made available. Also, it is evident that many institutions actually do provide support for students who perform poorly on a diagnostic test, but this is usually staff intensive and, as yet, the use of CAA in this area is not widely supported. Note too that the apparent decline in student performance is much as might be expected with the changing background, but the awareness of academic staff of what to expect from students is of concern.

**TABLE 2**

**Staff Questionnaire Responses**

| Question | % yes |
|---|---|
| Did the institution write its own diagnostic test? | 77% |
| If a national computer based test were made available would you use it? | 67% |
| Have deficiencies in mathematics worsened over recent years? | 86% |
| Are there any dedicated support facilities available, "walk in" centres, etc? | 67% |
| Is any use made of CAA material for supporting students? | 17% |
| How many academic staff are up to date with key issues at the school/university interface? | 20% |
| Does a staff development programme exist to ensure staff are informed? | 39% |
| Is there a need for national diagnostic testing? | 73% |
| Would you use a national database of diagnostic questions? | 73% |
| If a diagnostic environment were provided which automatically linked to self-study and support, would you use it? | 85% |
| Should such an environment be offered to schools to help students prepare for university studies? | 67% |

Copied from another paper

## The Survey of Student Responses

Over 100 students in the 10 universities were surveyed. As well as seeking general information from them and their attitude to the diagnostic test, they were asked about their prior knowledge of mathematical topics and subtopics listed in accordance with the classification given in Core Zero of the SEFI Core Curriculum (see Mustoe & Lawson, 2002). Students were given an enlarged form of the example table template below with the definitions given and asked to define their knowledge of each subtopic in one of three exclusive ways.

**TABLE 3**

**Student Questionnaire Responses**

| Topic | Subtopic | No | Practice | Acquaint |
|---|---|---|---|---|
| Diff'l Calculus | chain rule | | ✓ | |
| | stationary points | | | ✓ |
| Geometry/Trig | triangle solution | | ✓ | |
| | radian measure | ✓ | | |
| Prob/stats | exclusive events | ✓ | | |

Copied from another paper

**PRACTICE**: Student will have done several examples, even if not perfectly recalled for the diagnostic test. Would have had the confidence to tackle a question on the subject in the examination qualifying to get to university.

**ACQUAINT**: Teacher possibly did a few examples, but student would have been very hesitant to tackle an examination question on the subtopic.

**NO**: Really no knowledge. At best a teacher may have mentioned the idea, or it has been seen in part of a book but not studied.

At the end of the questionnaire, students were asked to give, in rank order, their preferred strategy of follow-up assistance to focus their learning in mathematics. The five options given were as follows:

1 to 1 tuition – paid for at 15.00 British Pounds per hour

use of 'walk-in' centre – with access to help materials and very occasional advice

support classes on diagnostic topics– possibly held at 1 p.m. or 5 p.m. weekly

written tests on diagnostic topics– marked by staff, every 3 weeks

WEB-based computer tests – point/click responses and feedback, always available

Early analysis suggest that support classes seem the most popular and the 1 to 1 tuition at 15.00 British Pounds per hour the least, but one wonders how different that would be were the tuition free.

## Conclusions

There is a general consensus of the need for some form of diagnostic testing of students' mathematical skills on entry to higher education. There are a wide number of approaches all with substantial merit. Having investigated the responses from both staff and student questionnaires the following conclusions are drawn.

Advise students of the diagnostic test and supply revision materials *before* arrival

With their admission information, advise students of the existence of a diagnostic test, and supply examples of the questions together with suitable revision materials, e.g. LTSN booklets. This will make it clear what information is being sought and will enable students to plan their own learning.

Make sure the purpose of the diagnostic test is clearly defined and provide follow up and support.

Not only should you explain to students the precise role you envisage for the diagnostic test at the outset; it needs to be made equally clear that the diagnostic test makes no summative contribution to any part of the degree programme.

Have a clear strategy for remediation. Provide support materials and a mechanism whereby students can assess their progress. (This could simply be repeated attempts at the diagnostic test.) Publish cohort results to enable peer reflection on performance.

## References

13. D. Lawson, M. Halpin, & A.Croft, *After Diagnostic Test – What Next?* MSOR Connections, **1(3)**, (2001), 19-23.
14. D. Quinney, *Diagnostic Testing in Mathematics*, 2001 http://www.keele.ac.uk/depts/ma/diagnostic/.
15. T. Hawkes, & M. Savage, eds, *Measuring the Mathematics Problem*, Report published under the auspices of the Learning Teaching Support Network (MSOR), Institute of Mathematics and its Applications, the London Mathematical Society and the Engineering Council, 2000.
16. LTSN MathsTEAM, 2003. http://www.ltsn.ac.uk/MathsTEAM.
    (A) *Diagnostic Testing for Maths*, ISBN 07044 23731.
    (B) *Maths Support for Students*, ISBN 07044 27358.
    (C) *Maths for Engineering and Science*, ISBN 07044 2374X.
17. 5. L. R. Mustoe, & D. Lawson, Mathematics for the European Engineer, a Curriculum for the Twenty-First Century, SEFI Brussels, ISBN 2-87352-045-0, 2002.

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# NEW FORMS OFTEACHING PROVOKE AND REQUIRE NEW FORMS OF ASSESSMENT

JOSEF BOEHM

In the very early nineties the Austrian Government purchased the general license of *DERIVE* for all Austrian General Secondary Schools. Several people – school authorities and experienced teachers – didn't want to leave the colleagues alone with this great challenge and change in mathematics education (as it has been done when the ordinary pocket calculators had entered school some years before). So the ACDCA (Austrian Centre for Didactics of Computer Algebra) was founded.

Four major projects were started and finished since then:

**CAS I: 1993 - 1994 DERIVE Project**

**CAS II: 1997 - 1998 TI - 92 Project**

70 classes on 44 Secondary Schools, (65 teachers, 680 female and 1570 male students)

**CAS II: 1999 - 2000 2. TI - 92 Project**

Electronic Learning Media in Maths-Teaching - Influence on Teaching, Learning, Curriculum and Assessment (83 teachers, 140 classes)

The first two projects investigated the teachers' and students' acceptance of new teaching methods using technology. They focused on changes in students' learning success, on changes in their attitude towards maths and science, on differences in knowledge and comparing the traditional teaching with the technology supported methods.

Next we tried to find new fields for investigation and research, and decided to work in the following five research areas:

- **Electronic Teaching- and Learning Media**
  Testing and evaluating existing software for using CAS and dynamic geometry in classroom.
- **TIMS Study - Quality Control for Maths Teaching**
  Preparing tests to control the learning success and running the tests in CAS and non-CAS classes.
- **Preparing a comment to the Upper Secondary Level Curriculum with special regard to a CAS-supported teaching**
  Teaching selected teaching sequences according to a recommended process and giving written reports.
- **Influence of CAS on the assessment-situation**
  Trying and observing new forms of assessments with one or more classes.
- **New Learning Culture with CAS**
  Preparing workstations for "open learning" for some selected items, as Direct and indirect proportion, Introducing the function concept, Simultaneous equations, Power- and root-functions, Calculus (differentiating), Preparation for the end-examination.
- **Influence of CAS on the assessment-situation**
  12 teachers (6 female, 6 male; 9 from General Secondary schools, 3 from Colleges for Business Administration) - 16 classes (aged 15 to 17)

I joined the last group, because during the last years when I taught technology supported (TI-92) I felt uncomfortable with the given situation which did in no way consider new and additional competences which we wanted the students to acquire and which couldn´t be assessed in the traditional tests which consisted more or less of recipe following calculations.

The situation until now was and in most cases still is: Fixed number of written tests, oral assessments possible and on demand of the students. Thus no flexibility in posing the task (usually 4 or five problems) and in changing the form of the assessment.

We found some fields worth to be investigated:

(1) "Continuous" carrying on the traditional form of tests including CAS.

(2) Problem solving tests supported by textbooks, notebooks, any materials....

(3) Total assessment time for year can be divided in shorter tests for basics and longer problem solving tests; the assessment times are fixed by the teachers together with the students.

(4) Presentations of selected chapters (problems) - individually or in groups.

(5) One part of the written tests can be substituted by a "project work" (= "Facharbeit")

(6) Cross curriculum written test

(7) Written test as group work

(8) Instead of a fixed number of one-hour-tests have more shorter tests - announced and not announced - to measure the increase of knowledge

(9) "Inner distinguishing" in the assessment situation.

(10) the "Must" / other task(s) for a better mark / extra credits for the gifted.

**My class: IIIc**, 27 not very bright – but cooperating and friendly - students (age 16 - 17).

They formed 7 groups for "Facharbeiten" – presentations (including preparing hand outs and home exercises for the colleagues). The presentations will be graded.

Usually we have two assessments (written tests) à 50 minutes per semester = 200 min /year. Additionally the teachers needs some "notes" about the students to give the final mark. If a student is in danger not to pass the year an oral assessment (15 minutes) is compulsory.

Helmut Heugl – the chair of the ACDCA, school inspector in Lower Austria – made possible an "experimental year" for all colleagues in our group so that we could work outside of the laws with respect to any of the 9 changes in assessment habits given above. We had to thoroughly and detailed explain what we were intending to do and we needed the agreement of students and parents.

We – the students and I - agreed on:

First semester: having 3 short tests (basics, with and/or without the TI-92) and one problem solving test (1 hour).

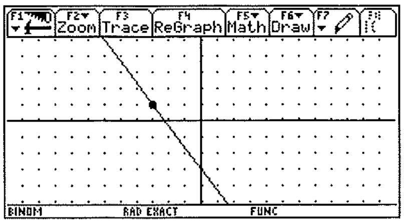
Second semester: 2 short basics tests à 20 minutes and one extended problem solving test with teamwork. One group presentation of a selected chapter

In the following I´ll show how we did in this demanding year.

**Additional Comment**. Resulting from an earlier "Pedagogical and Didactical Project" I allow my students – with a very few exceptions – the use of supporting materials during written tests (textbook, school- and home exercises, private notes, etc)

## Basics Test (15 minutes, without TI, without any additional support)

Give the equation of the represented linear function?




Give the equation of the line which is perpendicular to the given line and passes the origin?

What is the equation of the linear function, if the distance between the grids in horizontal direction represents 10 units?

(1 x 1 grid)

Given are two straight lines g: $6y - 2x - 12 = 0$ and h: $x + y = -2$. Find their intersection point graphically.

How can you decide if two lines are parallel seeing their equations?

Write down the equations of two parallel lines and underline the upper one.

Line g: $2x + 5y = -4$ and line h: $x - y = 5$ and three points: P(-2 / -1), Q(-5 / -10), R(3 / -2)

Find out if the points are lying on g and/or h.

## Basics Test - Power Rules (25 minutes, TI-92 allowed)

a) Write down the result as compact as possible and without denominator:

$$\left(\frac{a^2 b^x}{c^n}\right)^3 \left(\frac{c^{n+1}}{a^{x-1} b}\right)^{-2} =$$

(4)

b) Bring the expressions under one common root: $\frac{2x}{3y^2} \, 3\sqrt{\frac{3y^2}{4x}}$

(3)

c) $\sqrt[3]{3024 a^4 b^6 c^{20}}$; draw the root as far as possible. Describe the process how to reach the result. (4)

d) Calculate with the TI and give reasons for the result: $\frac{(12a^2 b^3 c^3)^4}{(18 a^3 b^2 c)^3}$

(4)

e) $\frac{x\sqrt{y} + y\sqrt{x}}{\sqrt{y} + \sqrt{x}} =$ Explain the calculator's result.

(5)

In parenthesis are the points given which could be reached. As you can see the CAS-calculator is used as an assessing tool. The students need some competence in treating the calculator, but the device doesn't give the (full) answer in all cases. In contrary, it opens new and sometimes unusual questions. A sound knowledge of the power rules is necessary to pass the test

Spread over the year the groups had to prepare their presentations. I provided material (most of it in English) and they had the task to give a vivid and well organized demonstration of the subject including working on blackboard and projection device (ViewScreen, transparencies), preparing handouts and providing examples as home exercise. I underlined that I had not the intention to repeat their chapter in the following classes (not enough time to do it twice) and that their stuff will be assessed as usual in written and/or oral tests.

## The Seven Presentations (Facharbeiten)

Application of a larger system of equations

Logistic growth

Complex numbers - Theorem of Vieta - Fundamental Theorem of Algebra

Repetition and extending the formulae for areas and volumes

Presentation of extended tasks from trigonometry

Introduction into Programming the TI-92

Compound interest - Present value - Future value

I include three sample pages of the presentations:

The first is an application of simultaneous linear equations. Two kinds of bikes must be assembled from several basic parts. The problem was not among the materials provided by me, the pupils invented it for their presentation. Very nice was the home exercise: the task was to produce a combination of three different cakes and it was necessary to find out the right amount of ingredients (sugar, eggs, flour, ....).

The second example is the a transparency which should lead to logistic growth (other growth- and decay models had been part of earlier classes). They started with the discrete model (recursive sequence) and then provided the continuous model (as a Black Box), referring to the prove which would be given later with means of Calculus.

The same group (4 boys) gave a second presentation: "Introduction into Programming the TI-92".

I made the experience that it is very hard – but necessary – for the teacher to remain as much as possible in the background keeping silent. The students are not very happy being interrupted and corrected at any occasion. It was a hard learning process for me, too. But it should be the students' hour(s) and not mine.

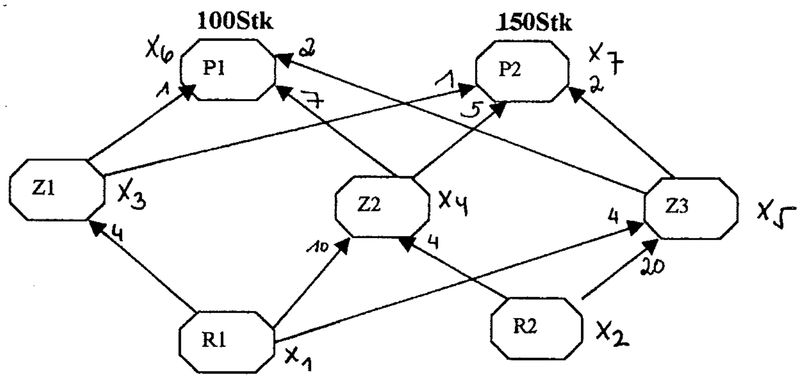
I recorded one presentation with a video camera and we had much fun later on, discussing and analysing this "performance".

. Schritte unterlegen für die Präsentation — Ausarbeitung 1

mehrstufiger Prozess - Verflechtungsmatrix

## Anwendungen des Gleichungssystems (GLS)

Ein Produktionsbetrieb fertigt aus den Rohstoffen **R1 (Schrauben)** und **R2 (Draht)** die beiden Endprodukte **P1 (Mountainbike)** und **P2 (Citybike)**. Bei der Herstellung entstehen die drei Zwischenprodukte **Z1, Z2** und **Z3 (Sattel, Gestänge** und **Räder)**. Die folgende Zeichnung zeigt in Pfeilrichtung gelesen, wieviele Einheiten eines jeden Produktes für die einzelne Herstellung notwendig sind.




*Die Zeichnung muss nun so gelesen werden:*
Man benötigt 4 Schrauben, also 4R1 um einen Sattel (Z1) herstellen zu können. Das gleiche, man benötigt 10 Schrauben (R1) um ein Gestänge (Z2) herstellen zu können,......

Wenn man dann für R1, R2, Z1, Z2,. die gewünschten Einheiten einsetzt, entsteht ein lineares Gleichungssystem:

Wie hoch ist der Gesamtbedarf von R1 und R2?

Stelle dazu die passende Gleichung auf und versuche sie zu lösen!

(1) $x_1 = 4x_3 + 10x_4 + 4x_5$ (Mengen an x1)
(2) $x_2 = 4x_4 + 20x_5$ (Mengen an x2)
(3) $x_3 = x_6 + x_7$ (Mengen an x3)
(4) $x_4 = 7x_6 + 5x_7$ (Mengen an x4)
(5) $x_5 = 2x_6 + 2x_7$ (Mengen an x5)
(6) $x_6 = 100$ (Mengen an x6)
(7) $x_7 = 150$ (Mengen an x7)

Bringe nun das Gleichungssystem in eine ordentliche Form und löse es! alle Unbekannten auf eine Seite!

$x_1 - 4x_3 - 10x_4 - 4x_5 = 0$
$x_2 - 4x_4 - 20x_5 = 0$
$x_3 - x_6 - x_7 = 0$
$x_4 - 7x_6 - 2x_7 = 0$
$x_6 = 100$
$x_7 = 150$

Matrix: von TI

$x_1 = 17500$ Schrauben
$x_2 = 15800$ Alluminium
$x_3 = 250$ Sättel
$x_4 = 1450$ Gestänge
$x_5 = 500$ Räder

## Das kontinuierliche Modell

Für das kontinuierliche Modell gibt es eine explizite Funktionsvorschrift, die sich wiederum aus der Analysis begründet:

$$B\,(x=t) = \frac{K}{1+ (K/B_0 - 1)\, e^{-cKt}}$$

Die Proportionalitätskonstante *p* für kleine Zeitintervalle kann als Näherungswert für die Wachstumskonstante *c* verwendet werden.

Damit ergibt sich mit K = 200, $B_0$ = 35 und c = 0,0015 die Rehformel:

$$B\,(t) = \frac{200}{1+ 4{,}71e^{-0{,}3t}}$$

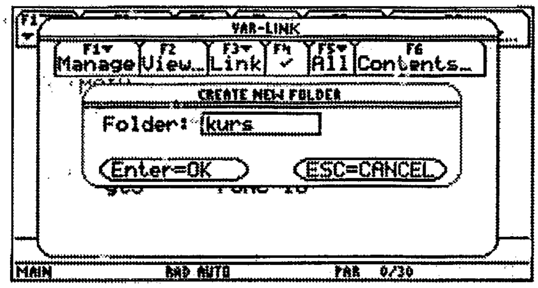
Vergleiche die Tabellenwerte der beiden Modelle!

| n/t | diskret | kontinuierlich |
|---|---|---|
| 0 | 35 | 35 |
| 1 | 43,66 | 44,52 |
| 2 | 53,9 | 55,75 |
| | | |
| 5 | 93,29 | 97,57 |
| 6 | 108,22 | 112,41 |
| | | |
| 10 | 161,44 | 161,98 |

Sample page 3

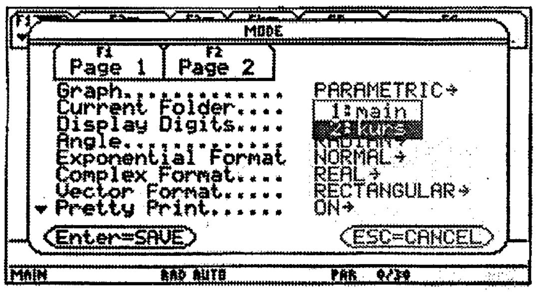
# Einstieg ins Programmieren

## ANLEGEN EINES ORDNERS



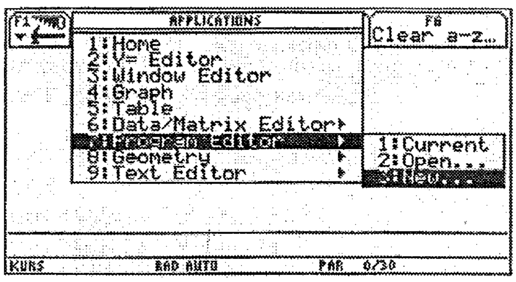

Drücken Sie:
2nd – VAR-LINK
F1 (Manage)
5 Create Folder
Benennen des Ordners
Zweimal Enter

## IN EINEN ORDNER WECHSELN




Drücken Sie:
Mode
Gehen Sie auf Current Folder
Wechseln Sie auf Kurs
Zweimal Enter

## STARTEN DES PROGRAMMEDITORS




Drücken Sie:
**APPS**
**7** Program Editor
**3** New
Geben Sie einen Namen für Ihr Programm ein

Im Bereich zwischen Prgm und EndPrgm müssen die von dem Programm durchzuführenden Befehle eingetragen werden.

**Short test 25 minutes** (including the stuff of one presentation!!)

$z1 = 3 + 4i$, $z2 = 1 - 2i$, $z3 = 4 + i$

Calculate: $z_1 \, . \, z_2 - z_3^2 =$

Given is the set of solutions $L = \left\{3, -1, \frac{1}{2}\right\}$.

Give two equations of degree 5 having L as set of solutions.

For which k have the parabolas a tangent in common:

$$y = k\,x^2 + x - 1 \text{ und } \quad y = -\frac{x^2}{2} + k\,x + 1?$$

Perform the task without and with the TI-92.

Find the solution without using solve():

$36\,x^6 + 132\,x^5 - 263\,x^4 - 638\,x^3 + 926\,x^2 - 770\,x + 1225 = 0$

We had discussed very deep the solutions of equations of higher degree. But I had not posed a question like question (2) before. The students knew about the existence of multiple solutions, and how they appear by factoring the equation polynomial, but I had not given the reverse question. Questions of this kind are important to really test, if – at least some of them – understood the "big ideas" behind.

Then the first "great day" was here: the Problem Solving Test. I was asked over and over: "Sir, is this true that we will have not seen the problems before. Is this really absolutely new for us? How can we manage this? .....".

When I set the problems I had a bit pity with them and started with an introductory example which was not unknown to them, because we had worked through a large range of examples covering exponential growth and decay.

The two following problems were really hard. We had talked a lot about recursive models, but this one was new to them and the third problem was really a problem, because they first had to understand the text and transfer it into a sketch (we are on a business school and not on a technical one!!). Then the sketch forms the base of a GOZINTO-graph which shows the interdependence of the singular parts and finally this graph must be described by a system of linear equations, which then easily can be solved by any device, followed by the final interpretation of the result.

I promised to mark the parts of example (3) separately, i.e. if they misunderstood the text and produced a wrong sketch but if then the graph follows this sketch, then the graph would be regarded as right, and if the system of equations follows from their (wrong) graph correctly, then this step would be regarded as correct, etc.

As you might assume, grading this test was not an easy task!! It turned out as "Problem Solving" for the teacher.

## The Problem Solving Test!!!

A1) The "Warming Up"

A Radium isotope has a half life of 11.7 days.

What will remain of 250g after 3 weeks?

What will remain of any mass after three times half life?

How long do you have to wait until 20g will remain of the given 250g?

A2) You take a loan about 150 000ATS at an interest rate of 5.75%. Try to find a way for finding the repayment after n years. You may assume that there are no extra payments between. (The interest rate is the growth rate of the loan for one year.)

What is the debt after 8 years?

Set up a (recursive) model for the case that we have a payment of 20000 ATS at the end of each year. How many payments are necessary?

Describe your way to find an answer.

What is the recursive equation?

Write down parts of the table or sketch the graph.

(Set the -values nmin und xmin = 0!)

Try to find payments such that you will have paid back the whole exactly after 12 years.

Give a report about your tries. (it is not sufficient to only write down a result!!)

A3) A threepod is assembled: each of the three legs consists of two halflegs, which are put together by 6 bolts. Each of the halflegs consists of two rods, screwed together by 4 bolts. The three complete legs are fixed on a plate using 18 bolts giving the threepod.

Produce a sketch of the threepod. And then set up the Gozinto-Graph.

There is one order: 15 threepods + replacement parts (additional 8 halflegs, 4 plates and 400 bolts). Find the system of equations to fix the production plan.

It was interesting that the very easy "Warming Up" was only solved by three students. The excuse of the class: "We didn't expect well known problems, so we didn't prepare for them!!".

Interestingly for me was the fact, that they performed rather well tackling the "problems". So finally the marks were not too bad and didn't differ from the marks given earlier on traditional written tests.

The first term ended and we (students and I) agreed on 4 questions which should be answered by them

## Questions to the students after the first term:

(1) Are my expectations fulfilled?

(2) What did I like / dislike until now?

(3) Has my attitude towards maths changed?

(4) Are there any changes in success in learning?

(1) 15 yes / 8 no (because most of them had expected to have more group work, but this was not the aim of the project!)

(2) They liked best the presentations, liked the preparation of the presentations working in teams, no single dislike!!!

(3) Yes and No, special comments on the group presentations and their positive effects.

(4) half / half

## 2nd term: 2 short basics assessments, one very short test on formulae

Formulae Test: Volume of a Cone

Surface of a Cylinder

Give a sketch of a segment of a sphere with two bases

Area of a equilateral triangle

etc.

## Basics Test (30 minutes with TI)

1) Triangle with given area A = 153, altitude $h_a$ = 17.3 and angle ß = 27.75°. Find the missing sides and angles.

2) Two masts are standing on a horizontal base plane in a distance of 37m. From the pedal points of the two masts the tops of the respectively other mast can be seen under elevation angles 26.03° and 42.87°.

a) What is the height of the two masts?

b) A rope is stretched tautly from one top to the other. What is the minimum length of this rope?

3) In a quadrangle with a, b, c and d, sides a and b form a right angle: a = 2.82; b = 3.17; c = 4.28; d = 5.12. What is the area of the figure? What is the size of the angle opposite to the right angle?

30 minutes seems to be very short, but it must be said that we developed a package of functions which allowed quick calculations with triangles and we could do without Sine- and Cosine Rule, because we had changed the "White Box" into a "Black Box" and from this moment, the student could focus on the problem and on the strategy how to solve the problem supported by a sketch. The results of this test was fine.

The second "Great Day" of our project approached. The students had been satisfied with the tests, their successes with the presentations and the good mood of us all, feeling a special class moving outside of the law, doing and trying things which nobody else in Austria was allowed to (except some other project classes).

The Group Assessment was unique, because only one colleague from another school joined me with this experiment. Encouraged by a talk given by Marlene Torres-Skoumal at the ACDCA-Summer Academy in Goesing about group assessment I wanted to experience this. We always hear that our society and the companies expect group competence from the students when they leave school, but we very seldom give the chance to train it in school.

The first question of all colleagues (and students, of course) was and still is: How did you form the groups?

I had the choice of two models:

(1) Try to form the groups according to their maths performance (bright to bright and weak to weak). But then I had to set different problems and didn't know how to grade fair in this case.

(2) Mix the groups. But then I had to face the danger that the weaker students in the group will "live" on the work of the others. How to evaluate the work of each single student?

Questions over questions ....

This is how I did:

I told the students that they should form groups of 3 or 4 students and try to distribute their mathematical resources (more gifted students) in the best possible way. I didn't want to have the two brightest mathematicians in one group. They should bring their proposal for approval one week before the Group Assessment. I was free to order changes. They did their best, I didn't accept one group. They had to separate two better students.

The second question is: "Which kind of tasks will you give?"

I answered very sincerely: "Believe it or not, I don't know at the moment. It is for me as challenging, demanding and thrilling as it is for you". And indeed, I didn't know until last evening before the test. Then I set down and after a while it was ready

## The Group Assessment.

26 pupils formed 8 groups ($2 \times 4$ and $6 \times 3$). 50 minutes were extended to 100 minutes.

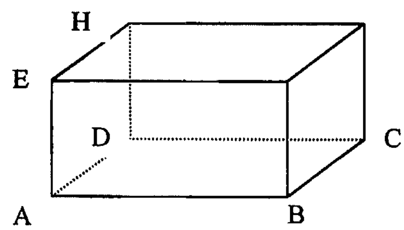
1) Given is a rectangular prism with

AB = a = 7cm

BC = b = 5 cm

AE = c = 4 cm.

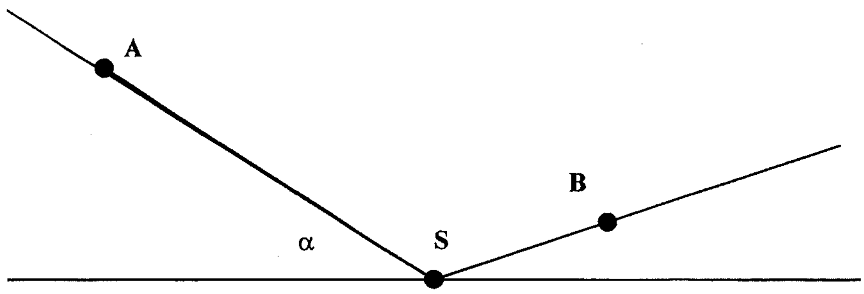
Find the distance from point H to the space diagonal EC.




2) From a point A lying on an incline falling under $\alpha = 32,1567°$ one can see a point B lying on the opposite incline under the depression angle of 22,3462°.

Having moved from A 150m downhill to a point A', you can see B from A' under a depression angle of 14,5883°. A' has a distance of 125m from the bottom of the valley.

In a map you can read off B´s height above sea level with 982m.




a) Neat drawing of the situation in measure 1 : 2500.

b) What is the height above sea level of points A and A'?

c) What is the inclination angle ß of the opposite slope (containing location B)?

d) What is B´s distance of the bottom of the valley?

3) Give a proof using an appropriate sketch:

In each quadrangle with diagonals perpendicular to each other the sums of the squares of two respective opposite sides are equal.

It was very fascinating observing the students organizing the work in their groups. First they read an discussed the problems. Then they tried to distribute the work according to their abilities and preferences. This took much more time than I had expected, but I saw that this was a very necessary and positive aspect. So I decided immediately to extend the working time from one to two hours. No one single student did only copy the other´s work or was not really busy on his part.

For the second hour I asked a colleague – teaching commercial subjects – to supervise the class. When she handed in the assessment papers she was very excited about the working style and attitude the students showed during the test. From this day we had one important voice more in our school concerning our new way in teaching mathematics.

Each group had to deliver one paper. All members of a group received the same marks. Finally only one group had a "5" (= "Not Sufficient") mark in Austria (1 – 5), which was accepted by the students without any claiming about grading too hard.

At the end of the year I asked for the students' opinion to 8 items:

## (1) Separation basic knowledge - problem solving

*positive, problem solving was for some too difficult, "problem solving was easier within the group", "fine, because of the applications in problem solving test", "good, I like to face problems!"*

## (2) Project - Presentation

*very motivating, the students appreciated to work on one item very concentrated using provided materials, "when we received the materials we never thought to be able to manage the task, we asked older students and friends from other schools, nobody could help us. Then we spent one afternoon together and step by step we worked through the materials. Now we are sure that we have a very sound knowledge about that stuff, because we have learnt that by ourselves".*

## (3) Total Year´s time for assessment

*was accepted by all students and very welcome*

## (4) Group test

*the students liked it, many of them wrote down, that they had talked a lot during the test and had a good working atmosphere and that they so were able to find at least partial solutions. They also expressed that they split the work according to the group members´ abilities, i.e. that even in a very short time they were able to organize themselves within the groups.*

## (5) Change in maths teaching

*the students recognized correctly that maths teaching didn´t change very much - that was not the aim of the project. But there were some notes that the lessons became funnier and more interesting.*

## (6) Change of attitude towards maths

*8 wrote that their attitude has improved, "although I am not a good mathematician, the many various ways to tackle a problem were very very interesting, great and motivating for me"*

## (7) Estimation of maths knowledge

*No remarkable comments*

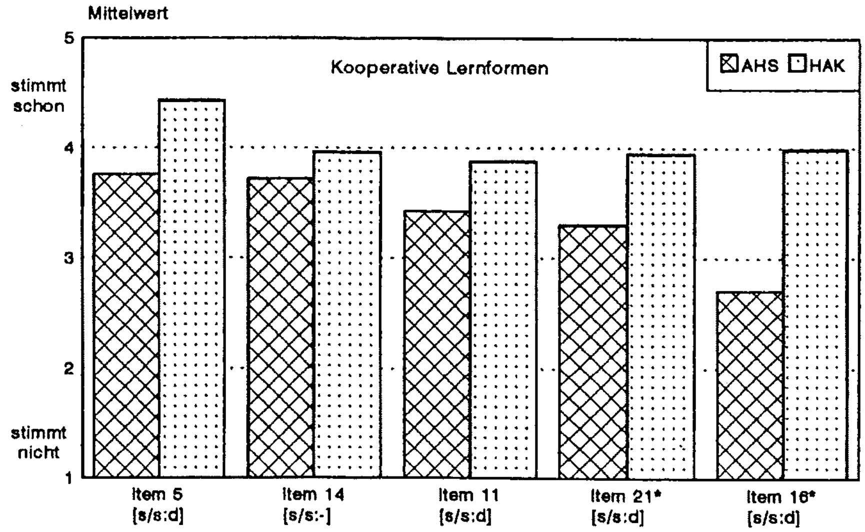
## (8) Global impression of the project

*The overall impression was excellent. All of them agreed in their wish to renew this project in the next year.*

Two results from the report given by the ZSE (Zentrum für Schulentwicklung = Centre for School Development) are following.

The ZSE collected questionnaires from all participating students and teachers and gave a very extended report.

Abbildung 8: Angaben der Schüler hinsichtlich kooperativer Lernformen getrennt nach Schulart ($N_{AHS}$=204, $N_{HAK}$=101)16




5 Ich arbeite gerne mit anderen Schülern gemeinsam an einer Aufgabe
14 Bei umfangreichen Aufgaben führt das Arbeiten in einer Gruppe schneller zu besseren Ergebnissen
11 Ich wünsche mir mehr Gruppen- oder Partnerarbeit im Mathematikunterricht
21 Am liebsten behandle ich umfangreiche Aufgaben <nicht> alleine
16 Bei Gruppenarbeit arbeitet meist <nicht> nur der beste Schüler und die anderen schauen ihm zu

---

HAK = Business College, AHS = General Secondary School

(5) I like to collaborate with other pupils solving a problem

(14) Working on extended tasks group work leads quicker to better results

(11) I'd like to have more group- or partner work in maths teaching

(21) Treating extended problems I prefer working not alone

(16) In group work in most cases not only the best student works and the others are watching

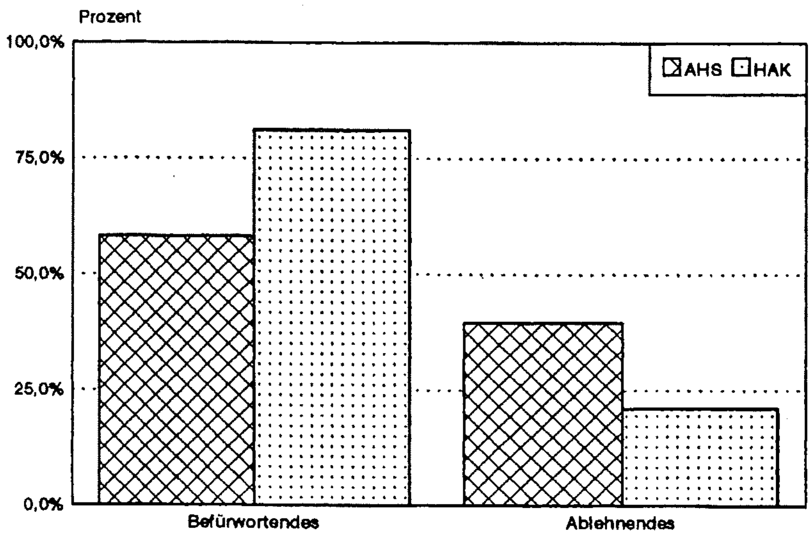
1 – I completely disagree, 5 – I completely agree

It was very interesting for all participating teachers that the vocational schools showed quite other results than the general secondary schools with respect to their attitude towards cooperative teaching (and assessment) forms.

One reason might be that our students (Business College) are more used to collaborate work. Another one could be that our students were one year older than the others.

The second diagram shows the acceptance of the new forms of assessment. The first two boxes are for "I'd like to do it once more", the second box represents the percentage of students who did not like these new forms. Here again the vocational schools show a significant preference for the new forms.

**Abbildung 3: Befürwortende und ablehnende Stellungnahmen zum erprobten System der Leistungsfeststellung getrennt nach Schulart ($N_{AHS}$=182, $N_{HAK}$=90)**




The last word to one of my students:

*"Overall I found it a hard maths year, but it was very informative. If I had to choose between an "ordinary" math teaching and this one, undoubted I would take this one. And in my opinion this should be introduced at all schools and in other subjects, too, because one will learn how to work independent AND in a group, as well".*

This affirmed my position and I was sure to be not only on an interesting but also on a right way.

There was a 4$^{th}$ ACDCA-project in 2001/2002 (CAS IV).

We defined four main research fields:

Supervision (experienced teachers and beginners) / English as working language in maths education

Teaching Materials under Special Consideration of Technology

Quality Standards in Maths Education

Responsible Technology Supported Learning

Extended reports on all ACDCA-projects (CAS I – CAS IV) can be found and downloaded at www.acdca.ac.at.

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ACDCA, T³-Austria and DERIVE User Group
<nojo.boehm@pgv.at>

# MULTIPLICATION THINKING: PROCEDURAL RATHER THAN CONCEPTUAL KNOWLEDGE AS A SOURCE OF DIFFICULTIES FOR TEACHERS AND THEIR STUDENTS

GEORGE BOOKER

Multiplicative reasoning and the fraction and ratio ideas that grow out of it are the key to the development of mathematical ideas in the secondary school and in tertiary study. Yet research in Australia, New Zealand and elsewhere suggests that many students are not gaining these ideas in the middle years of schooling and are consequently avoiding or failing the more advanced mathematics courses in the latter years of high school. Research has also suggested that inadequate conceptual and content knowledge in middle year teachers may be a contributing factor. This paper reports on the difficulties experienced by undergraduate students taking a foundational mathematics education course that provides a major focus on conceptual, practical and theoretical investigations of multiplication, common fractions, decimal fractions and proportional thinking.

Multiplicative thinking, and the mathematical content and processes that develop from it (referred to as a 'multiplicative conceptual field' Vergnaud 1995, 41-42) have been recognised as crucial to the development of most of the fundamental ideas of higher mathematics (Mulligan 2002, 497; Harel & Confrey 1994, viii). Yet, there are not "smooth, continuous paths from early addition and subtraction to multiplication and division, nor from whole numbers to rational numbers" (Hiebert & Behr 1988, 8). In particular, multiplication simply viewed as repeated addition and rational numbers considered as pairs of whole numbers inhibit the growth of more advanced processes. A break from earlier additive concepts and a re-conceptualisation of number is needed to allow the development of proportional thinking, algebra and the more advanced topics that build on these fundamental ideas (Ma 1999, 116).

Nonetheless, multiplicative thinking begins when the initial multiplication concept is generalised from addition. Indeed, it is this need to build from earlier additive notions, rather than real situations that can unambiguously give rise to multiplication in the manner used for addition and subtraction concepts that is often the cause of difficulties for many children and adults (Booker et al 2003). In turn, initial ideas of multiplication lead to a related conception of division in terms of sharing and repeated subtraction to allow the development of sophisticated algorithms for dealing with larger numbers and decimal fractions. They also underpin initial fraction ideas from the concept of equal parts out of the total number of parts to permit new numbers based on proportional reasoning rather than counting and relative size. It is this thinking that allows renaming among fraction forms from improper fraction and mixed numbers to equivalent common fractions, decimal fractions and per cents.

The rational number concept, rates and ratios that grow out of these conceptions are all fundamentally multiplicative in nature and are needed to allow the development of algebraic concepts and processes. They provide the reasoning that underpins the new notation as well as meaning for the complex relationships and functions it is used to represent. A further extension to number notation, via the concept of exponents, is also essential. These concise representations give meaning to the large numbers that proliferate in contemporary society and allow them to be readily contrasted and compared. Nonetheless, just as multiplication needed to be extended from its initial meaning of repeated addition, in time exponentiation will need to be broadened beyond repeated multiplication (Greer 1995, 67). Such an extension is essential to provide a basis for the exponential and logarithmic functions needed to analyse and describe patterns of growth and decay in environmental and social factors that a rapidly changing society is bringing about (Usiskin 2001, 81; Confrey 1995, 295-296).

Making these extensions is not easy. Not only do many students remain wedded to the initial concepts developed with whole numbers (Irwin 2003; Siemon & Stephens, 2001) intending teachers also have many misconceptions and underdeveloped ways of thinking that hinder the growth and use of multiplicative structures for themselves and their students (Anghileri 1999; Kierin 1995, 396).

## Student teacher difficulties with multiplicative ideas

The development of multiplication moves from an initial repeated addition conception and use of equal groups to arrays that establish multiplication as an operation in its own right:

| | | |
|---|---|---|
| 4 objects in each row | • • • • | 4 |
| 3 rows | • • • • | x 3 |
| 3 fours are 12 | • • • • | 12 |

The array also shows the matching fact:

| | | | | |
|---|---|---|---|---|
| | • • • | | | |
| 3 | • • • | | • • • • | 4 |
| x 4 | • • • | | • • • • | x 3 |
| 12 | • • • | is the same as | • • • • | 12 |

This conception of multiplication then allows the multiplication basic facts to be readily constructed by linking to earlier knowledge through the visual patterns provided by the arrays for each pair of facts. In turn, combining these facts with numeration and addition provides a meaningful algorithm for multiplication while linking them to the division concepts provides a basis for division with larger numbers and decimal fractions.

## A procedural understanding of the multiplication concept

In contrast, many student teachers have arrived at university with a set of techniques for finding answers to calculations acquired in a rote manner while they were at school. There is little understanding of the need for concepts, let alone what they might be, and their procedures are often confused. Consequently, they are unable to give stories to illustrate any meanings underpinning the operations and when asked to explain what multiplication might give examples like:

What is **multiplication?** finding the corect ansuer but mulltiplying 2 number to geather

Multiplication is increasing a number in lots or groups by another number.

Multiplication is adding a number together to find a result eg. 39 x 3 39 + 39 + 39

*Descriptions of multiplication are simply ways to obtain an answer*

This lack of understanding of the meaning of multiplication other than as rules to find an answer then leads to difficulties in discerning errors or providing appropriate ways of thinking when assisting young children to come to terms with the algorithm.

## A procedural understanding of the multiplication algorithm

When asked to analyse a student's error and show a more appropriate way of thinking, the focus is chiefly on the steps that have not been followed and all that occurs is an insertion of the 'missing steps' into the process without any sense of the underlying meaning of the computation:

A child obtained the answer

```
 ⁵49
 ×86
 374
```

What was his error? Because he didn't cross out his 5 which should have been directly above the 4, same size and then crossed out when he used it. However he calculated the 5 in again into his second sum so the answer is increased to what it should be.

*The analysis of the error made simply looked at the procedures followed with each step and made no assessment of the overall multiplication.*

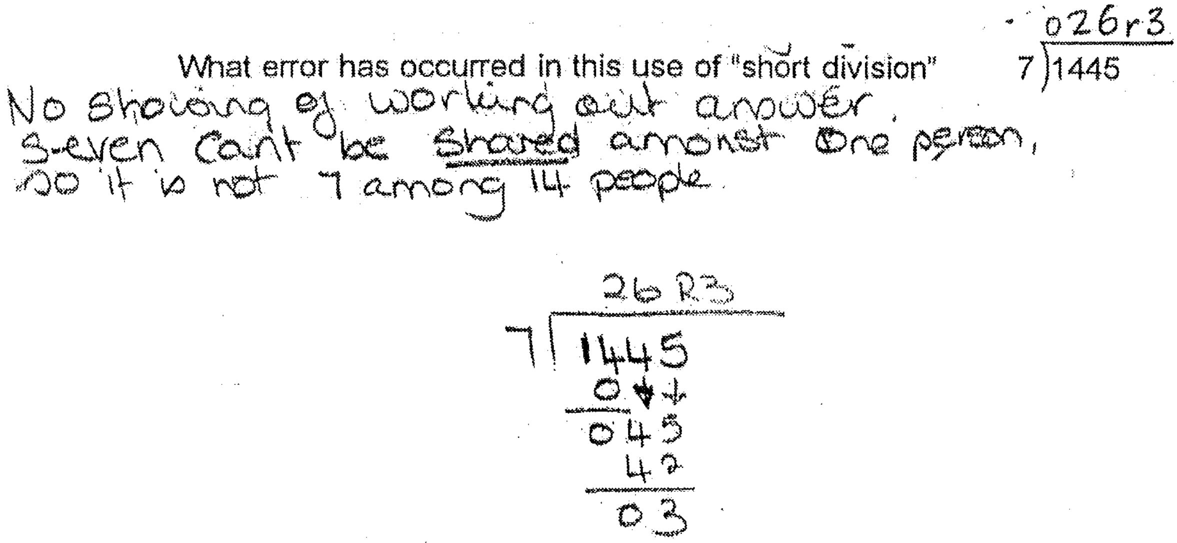
Show the full language and recording that would be used to provide a correct result:

| Language | Recording |
|---|---|
| nine multiply six is fifty-four | 5 |
| four multiply six is twenty-four | 49 |
| add the five is twenty-nine | ×86 |
| nine multiply eight is seventy two | 294 |
| four multiply eight is thirty-two | 392 |
| add the seven is thirty-nine | 4214 |

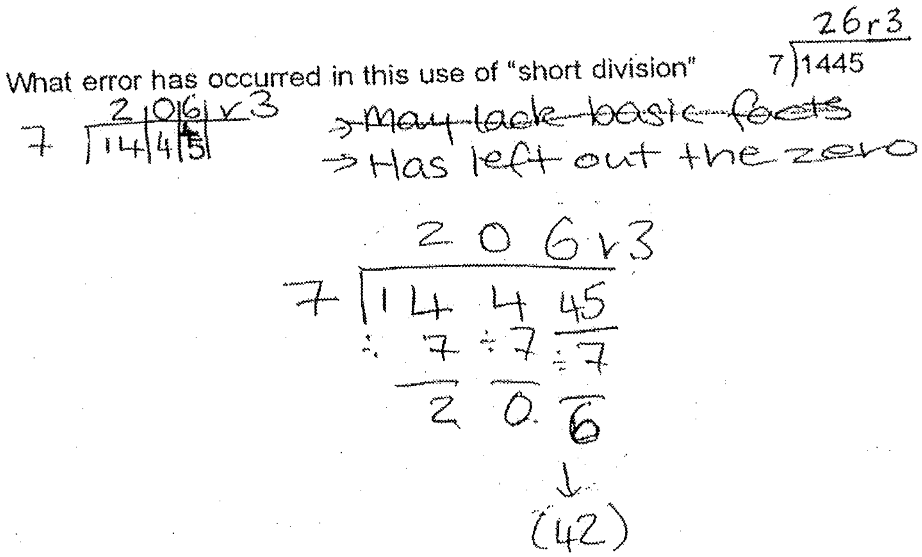
*The multiplication was completed, with no reference to the overall sense of the multiplication or the place value ideas that underpin it*

## An inability to link and use multiplication in division

Division is frequently seen as the application of steps without meaning, usually with no recording to show the process. When recording is used, it may simply be a justification of the steps that have been followed:




*While this student teacher was able to discuss the error, confusion over the notion of division as sharing is still apparent and her own working actually repeated the error the child made.*

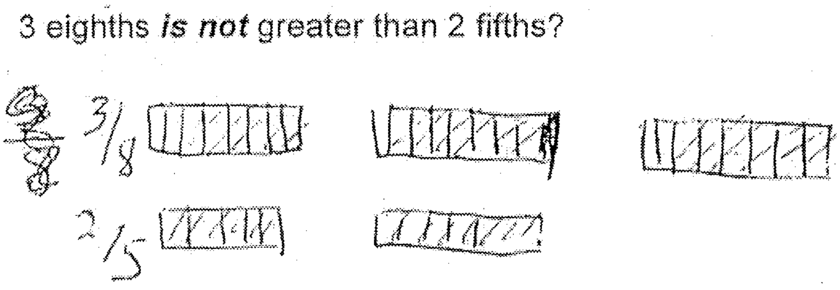



*The error is described mechanically as 'leaving out the zero' and his own working has merely justified each digit in the answer by a division process recorded more akin to subtraction*

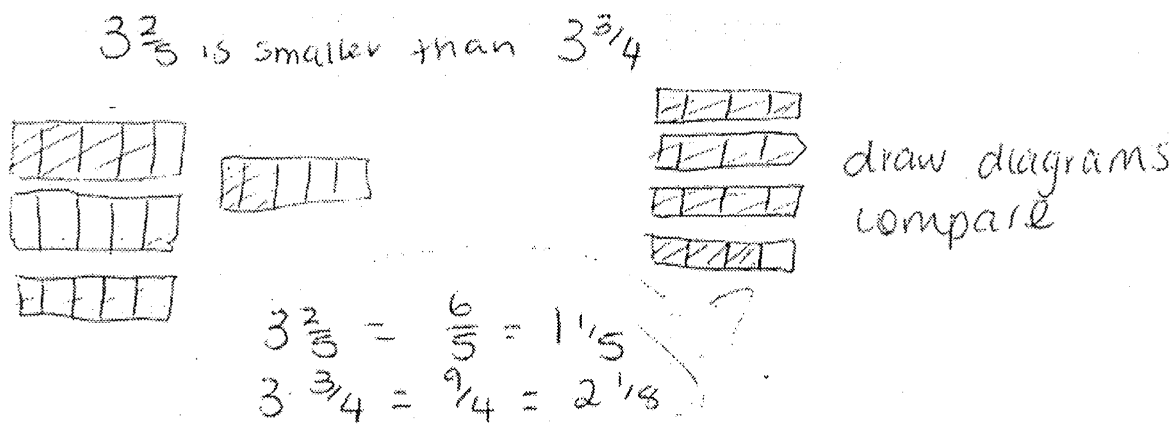
While some of these difficulties could be by-passed in their own lives by the use of calculators, this does not provide a means of teaching meaning to the children whom they will teach. Nor will it allow them to extend multiplicative ideas to the fractions and ratios that will occur in real life and in further mathematics.

## Viewing fractions as two whole numbers, subject to rules that are frequently confused

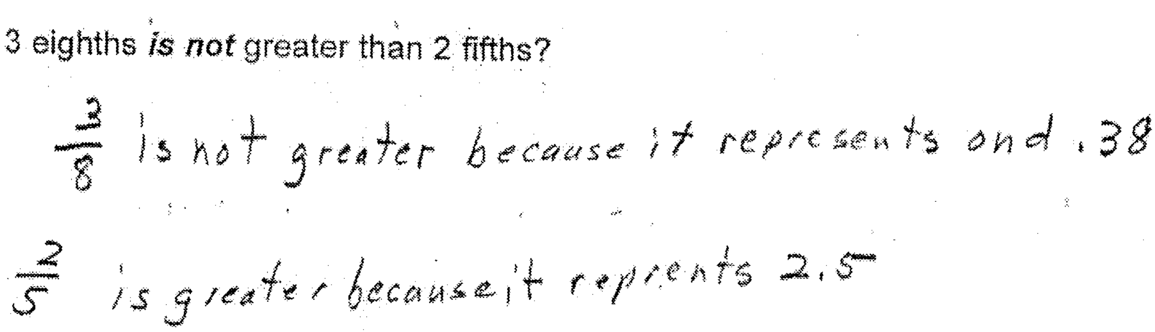
When asked to compare or rename common fractions many misunderstandings surface, even when diagrams are used to "show" the fraction amounts:




*A literal interpretation of the two numbers in the fraction as3 eights and 2 fives*



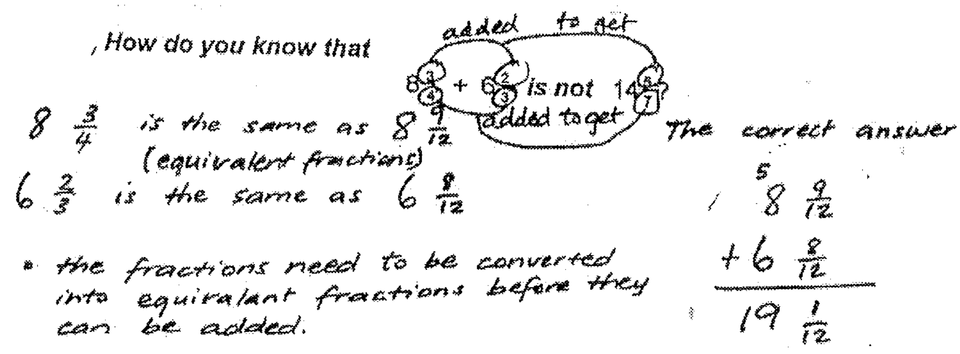

*While diagrams are used to show 3 ones and the fraction parts, symbols are relied on to provide an answer. This restricted understanding of mixed numbers then leads to multiplying 3 with 2 and 3 with 3*



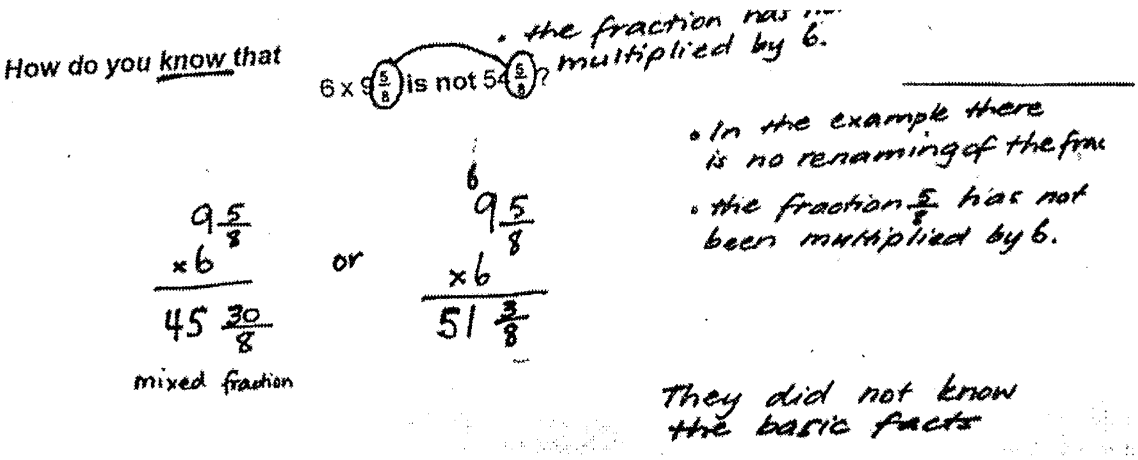

*When meaning is not present, the symbols may be used in a "convenient" way to justify an answer*

In turn, little understanding of fractions and the "rules" used to manipulate them cause further difficulties when computation is called for:

## Confusion with other procedures




*9 twelfths and 8 twelfths have been added to get 17 twelfths, but this has been interpreted as 5 and 1 twelfth rather than 1 and 5 twelfths*

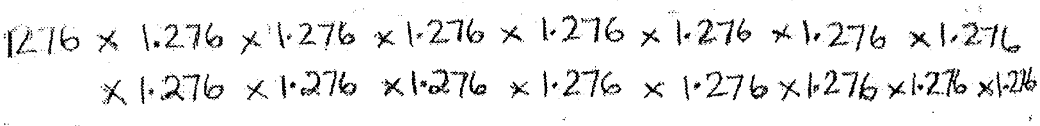



*A basic multiplication fact error has given 6 nines as 45 not 54. When 6 and 5 eighths were multiplied30 eighths has been interpreted as 6 an 3 eighths rather than 3 and 6 eighths*

## Confusion between addition and multiplication concepts

When multiplication is simply viewed as repeated addition, this often transfers to misunderstandings with exponents:

Write 1.279 x $10^{15}$ in full



## Conclusion

Many students arrive at university planning to become primary teachers. While their entrance scores might be high, this often hides an inability with elementary mathematics. They may also fear the subject and dread having their shortcomings revealed. Despite a program that not only attempts to show them how to help young children learn and enjoy mathematics but also builds their own knowledge and competence, changing from a rule-based view of mathematics to a more conceptual one is neither simple nor guaranteed. Yet, as Ma (1999, 147) points out, "improving teachers' subject matter knowledge and improving students' mathematics education are interwoven and interdependent". If student teachers are still exhibiting fundamental misconceptions about multiplication and its use in building division and fraction ideas, their ability to extend the thinking of the children they teach to ratio, proportion and simple functions will be very limited.

## References


1. Anghileri J, 'Issues in teaching multiplication and division' in Thompson I (Ed) *Issues in teaching numeracy in Primary Schools* Buckingham: Open University Press, 1999
2. Booker G, Bond D, Sparrow L & Swan P *Teaching Primary Mathematics [Third Edition]* Sydney: Pearson Education Australia, 2003
3. Confrey J, 'Splitting, Similarity and Rate of Change: A New Approach to Multiplication and Exponential Functions' in Harel & Confrey (Eds) *The Development of Multiplicative Reasoning in the Learning of Mathematics*, Albany: State University of New York, 1995, 295-296.
4. Greer B, 'Extending the meaning of multiplication and division' in Harel & Confrey (Eds) *The Development of Multiplicative Reasoning in the Learning of Mathematics*, Albany: State University of New York, 1995, 67
5. Irwin K, An evaluation of the numeracy exploratory study (NEST) for years 7 through 10 in 2002. Wellington: Ministry of Education 2003
6. Harel G & Confrey J, The Development of Multiplicative Reasoning in the Learning of Mathematics, Albany: State University of New York, 1995,viii
7. Hiebert J & Behr M, Introduction to *Number concepts and operations in the middle grades* Reston: National Council of Teachers of Mathematics 1988, 8
8. Kierin T, 'Multiple views of Multiplicative Structure' in Harel & Confrey (Eds) *The Development of Multiplicative Reasoning in the Learning of Mathematics*, Albany: State University of New York, 1995, 396.
9. Mulligan J, 'The role of structure in children's development of mathematical reasoning' in Barton B, Irwin C, Pfannkuch M & Thomas M (Eds) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, New Zealand) Sydney: MERGA, 2002, 497
10. Siemon D & Stephens M 'Assessing numeracy in the Middle Years of Schooling - the shape of things to come' in Mathematics Shaping Australia (Proceedings of the 18th Biennial conference of the Australian Association of Mathematics Teachers) Adelaide: AAMT (CD-Rom) 2001
11. Steen L (Ed) Mathematics and Democracy Washington: NCED, 2001
12. Usiskin Z 'Quantitative Literacy for the next generation' in Steen L (Ed) Mathematics and Democracy Washington: NCED, 2001, 79-86
13. G Vergnaud, 'Multiplicative Conceptual Field: What and Why?' in Harel & Confrey (Eds) The Development of Multiplicative Reasoning in the Learning of Mathematics, Albany: State University of New York, 1995, 41-42.



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# UNDERSTANDING WHAT YOU ARE DOING: A NEW ANGLE ON CAS?

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Powerful Computer Algebra Systems (CAS) are often used only with reluctance in early undergraduate mathematics teaching, partly because of concerns that they may not encourage students to understand what they are doing. In this exploratory study, a version of a CAS that has been designed for secondary school students was used, with a view to considering the value of this sort of student learning support for first year undergraduate students enrolled in degree programs other than mathematics. Workshops were designed to help students understand aspects of elementary symbolic manipulation, through the use of the Algebra mode of an algebraic calculator, the Casio Algebra FX 2.0. The Algebra mode of this calculator allows a user to undertake elementary algebraic manipulation, routinely providing all intermediate results, in contrast to more powerful CAS software, which usually provides simplified results only. The students were volunteers from an introductory level unit, designed to provide a bridge between school and university studies of mathematics and with a focus on algebra and calculus. The two structured workshop sessions focussed respectively on the solution of linear equations and on relationships between factorising and expanding; attention focussed on using the calculators as personal learning devices. Following the workshops, structured interviews were used to systematically record student reactions to the experience. As a result of the study, the paper offers advice on the merits of using algebraic calculators in this sort of way.

Computer algebra systems (CAS's) have been available on computers for at least two decades and on hand-held calculators for around a decade, and yet have not been widely accepted into the early undergraduate or senior secondary years of schooling.

A possible reason for this slow introduction of a powerful new technology may be an ambivalence among mathematics teachers regarding the place of symbolic manipulation in elementary algebra. Indeed, as Heid and Edwards note, "The sheer power of CAS may have slowed their introduction into mathematics instruction." [2, p.135]. Once the necessary syntax has been learned, modern CAS's can be used to perform all of the necessary symbolic manipulation expected of unsophisticated students. Many teachers have expressed concern that students will "not really understand what they are doing" if they use a CAS to perform what was traditionally completed by hand.

Nor are these sentiments restricted to teachers. In a recent study, Povey and Ransom [6] reported similar unease among undergraduate mathematics students regarding various kinds of information technologies. They noted two themes, both of central importance to this paper, one concerned with understanding mathematics and the other with who is in control of the mathematics. In reflecting on the feelings of many students, they noted:

> The implication is that if you don't know 'how the computer did it', then you don't 'understand' and it is out of your control. This particular choice of words is interesting because, of course, computers and calculators do not do calculations the same way that people do. Even when people realise that this phrase is not to be taken literally, they can find it hard to let it go because it serves as a metaphor for 'understanding'. [6, p.57]

Most concerns with CAS seem to be related to the issue of whether students ought be permitted to use it in their everyday mathematical work. Less extensive uses of CAS involve educational or instructional uses, which are the focus of the present paper. In exploring four possible roles for CAS in the secondary school, Heid and Edwards [2] identified one of these as:

> … to create and generate symbolic procedures, giving the user access to symbolic procedures of almost any "chunk size." For example, the CAS can perform traditional symbolic manipulations in a step-by-step fashion, with students issuing commands to transform an equation until it is solved. [2, p.131]

Reflecting this possible role, Kutzler [4] described in detail how a hand-held CAS system can be used to solve a linear equation with a succession of equivalence transformations. Similarly, Klein and Kertay [3] also described some ways of doing this with simple linear equations and a TI-89 calculator, observing that the most popular approach with their students involved the use of an applet downloaded into the calculator. The purpose of the applet was to allow students to choose various symbolic manipulations for themselves and explore their consequences.

Consistent with the concerns that excessive uses of CAS may even be harmful, educational uses of CAS are concerned with the development of what Pierce [5] has described as 'algebraic insight' and what Arcavi [1] has described as 'symbol sense'. This paper describes an exploratory study of some ways in which a hand-held CAS might be used to help students in these directions.

## Using CAS for learning

Most CAS devices have been programmed on the assumption that users are already confident with elementary algebra, and hence tend to automatically provide a simplified version of any algebraic results. Presumably, the reason for this is that authors of CAS programs design the software to produce the results most likely to be useful to typical users. If users are mostly advanced undergraduates, or professional users of mathematics, such a strategy seems sensible; however, if the users themselves are not yet fluent with symbolic manipulation techniques or their meanings, the situation may be different.

The present study uses the Casio Algebra FX 2.0 calculator, which is an algebraic calculator (i.e., incorporating a CAS) designed for a relatively unsophisticated audience of secondary school students. A distinct difference between this calculator and other CAS devices is that it offers students an 'Algebra' mode, intended to provide details of the symbolic manipulation steps used. This detailed step by step approach seems ideally suited to students still trying to master, what may be considered to be, very basic algebraic processes.

To illustrate the difference, consider the simple linear equation, $3x - 4 = x + 2$. A standard CAS system allows for students to 'do the same thing to both sides' in a systematic manner in order to duplicate the thinking associated with traditional by-hand procedures. Both Kutzler [4] and Klein & Kertay [3] describe such a procedure, used to help beginners understand the solution of simple linear equations. Kutzler [4] describes the use of a CAS for this purpose as 'scaffolding', and outlines a theoretical rationale for it:

> Solving an equation by applying equivalence transformations means the alternation of two tasks: Choosing an equivalence transformation, then applying it to the equation. ... The main advantage of the ... approach is that students can fully concentrate on the higher level task of choosing an equivalence transformation, then let the calculator apply it, then study its effects. Once they are comfortable with choosing equivalence transformations, they should solve equations by hand. [4, p.25]

Figure 1 shows the kind of facility offered by the calculator in the CAS mode of the Algebra FX 2.0, before and after the operation of subtracting $x$ from each side of the equation has been completed. On the calculator, the user enters only the operation '$-x$', which the calculator interprets as 'subtract $x$ from the previous Answer', in this case, subtracting $x$ from each side of the equation previously entered:

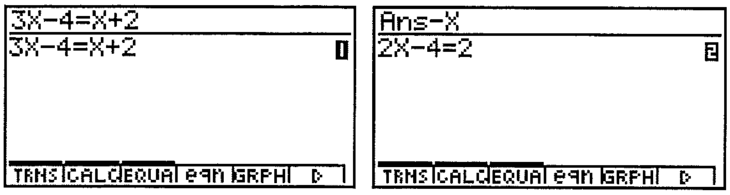


*FIGURE 1.* Using CAS with automatic simplification

Further steps of adding 4 to each side of the equation, and then dividing each side by 2 yield the conventional solution on the calculator, as shown in Figure 2. The final screen dump conveniently summarises the various solution steps.

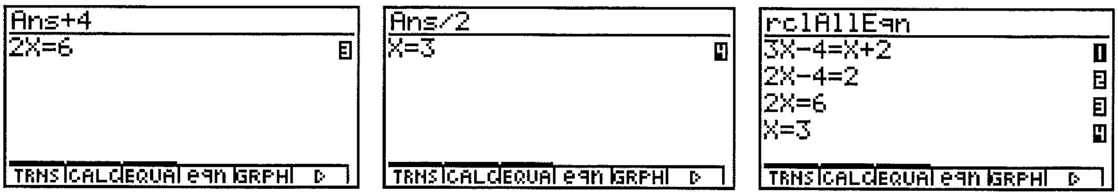


*FIGURE 2.* Completing the solution with automatic simplification

The major focus of this study involves the use of the calculator's Algebra mode, for which automatic simplifications are not provided, and users have to decide for themselves which symbolic steps are to be performed next. To illustrate the difference, consider the same equation as previously, in Algebra mode instead of CAS mode. As Figure 3 illustrates, the calculator responds correctly to the command to subtract $x$ from each side, but leaves the result unsimplified, until the *simplify* command is used.

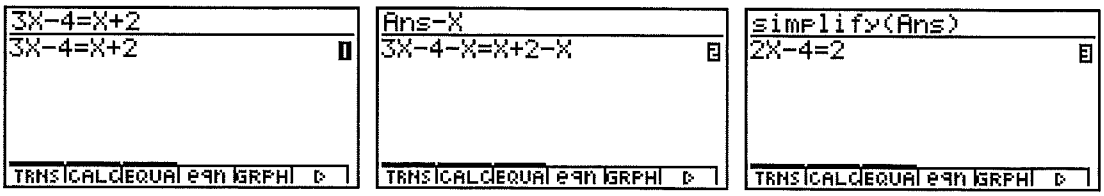


*Figure 3:* Using Algebra mode to solve an equation

While an experienced user might regard this as unnecessarily tedious, a student struggling with the logic of the process of doing the same thing to both sides of an equation is provided with an opportunity to really see what is happening. The screen dumps in Figure 4 show a final summary of the operations used to solve the equation in this way.

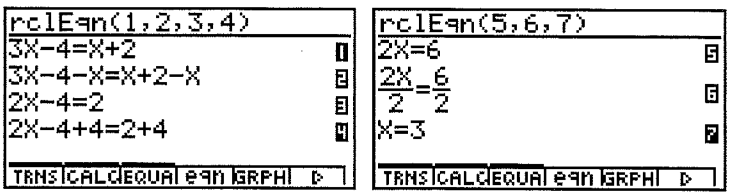


*Figure 4*: Summary of the solution in Algebra mode

A distinct advantage of performing the manipulations with the calculator is that they will always be correct, even if a poor choice is made. For example, a common misconception is that numbers or variables can be 'removed' by 'taking them away'. The consequence of such a move to solve the equation $2x = 6$, in the final step of the solution of $3x - 4 = x + 2$, can be seen to be incorrect, or at least unhelpful, when the subtraction operation is performed on the calculator, as shown in Figure 5.

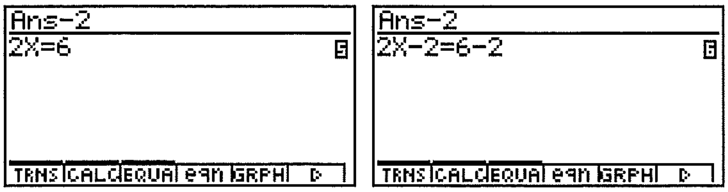


*Figure 5:* A poor choice of equivalence transformation is accepted and executed

Several authors [2, 3, 4] regard this inherent capacity of a CAS as a distinct advantage for learners. The particular additional advantage offered in the present context by Algebra mode is that the calculator leaves the result unsimplified, so that the result is completely transparent to the learner, and its limitations clear.

As for equations, a CAS will automatically provide a simplified version of an algebraic expansion. Figure 6 shows the binomial expansion of $(3x - 5)(2x + 7)$ in CAS mode. While this provides an immediate result, suitably simplified in descending order of powers of the variable, it seems unlikely to be of much help to a student who has not yet understood the processes involved. Indeed, such a screen seems to provide no more assistance to a student grappling with this idea than looking up the correct answer in the back of a textbook.

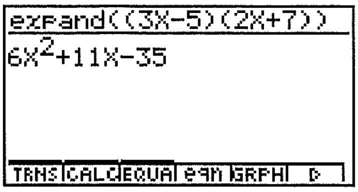


*Figure 6:* An expansion in CAS mode

In contrast, the first two screen dumps in Figure 7 show that the Algebra mode of the calculator provides an opportunity for students to see precisely what is involved in such an expansion. The simplified version is only available after the user decides to ask the calculator to provide it, as shown in the second screen. In addition, Figure 7 shows that the calculator in Algebra mode uses less conventional algebraic notation ($xx$ instead of $x^2$) in order to make clear what operations are involved in expanding some expressions.

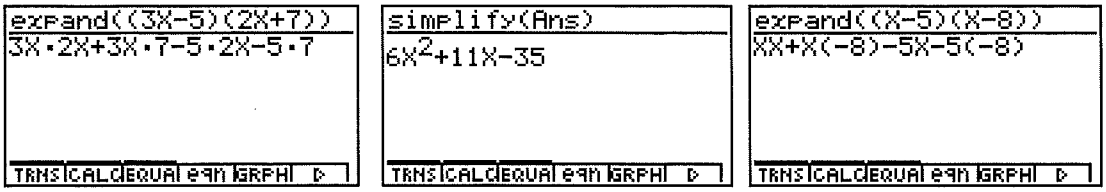


*Figure 7:* Expansions in Algebra mode, with unsimplified results and use of $xx$ instead of $x^2$

## Design of the Study

These potentials for learning were the basis of an exploratory study involving students in a first year undergraduate mathematics service unit at Murdoch University. Students in this unit are not mathematics majors, but are usually hoping to major in science or computer science. Many students have not done mathematics for some years; many of them are mature age students, returning to study. Most of these students have weak algebraic backgrounds and take the unit as a transition to undergraduate mathematics. Students are expected to make routine use of graphics calculators in the unit, and are helped to do so, but do not generally have access to an algebraic calculator with CAS capabilities.

Two voluntary workshops using CAS calculators were organised early in the semester as an extra for students who felt that they did not really have the required background for even this basic service unit. The first of these focused on solving simple linear equations by exploring the strategy of 'doing the same thing to both sides' of the equation, while the second was concerned with expanding and factorising binomial expressions. The first workshop included about 24 of the 80 enrolled students, in two separate groups of 12, a few days apart early in the semester, as well as two of the authors. Students were provided with a Casio Algebra FX 2.0 calculator during the workshop and the group had access to an overhead model of the same calculator. The second workshop was conducted mid-way through the semester, involving about 10 students and the first author. All workshops were quite informal in nature.

For each workshop, a series of relevant algebraic exercises was devised and given in advance to students. In the workshop itself, students and instructors together used the calculators to explore the algebraic procedures and their meanings.

Some weeks after the second workshop, interviews were conducted with eight students from the group of students who had attended the workshop on solving equations; some of these students also attended the second workshop on factorising. Students volunteered to be interviewed and for the interviews to be tape recorded. The main focus of the interviews and this paper is the first workshop.

The interviews were somewhat structured but allowed students some freedom to make comments and give their opinions and thoughts on related matters. Each student was asked why they had attended the workshop, how many equations they could solve in their head from a given list, whether they felt that the workshop had helped them and why or why not it had helped. Other comments were sought concerning the timing of the workshop. The intention was to get feedback about their perceptions of the usefulness of using the algebra mode on the calculator, how they thought the workshops could be improved and their opinions on the timing of the workshops in the semester.

## Results

At the time of the workshops, it seemed that the students in the workshops naturally fell into three main groups. The first included those for whom using the calculators was useful for learning the steps of how to solve equations when 'doing the same thing to both sides'. A second group comprised those who found that the use of the calculators helped them to understand and clarify what they had already been doing previously. A third group consisted of those who were ambivalent about the use of calculators in a situation where they thought that they should be solving the equations by hand, akin to the sentiments reported by Povey and Ransom [6].

The student interviews confirmed that these were the main groups with some overlap. A brief description of these three kinds of responses is given below, with some supporting comments from selected students.

### *Useful for learning the process of solving equations*

The students that fell into this category had little confidence about solving equations beforehand; they were comfortable about solving them in their heads only while the variable was on one side of the equation. They had weak mathematical backgrounds and were unsure of the procedure. The working through of all the steps was useful to them as this highlighted how they needed to be thinking at the various stages. For example, one student said:

> So I could understand how it works, the idea of how it works. When it showed the steps it was useful. I always did it another way-put the $x$ on one side but I couldn't do it all the time so need another way. The calculator helped to see the other way.

Another student also reacted positively to the details provided by the Algebra mode of the calculator, seeing a parallel with doing the same sorts of things by hand:

> Yes, I think the most beneficial thing to me was being able to see the process. The important thing for me in learning and re-learning is being able to actually fully write everything out so I can follow it all the time. ...I have got to have all the bits to follow. I can look at that now and it's just there but it wasn't always like that.

In a later remark, this student seemed to appreciate that they were doing the necessary thinking by themselves, while the calculator was merely doing as it was instructed:

> Doing what you would have been doing by hand, thinking it through, but without having to write it all on paper and use up sheets. Seeing every step made a huge difference to me.

### *Clarifying what the procedure does*

It appeared that some students had been using a procedure of 'changing sides and changing the sign' for their solving of equations. Unfortunately for them this does not cater for the more complex tasks of multiplication and division and so the method as a whole has flaws; indeed, it is a good example of what Skemp [7] referred to as 'instrumental understanding'. The workshop seemed to help such students to see the reason behind some of the steps they had been taking and to draw attention to some of their misconceptions. Thus, one student remarked:

> I never understood what it was about before. I didn't take away, I just did the opposite on each side. I followed the rules of changing sides and changing signs but I didn't know *why* until this session.

Similarly, another student spoke about moving something to the other side of an equations in order to add or subtract, and was surprised to learn of the connections with doing the same thing to both sides:

> I was taught it this way. I would not think about doing the same thing to each side, but you know it ends up the same.

While it seems that such students have benefited from the workshop experience with the calculators, we are reluctant to conclude that a brief experience of this kind can change a long-standing habit. Indeed, when asked in the interview to describe what she would do to solve $3x = -5$, this student said that she would 'do the opposite thing'.

### *The calculator is doing all the work*

The reluctance of some students to embrace the calculator use seemed to be based on the notion that the only way that you should do algebra is by hand, reminiscent of some of the students described by Povey and Ransom [6]. They were resistant to use the calculator for several reasons which included the fact that they would not be able to use them in the exam at the end of the unit; they thought you should solve the equations by hand before using a calculator; they were also concerned that a calculator could give you the wrong answer. Thus, one student noted, when asked if the workshops had been helpful to her:

> No. We can't use the algebra calculator. I can't see the point if we can't use the calculators. I suppose it was a little bit. [helpful]

Another student felt that the calculator was not helpful, and seemed convinced that by-hand algebraic manipulation was the only credible evidence of understanding:

> It wasn't showing you on paper how to do the steps. It was just banging numbers in the calculator which doesn't show you anything. Unless you have a fundamental idea of how to shift things around you can't use the calculator. Maybe down the track a bit when you can see that they are getting the concepts then you can use this as an adjunct for working things out, but you still need to know how to shift them across there. You can use it to check the hard ones. Once you've got the concept of equations by structuring it yourself then you can use it to check.

This same student showed an awareness of the pitfalls of calculator use when later asked about using the graphics calculators to solve equations using a numerical 'solve' command rather than using the algebraic steps on the calculator. In the interview, it was clear that he was motivated to learn how to solve equations algebraically, so that he would not be forced into relying on the calculator. When referring to routine use of the numerical solve command, he said:

> That would be fraught with danger. You need to be able to look at it and see if the answer is reasonable. Can't just take what the calculator or computer tells you.

Reluctance of this kind is not surprising; nor indeed is it unhealthy, in our view. It may merely reflect a strong view that equations ought to be solved by hand by humans whenever possible. It may also suggest, however, that we needed to make clearer to students the roles calculators might play in developing their understanding of solutions.

## Conclusions and implications

This was an exploratory study, quite informal in nature, and we make no strong claims for generalisability of our observations. Nonetheless, student reactions were sufficient for us to conclude that further work of this kind seems to be warranted. The students in this unit have particularly weak algebraic backgrounds, and yet will be required to develop considerable manual competence with algebraic manipulation in the early years of their undergraduate study. As mature learners, some of whom are well beyond high school age, the students seem able to learn from this kind of use of CAS on a calculator, especially when the details of the calculator's work are available to them through the Algebra mode.

Student reactions also made it clear that care is needed to conduct work of this kind early in (or even prior to) the semester, as the work fills in gaps in students' mathematical backgrounds rather than is a part of the learning outcomes of the unit. The relationship of the calculator to their other work (utilising a graphics but not an algebraic calculator) needs to be made clear, as does the purpose of the workshops, so that students realise that the calculators are to be used as an aid to learning how to solve equations rather than a substitute for doing so. Future workshop activities need to be carefully planned to optimise student learning with the calculators, especially as there is not yet a plan to allow students to have uncontrolled access to this sort of technology.

## References

1. Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For The Learning of Mathematics, 14*(3), 24-35.
2. Heid, M.K. & Edwards, M.T. (2001). Computer algebra systems: Revolution or retrofit for today's mathematics classrooms? *Theory Into Practice, 40*(2), 128-136.
3. Klein, R. & Kertay, P. (2002). An introduction to simple linear equations: CASs with the TI-89. *The Mathematics Teacher, 95*(8), 646-649.
4. Kutzler, B. (1997). *Solving Linear Equations with the TI-92.* Hagenberg (Austria): bk teachware.
5. Pierce, R. (2001). Using CAS-calculators requires algebraic insight. *Australian Senior Mathematics Journal, 15*(2), 59-63.
6. Povey, H. & Ransom, M. (2000). Some undergraduate students' perceptions of using technology for mathematics: Tales of resistance. *International Journal of Computers for Mathematical Learning, 5*, 47-63.
7. Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

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# USING ACROTEX TO PROVIDE A STRUCTURED LEARNING ENVIRONMENT FOR MATHEMATICS

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**Abstract**

This paper reports the results of an investigation into the feasibility of creating interactive web-based material for a second year mathematics course (MATH2001 – Vector Calculus and Complex Variables). MATH2001 is a course designed for students with a broad range of interests. It is one of the basic analysis courses for mathematics and science students, in addition to being a service course for engineering students. The enrolment is typically around 700 students, some of whom benefit from being able to gain easy access to those ideas from first year calculus and algebra courses that are assumed knowledge for MATH2001. Given the size of the class, the provision of on-line material seemed a practical way to offer such additional assistance. We wanted to link any new material to the existing course material, which included comprehensive lecture notes, exercises and solutions, the source files of which are in LaTeX. We report on our search for a suitable tool with which to do this, and give reasons for our eventual decision to use AcroTeX. Using AcroTeX, we have produced solutions to tutorial exercises that are hyperlinked pdf files. In these solutions, whenever a step depends on some concept covered in previous work, there is a link to an explanation of that concept. This material can be viewed from http://www.maths.usyd.edu.au/u/UG/IM/MATH2001/

Our investigation also included an extensive survey of existing web sites, since we were interested in whether any existing material would be useful for our students. Some of the results of this survey are presented. We have included a number of sites that are useful for undergraduate mathematics students in general, as well as students of vector calculus and complex variables.

## 1 Introduction

At the University of Sydney, science and engineering students typically enter first year having studied two years of calculus at high school. In first year they study further calculus, including the calculus of functions of two variables, as well as linear algebra. The second year course Vector Calculus and Complex Variables (MATH2001) follows on from the first year courses and is designed for students with a broad range of interests, including mathematicians, scientists and engineers. The enrolment in MATH2001 is typically 500 to 800 students.

In 2001 we embarked on a project to investigate the feasibility of creating more interactive and varied on-line materials for MATH2001. We were motivated by several factors. Firstly, in such a large enrolment there are invariably some students who need to be reminded from time to time of those ideas from first year that constitute the assumed knowledge for MATH2001. Secondly, almost all engineering students take MATH2001, and the Engineering Faculty at the University of Sydney had indicated that more web-based course material would be appreciated. And of course, students increasingly expect that they will be provided with extensive on-line resources.

Since we already had much course material on the web, in the form of lecture notes, tutorial exercises and solutions, we wanted to link any new material to the existing material. We decided, therefore, to provide on-line solutions to tutorial and assignment exercises, with hyperlinks to definitions, examples, prerequisite topics and additional explanations, and to produce on-line quizzes.

## 2 Searching for a suitable tool

It remains quite a challenge to publish mathematical content on the web. Despite the release in February 2001 of MathML 2.0 as a World Wide Web Consortium (W3C) recommendation (see [1]), and the increasing browser support for MathML, difficulties persist. The periodically released "status reports" from Design Science [2] discuss aspects of browser support and other MathML issues.

Different approaches to publishing mathematical documents on the web, and the corresponding difficulties, are well documented. Ian Hutchinson [3] categorises the various approaches into five generic types, and discusses the relative advantages of each. The Math Forum at Drexel University has several pages [4] attempting to clarify the issues.

The problem stems from the fact that HTML, the standard language of the web, does not easily support mathematics. Since our course material contains a large number of complicated equations and a significant number of diagrams, the prospect of using native HTML was unattractive. Although HTML is the most browser independent format, every equation and diagram is stored in a separate image (often gif) file. This makes equations and diagrams difficult to manage. Recent progress on the 'HTML+MathML platform' (see [2]) may possibly remove the multiple image file problem for equations, but our limited tests indicated that browser support remains a problem. Also, we had a large amount of existing course material in LaTeX form, and we were unable to find a suitable LaTeX to MathML converter. We were thus led to concentrate on solutions other than HTML.

The most suitable tool for our purposes was found to be AcroTeX. See [5]. (In earlier versions this was named Webeq, not to be confused with the WebEQ MathML equation tool).

AcroTeX is an abbreviation for the "AcroTeX eDucation Bundle", a collection of packages for inclusion in a LaTeX source document. LaTeX is the most commonly used and arguably best program for typesetting mathematics. AcroTeX itself uses the LaTeX *hyperref* package to enable hyperlinks. AcroTeX can incorporate quizzes and objective questions, in which equations and formulae can be given as answers. Coloured image overlays are also permitted, providing an attractive finished product, as illustrated in the demonstration file available from [5]. The latest release (October 2002) also permits Javascript to be embedded in the LaTeX file allowing for further interactive content. A LaTeX source document using AcroTeX can be directly converted to a pdf file, for example using pdfTeX . However, if the source contains postscript utilities, such as pstricks diagrams, then an intermediate postscript file must be created. In this case, some auxiliary software is needed to produce the corresponding pdf file. We used Acrobat Distiller, which was available through a low cost site agreement with the University of Sydney. The resulting pdf file may contain multiple diagrams, equations and hyperlinks to associated pdf files or URL web addresses. No additional image files are needed as in the HTML approach. At the user end, these pdf files may be viewed using a free reader such as the Acrobat Reader, in which case the links are active, the file is searchable via [Ctrl][F], and other Acrobat tools may be used.

## 3 The new on-line material for MATH2001

In MATH2001 the lecture notes provide all the basic definitions, concepts and techniques that the students have not encountered previously. The existing tutorial examples are designed to illustrate and reinforce this new material, but do not emphasise material from previous years. We have now enhanced the solutions to these tutorial examples using AcroTeX to produce hyperlinked pdf files. A typical example of a new concept introduced in MATH2001 is the curl of a vector field. This builds on more elementary ideas, such as the definition of a vector field, the vector product and even the calculation of partial derivatives. When the curl is encountered in a tutorial example on-line, links and explanations are provided for the essential concepts covered in earlier courses. Each such topic has a description in a pdf file, based on the content of *the little blue book* [6], a summary of first year mathematics at the University of Sydney. Furthermore, each tutorial is linked to the existing lecture material, broken into topics each with its own pdf file. Once the material has been composed and linked in this way, it is also available as on-line course material for other future courses, including those at a more advanced level.

These new hyperlinked tutorial solutions and supplementary material were made available to students in Semester 1, March 2003, at
`http://www.maths.usyd.edu.au/u/sandrab/math2001/tutesols.html`.

In addition to the hyperlinked tutorial solutions we are also producing quizzes which students can attempt, in their own time, over the web. Using AcroTEX , we were able to produce these relatively easily from multiple choice questions which already existed in various TEX documents, such as past exams for MATH2001.

The use of AcroTEX was not entirely without problems. We found that it was important to use Postscript Type 1 fonts to obtain good quality screen presentation of the resulting pdf files, and Type 1 are not necessarily the default fonts. The main problem, however, was due to the reliance of AcroTEX on pdf files. When viewing pdf files across the web, blank pages sometimes appear, and pages may hang. This problem is quite widespread and dependent on different browser and Acrobat versions and setups. The problem seems most noticeable when using a dialup modem connection. A number of remedies are suggested by various web sites (for example [7]), but we have not been entirely successful in solving the problem.

## 4 How to create an interactive pdf document

Using the *hyperref* package it is extremely easy to include links to websites within LATEX documents. The *hyperref* package is automatically included with AcroTEX , but can also be loaded by itself. Here is a very simple interactive document, with a link to the School of Mathematics and Statistics website at the University of Sydney:

```
\documentclass{article}
\usepackage[colorlinks=true]{hyperref}
\begin{document}
\href{http://www.maths.usyd.edu.au/}
{\underline{School of Mathematics and Statistics}}
\end{document}
```

A pdf document created from this consists of the underlined words
`School of Mathematics and Statistics`
and clicking on those words in the pdf document takes you to the website.

### 4.1 How to create quizzes

Creating quizzes using AcroTEX's *exerquiz* package is also very easy. They can be included in any pdf document by including the *exerquiz* package in the preamble, but this appears to work best if AcroTEX's *web* package is also used. There are essentially three types of multiple choice quizzes each of which is created in almost the same way. Each quiz needs to be started by clicking on the words Begin Quiz. The answers can be changed until the words End Quiz are clicked.

The first type of quiz does not show which answers have been selected by the user (thereby keeping the pdf file reasonably small), although if an answer is changed the user is asked for confirmation. The second type shows which answers have been selected and the third type shows the answers and includes an option for correcting the answers. Each type of quiz creates a progressively larger file and hence a longer download time.

The quiz is an environment, and for the first type of quiz the contents are enclosed inside the following `\begin`, `\end` commands:

```
\begin{quiz}{quizlabel}

\end{quiz}\qquad\ScoreField{quizlabel}
```

The label {quizlabel} and \ScoreField{quizlabel} enable the scores to the quiz to be calculated. There can of course be more than one quiz in the document. There is an opportunity to include a preamble to the quiz before the questions are enclosed in \begin{questions}\end{questions} commands. The questions are enumerated using the \item command. The possible answers are enclosed in \begin{answers} and \end{answers} commands. The number of columns for the questions is written after the \begin{answers}. The answers are enumerated with \Ans1 for the correct answer and \Ans0 for incorrect answers. The columns are separated by the & symbol.

The second type of quiz is created by starting with \begin{quiz}* rather than \begin{quiz}.

The third type is created by including the option \eqButton{quizlabel} after \ScoreField{quizlabel}.

The following code illustrates the use of AcroTEX's *web* and *exerquiz* packages. It produces a quiz of the third type with the possible answers for Question 1 in a single column and those for Question 2 in two columns. Both the dvips and pdftex driver options are shown. The AcroTEX manual [8] describes various other options.

```
\documentclass{article}
\usepackage{amsmath,amssymb}
%\usepackage[dvips,designi,nodirectory,tight]{web}  % dvips
\usepackage[pdftex,designi,nodirectory,tight]{web}  % pdftex
\usepackage{exerquiz}
\newcommand{\V}[1]{\mathbf{#1}}
\newcommand{\dint}{\displaystyle\int}
\begin{document}
\centerline{\textbf{MATH2001 \qquad Quiz}}

\medskip
\noindent Click on Begin Quiz when you wish to start the quiz.
You may change any answers until you have clicked on End Quiz.
\begin{quiz}*{quiz1}
\begin{questions}
\item %Question 1
The equation of the straight line segment $C$ from
$(2,-3,4)$ to $(3,3,3)$ is
\medskip
\begin{answers}{1}   %1 column
\Ans0 $(2-t)\V{i}+(-3+6t)\V{j}+(4+t)\V{k}$\\
\Ans0 $(2+3t)\V{i}-(3+3t)\V{j}+(4+3t)\V{k}$\\
\Ans1 $(2+t)\V{i}+(-3+6t)\V{j}+(4-t)\V{k}$\\  %correct
\Ans0 $(2-t)\V{i}-(3+6t)\V{j}+(4+t)\V{k}$\\
\end{answers}

\item %Question 2
Let $\V{F}=3xyz\,\V{j}+xz^2\,\V{k},$
and let $C$ be the curve
$\V{r}(t)=\V{i}-3\,\V{j}-t^2\,\V{k};\ \ t:0\to 3 .$
Then the work done by $\V{F}$ along $C$ can be calculated by the
formula

\begin{answers}{2}   %2 columns
\Ans0 $\dint_0^1(-2t^5)\,dt$ &
\Ans1 $\dint_0^3(-2t^5)\,dt$ \\  %correct
\\
\Ans0 $\dint_0^1(2t^5)\,dt$ &
\Ans0 $\dint_0^3(2t^5)\,dt$
\end{answers}
\end{questions}
```

```
\bigskip
\end{quiz}\quad\ScoreField{quiz1}\eqButton{quiz1}
\end{document}
```

The above code is processed as a LaTeX document, and a pdf file is created. The pdf file needs to be opened in Acrobat and then saved to automatically embed the quiz's Javascript in the file before it can be placed on a web site. There are many options for changing the format of the quiz and collecting the scores of the quiz. The AcroTeX manual [8] is well written and easy to understand for anyone familiar with LaTeX .

## 5 Some useful web sites

There is, of course, an enormous amount of material on the web, and much that is valuable for students of mathematics. The sheer volume of material available is somewhat overwhelming, however, and it is time-consuming to search for suitable material. As part of the project reported in this paper we carried out an extensive survey of existing web sites, with the intention of producing a list of sites that we could recommend to our students as useful resources and learning aids. What follows is a list, in no particular order, of some of the most useful sites we found.

`http://www.math.temple.edu/~cow/`

The COW (Calculus On the Web) library is a collection of interactive modules, each of which provides students with the opportunity to learn and practice an individual technique or idea. The site began as a calculus site, but has recently added modules on linear algebra and number theory. The calculus modules comprehensively cover the subject matter taught in most first and second year calculus courses at Australian universities. There are literally hundreds of modules, each of which consists of a number of problems. Answers can be typed in, and submitted for checking. There is a "help" button for each problem, which provides very useful explanations of the concepts involved. It is an impressive site, and would be a valuable resource for any tertiary mathematics student.

`http://www.geocities.com/jtaylor1142001/index.html`

This site is a hyperlinked set of solved and self-test problems, with a glossary. It is a comprehensive site covering elementary functions, calculus, and vectors. It includes many topics that are covered in most first year university mathematics courses, such as the Chain Rule, Mean Value Theorem, dot and cross products etc. It is constantly expanding.

`http://www.math.odu.edu/~bogacki/lat/`

The `Linear Algebra Toolkit` at this site covers basic procedures, such as Gauss-Jordan reduction, determinants, linear independence etc. It is an interactive site and looks to be useful for linear algebra students at first and second year university level.

`http://www.algebra.com/`

The Algebra Homework site supplies many solved problems, in addition to a "Forum/Help Center". It ranges from elementary algebra to vectors, matrices, groups, rings and fields.

`http://www.univie.ac.at/future.media/moe/`

The `maths online` site at the University of Vienna describes itself as a "gallery of multimedia learning units". It has over 60 interactive mathematics tools, such as calculators, equation solvers, plotters etc., as well as several interactive tests on set theory, algebra and calculus, and a downloadable 'HTML formula tool', for putting formulas onto the web. There is a page with links to interactive learning material, ordered by topic.

`http://mathworld.wolfram.com/`

Eric Weisstein's site claims to be "the web's most extensive mathematics resource". It is basically an encyclopedia of mathematics, where each entry is hyperlinked to other relevant entries and to reference books. It is comprehensive and detailed.

`http://mathforum.com/`

This Drexel University site is useful for both students and academics. It includes a collection of internet resources for mathematics subjects, as well as links to resources for mathematics education, and several pages devoted to "Key issues for the mathematical community."

For a more extensive list see "Mathematics on the Web" on the homepage of one of the authors, `http://www.maths.usyd.edu.au/u/cec/`.

## 6 Discussion

There is an increasing expectation that academics will provide on-line material and other resources for students enrolled in their courses. The demand comes not only from students, but also from universities, and will only increase as the well-documented trend for tertiary students to spend less time on campus and more in paid employment continues. This can be problematical for mathematicians, because of the difficulties involved in putting mathematics on the web. Using HTML, where every equation is stored in a separate file, can produce very nice pages (see the COW site mentioned above), but is difficult and time-consuming, and hence very expensive. The use of AcroTEX provides one solution to the problem. If one already has course material whose source is in LATEX , then it is relatively cheap and easy to produce hyperlinked pdf files, and to incorporate interactive quizzes. There remain the browser-related problems associated with displaying pdf files. However, since we made our hyperlinked pdf files available for students there have been about 4000 fetches, and no problems have been reported by students.

Finally, since there already exists much useful material on the web, one can easily construct a "virtual library" of sites relevant to a particular course, and make this available to students as an on-line resource.

## References


[1] *What is MathML?*, http://www/w3.org/Math/

[2] *Math on the Web: A Status report*, http://www.dessci.com/en/reference/webmath/status/

[3] I. Hutchinson, *Approaches to WWW Mathematics Documents*, http://silas.psfc.mit.edu/tth/webmath.html

[4] *Math Typesetting for the Internet*, http://mathforum.org/typesetting/

[5] *The AcroTeX eDucation Bundle*, http://www.math.uakron.edu/~dpstory/webeq.html

[6] S. Britton, C. Coleman, J. Henderson, *the little blue book*, published by the School of Mathematics and Statistcs, University of Sydney, 1999.

[7] http://www.adobe.com/support/rechdocs/29776.htm

[8] *The AcroTEX manual*, http://www.math.uakron.edu/~dpstory/acrotex/webeqman_p.pdf



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# EXPERIENCES IN TEACHING A SCIENTIFIC COMPUTATION COURSE WITH MATLAB

BOB BROUGHTON

Modern software tools such as MATLAB provide both basic programming facilities and sophisticated functions that allow the teaching of scientific computational skills. The challenge is thus to teach these skills to students with poor programming backgrounds. A course in computation and modelling using MATLAB was introduced in the engineering intermediate year. It is to be complemented by a further course at the first professional year. A second year MATLAB course is also running. The whole MATLAB programme is in the process of being rationalised. This programme will necessitate the development of instructional methods that effectively involve large classes of students in problem solving techniques encompassing decision making, transforming a problem into a model and thence into a computer process to provide solutions. Knowledge of numerical methods together with appreciation of floating-point arithmetic and associated computational error will be involved. One aim of the MATLAB courses is to make students technologically capable. In terms of the various education taxonomies, the course seeks to provide a student with a process to encourage knowledge and understanding leading to a level of mastery that will enable application of the concepts in new and concrete situations. In essence the student is invited to construct, from given basics, an understanding of the methodologies involved in using MATLAB as a tool in scientific computation. The "MATLAB in Scientific Computation" course has run 5 times in the past 3 years including as a very concentrated summer course. The course is very intensive and whilst the basic structure has remained essentially the same in terms of being a lab based, hands-on, hierarchical approach, the intimate involvement of the teaching staff in the course and the feedback obtained both verbally and by observation, has prompted changes in the instructional process.

As a lecturer and a sports coach I have always been interested in how to convey my message so that the receiver gains an understanding of the material and can go on and make use of it. In attempting to achieve this I have looked at ways of presenting material in conjunction with assessment processes since I believe you cannot do one without the other if you actually wish to get somewhere. This view is supported by Smith and Wood (5). Not that I knew that at the time, but working with a postgraduate student and taking study leave gave me time to read extensively in the realm of education theory and mathematical education theory in particular. There I found a vast amount of material, a good amount of it theoretical and clearly written by people who had either never taught or certainly never taught large classes. But there were a lot of pertinent articles well worth the read.

Over the years I have taken on the Keller Plan (3) style of teaching, introducing it with a colleague way back some 20 odd years ago in a level 200, numerical methods course. The style was an instant success and the 60 students in the class actually demanded the course questionnaires to fill in. I carried the Keller Plan approach on until recently when I went on leave and believe it was a very good way to teach and assess this type of course. I tried a half-way approach with a large basic first year course for two years with modest success and a came away from that experience with a lot of ideas for teaching first year mathematics to poorly prepared students. Of course one does not get a lot of support from most of ones colleagues who tend to think you are mad for putting in so much effort. However the very positive comments from alumni about the numerical course in particular give one cause for hope and indeed one wonders if the survey process shouldn't be taken amongst alumni.

In the course of all this I have become aware of Ramsden's(4) shallow and deeper learning concepts and Biggs'(1) five levels of learning. These I give as follows, quoting Biggs:

**TABLE 1**
*Levels of learning*

| RAMSDEN | BIGGS | |
|---|---|---|
| | Pre-structural | The task is not attacked appropriately; the student hasn't understood the point. |
| Shallow Learning | Uni-structural | One or more aspects of the task are picked up and used (understanding as nominal). |
| | Multi-structural | Several aspects of the task are learned but are treated separately (understanding as knowing about). |
| Deep Learning | Relational | The components are integrated into a coherent whole with each part contributing to the overall meaning (understanding as appreciating relationships) |
| | Extended Abstract | The integrated whole at the relational level is reconceptualised at a higher level of abstraction, which enables generalisation to a new topic or area, or is turned reflexively on oneself (understanding as far transfer, and as metacognition). |

I find I can relate to these and see how my ideas fit in. There is some opinion as to the levels commonly achieved and I would have to agree that as a generalisation we struggle to have our students attain the multi-structural level and certainly deeper learning or the relational or extended abstract levels are not readily attained by many of our students.

It was with these concepts that I came to the level 200 course in MATLAB and Scientific Computation that I have now given 5 times including once this year as a summer course. This is a somewhat crash course and the problems I faced were:

- clients who can no longer be relied on to have any programming background
- exactly what material should be in this course
    1. what areas of MATLAB should be covered
    2. what aspects of scientific computation should be covered
- how to teach the course
- how to assess the course

If I was to aspire to having students get to the multi-structural level and for some to take the next step into deeper learning I would have to consider very carefully my goals for the course and the expectations of the clients from such a course. I was heavily involved in the teaching of the course and had close contact with the students and with the tutors I used. I have been fortunate in being able to use the same tutors each time. This interaction has enabled me to modify the course.

My first decision was to list what I thought would be a good basic course in MATLAB at the second year level. I then discussed my ideas with colleagues and looked for a textbook. I found I had a reasonable programme and an excellent book, Chapman (2) to base it on. I then looked at how to structure that material.

My thought processes now went along the lines:

- Lay down some guiding principles.
    1. Stick to one topic or area per session
    2. Try to minimize the amount of non-essential material for a session so that the new concepts take centre stage.
    3. This course will not be a spectator course, the students will be expected to fully participate.
- Figure out how to actually implement these
- Figure out how to assess the course

I addressed the overall issue of structure by organising the course in a series of labs with homework sessions, a written test, a lab test, and a project. The philosophy behind the course is that each lab addresses a particular topic and no attempt is made to throw too many concepts at the students in one session. As the course progresses students are expected to use material from previous labs including M-files they have already constructed. This satisfies the minimization principle. Introductory lecture sessions precede the lab work. These lectures can be from 15 minutes to an hour depending on the topic. The students receive a lab project and are also provided with details of the next lab so that they can prepare for it.

Students use the MATLAB DIARY process to create a file for submission. As the course progresses students are requested to electronically submit M-files through the Departmental electronic submission software. These files are run through test programs and marks awarded for various aspects including well-commented, readable M-files, and well-presented and informative output.

The current structure is given in table 2.

**TABLE 2**
*STRUCTURE OF COURSE - LAB SESSIONS*

| | |
|---|---|
| Lab1 | An Introduction to MATLAB:<br>MATLAB environment & MATLAB as a scientific calculator: Variables; Use of the editor and introduction to DIARY files and script M-files. A simple M-file is constructed with Lab number, name and date. A simple sequential piece of code involving mathematical formula, for example the range of a rocket. |
| Lab2 | IEEE Floating Point Arithmetic and Errors:<br>In this lab the floating-point number system is discussed along with errors. A set of notes is provided since this material is not available in texts. The lab session consists of investigating calculations.<br>The work is continued in a homework assignment. |
| Lab3 | Basic 2D plotting:<br>The work of previous sessions is used to provide output for 2-D plots. Labelling, titles, axis features are used. |
| Lab4 | Matrix Algebra and MATLAB:<br>This lab is devoted to the use of MATLAB in matrix algebra and array manipulation and involves basic input, operations, matrix functions, partitioning, solving linear systems. |
| Lab5 | Programming in MATLAB – Branching:<br>This lab is devoted to if .. else .. end programming structures.<br>Students are encouraged to draw structure diagrams or equivalent design sheets for problems that involve decision making. |

| | |
|---|---|
| Lab6 | Programming in MATLAB - Looping *for/while*:<br>Introduction to the basic iterative process. The idea of an iterative process is described mathematically (using geometric and approximation ideas) and stopping criteria brought in.<br>The *while* loop is used to allow repeated input as in the text |
| Lab7 | Programming in MATLAB: The Secant Method and *eval* command:<br>The secant method is introduced from the approximation concept -.<br>Develop the iterative algorithm using the *eval* command for a string function - very useful for solving small problems.<br>Implement the secant iterative algorithm. |
| Lab8 | Programming in MATLAB - User Defined Functions:<br>Continue with the secant method pointing out that there is a set body of code that is problem independent that has inputs (function, tolerance, maximum number of iterations, initial values) and outputs (solution, indicator flag, number of iterations and function evaluations taken).<br>This leads to defining the core code in a user function - introduce the function M-file.<br>Input argument lists, output argument lists and the need for a *help* set of comments.<br>In addition the *feval* function is introduced together with how to use it in a function M-file environment. Introduces the new MATLAB handle @ notation for passing function names in the input argument list. |
| Lab9 | Programming in MATLAB - User Defined Functions – Advanced:<br>Again continue with the previous work and introduce the concept of *global* variables and the use of the MATLAB *nargin* and *nargout* variables for checking the argument lists and for the automatic setting of parameters.<br>In the homework assignments various plotting techniques are introduced - bar graphs, plotting several graphs on the same plot; subplots.<br>NOTE: The lab/assignment work for the iteration processes includes practical problems where the problem equations are described in terms of its own notation and students have to make the connection between the variables used in the program code and the problem variables. |
| Lab10 | 3-D and Surface Plotting 1:<br>3-D and surface plots are introduced. The concept of a mesh, contour plots, parameterised functions. |
| Lab11 | 3-D and Surface Plotting 2:<br>Continues lab 10 but with a multivariable calculus approach to verify a function of 2 variables as having various properties. Consider say a surface representing heat flux. Consider quiver plots and contour plots with respect to the function. Introduces the gradient vector, maximum and minimum. |
| Lab12 | 60 minute written test on MATLAB basics and floating point systems and errors.<br>Followed by a programming test - about 2 hours of a basic problem covering work up to Lab 7. |

The final project is a surface plot generated by a numerical process. For example interpolation of a set of data or generation of data from the solution of Laplace's equation for say heat on a thin plate. The Laplace equation project involves clever use of MATLAB's matrix algebra commands to set up equations. I use this project to illustrate the benefits of using sparse matrices.

The students are given explicit directions as to the nature of the next session. These are closely linked to the textbook so that students can familiarize themselves with the work in the next session. Of course one can lead a horse to water but not force it to drink and those students who have spent a little time looking through the material in advance have found it has paid off. This is an area that needs work on since, as one might expect, it is the poorer students who don't bother doing the preparation.

The introduction can take up to 50 minutes but usually it is between 20 and 30 minutes after which we move to the labs. The student is then given a lab assignment to complete during the lab session. Students are encouraged to talk to one another, the tutors and myself and I tend to keep an eye out for problem areas so I can run an impromptu help session.

The course has evolved into a spiralling, hierarchical form whereby concepts are introduced and then reappear again and again so that students keep coming across previously introduced pieces of work. This is not too difficult to imagine with concepts such as input/output, branching, looping, functions, stopping criteria, testing floating point results, and so on. This does mean that the work is reinforced and is not a one-off treatment. The underlying theme is to introduce one principle concept per session as indicated in table 2.

Session one introduces MATLAB as a scientific calculator. There is no attempt to consider arrays. The idea of putting these basic commands in an M-file is introduced and the student writes a couple of basic M-files. One is to identify the student and the work, and the other to introduce the layout I require (and reinforced in the textbook) for M-files. A sample problem I use because it has formulas with a good variety of functions is the constant thrust rocket problem. There is a steep learning curve and we struggle with the 3 hours often being exceeded (but I do make up for this later in the semester at a time when students are grateful not to be kept late).

The next session is the one that students find very difficult but I feel scientific computation cannot be done without a knowledge of floating point arithmetic. The catastrophic effects of not knowing this have caused deaths (missiles), ruined space projects (destruction of a space probe) with billions of dollars wasted, caused problems in finance with fraud and false economic indicators. So it is not to be ignored. What I am hoping to achieve with minimum exposure is a basic grasp of what IEEE floating-point arithmetic means in terms of the calculations performed and what possible implications there are. Some main issues here are issues of loss of significance and testing values for equality, in particular the zero case. So we do some simple problems and look for explanations.

Next comes basic plotting which includes the introduction of 1-D arrays and vectorization processes for functions. This is followed by a session dedicated to matrix algebra. At this stage I would just point out that I originally started with the notion that assessment would be on the labs plus a series of projects. It was clear I overdid the projects and in discussion with students modified the process. I also introduced a short written test and a basic programming test to sort out the grading. I sought to identify the levels of competency.

The summer version was held over 4 weeks with 3 sessions a week and I was forced to consider just how to cope with the marking and the lack of time for projects. It was actually a big help because I implemented regular small home-works in place of the larger projects. The homework reinforced the lab session(s).

The same principle was applied across sessions with students using the code set up previously as a base for a new concept (where applicable). For example the branching structures made use of the rocket problem by simply imposing conditions on the flight angle. Looping was introduced with a simple iterative process for solving a non-linear equation arising from a particular modelling problem.

The essential nature of the numerical iterative algorithm, which is so common in scientific computation, was introduced. The floating-point work was reinforced with the convergence criteria and practical considerations for stopping were discussed. The looping also carried on the branching work. The next looping session brought in the secant method, useful for the natural resources engineers in the class, which could use the previous work on iterative loops and solves the same problem, thus avoiding the bulk of the data entry for that problem. Students were encouraged to see that the same code could solve a variety of problems by relating the variables to the problem variables.

The next session on functions now took on a predictable nature where the concepts introduced in the previous session were turned into the use of user-defined functions. Again the whole point here was that in fact the students had very little data entry to do but a lot to think about in utilizing their work for the new concept. Students find the concept of user–defined functions difficult to grasp and take a long time to make what are small changes to their code. So I find it important to keep the data entry to a minimum while they grasp the concepts involved.

Throughout the course I try to connect students with their mathematics coursework so that mathematics is not seen as compartmentalised. I want students to see how the mathematics they learn appears in other contexts and they should think more laterally. My approach to iterative processes as examples of approximation has opened many eyes. First year and school mathematics can be readily woven into such a course. It is important to have students make connections between what they learn in different courses.

In terms of the levels of learning I have been gratified to see a reasonable number of students progress through the course from a stumbling and poorly coordinated grasp of what constituted good solutions to work, to being able to bring together what they have been learning in excellent presentations for the programming test and the final project. It would be too much to expect all students to achieve this but the number of students, who surprise themselves and me, with what they achieve, is extremely rewarding.

So there we are basically. I have left a few parts out - if anyone wants to talk to me more about the course I am happy to do so.

We are in fact discontinuing it in favour of new initiatives with the engineering school whereby a complete 2 year package in MATLAB is being introduced for engineering students - 171 is now compulsory for most engineers and is to be followed by 271 next year, also compulsory. So 280 will go into 271. We will continue the summer version because we believe it has a role to play. This year the summer course (282) attracted 24 students and the roll in the first semester course rocketed to 45 (from 30). The students have written or spoken glowing testimonials to the course, and postgraduate supervisors are very pleased with the results in their students. So we must have done something right.

## References


1. J. Biggs, Enhancing teaching through constructive alignment, Higher Education, 32(1996), 347-364.
2. S. J. Chapman, *MATLAB Programming For Engineers*, Brooks/Cole, 2nd ed., 2002.
3. F.S. Keller & J.G. Sherman, *PSI: The Keller Plan Handbook*, W.A. Benjamin, 1974.
4. P. Ramsden, *Learning to teach in higher education*, Routledge, 1992.
5. G. Smith & L. Wood, *Assessment of learning in University Mathematics*, International Journal of Mathematics education in Science and Technology, 31(1) (2000), 125-132.



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# Pastoral Care in Large Classes

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**Abstract**

Recent enhancements in flexible modes of delivery in higher education have given academics involved in teaching large classes a greater choice of methods to promote more effective learning for their students. However, it is still difficult for the academic to give the personal attention to each student that can occur in smaller classes. In this paper we describe and evaluate a range of approaches that contribute to the provision of pastoral care for large groups of students.

## 1 Introduction

The provision of pastoral care is an important part of teaching large classes in higher education but is difficult to implement in practice. We have designed a course structure in which pastoral care is implicitly derived from strategies introduced into the developed curriculum.

There are three areas of design that have been targeted to develop student self-worth and social identity in order to facilitate pastoral care within a large class environment. Firstly, we have looked at assessment items that encourage students to be personally involved in the course, such as free experimental work in a statistics course, reflective journals, and poetry writing. Secondly, we have replaced standard mathematics tutorials with peer-assisted study sessions (PASS), where second-year students act as facilitators to encourage learning and which also promote the development of friendships and networks in a small group setting. Finally, we have developed software tools which provide easily accessible profiles of students, including student photos and autobiographies, to allow the academic to build a comprehensive understanding of the class and its needs.

The approaches discussed in this paper are based on the experience of teaching a statistics service course with 500 students. Most aspects are readily transferable to similar courses in calculus and indeed to broader science courses.

## 2 The Student

An important first step towards pastoral care in a large class is ensuring that each student feels that they are personally involved in the class and the course. We have adopted a number of strategies for achieving this aim.

### 2.1 Surveys

A simple but effective technique for initiating a sense of belonging in statistics classes is to ask the students complete a personally directed survey in the first week of semester. Questions cover a range of topics, including

- "How tall are you in centimetres?"
- "What is your heart rate (beats/minute) as you are entering this data?"
- "How long is the sole of your left shoe in centimetres?"
- "Which country were you born in?"

- "How many hours of paid work did you do in the last week?"
- "How many total hours did you sleep between the most recent 3pm Friday and 3pm Sunday?"
- "What is your favourite pizza topping?"
- "On a scale from 1 to 10, how attractive do you think you are to members of the opposite sex?"

For ease of processing in a class of 500 this survey is completed by students via a web interface. The resulting data is then available dynamically for analysis. This data is used throughout the semester as examples in the lectures, allowing the students an appreciation of the importance of their contribution to the class and also how their responses compare to those of other students. Sex differences between variables such as sleep and self-perceived attractiveness also carry important humour into the lectures. The data is also used in practical classes, so that students can look deeper and get further involved with their class, while also developing important processing and analytical skills.

## 2.2 Projects

An important part of encouranging students to feel that they are valued members of a large class is to give them input into their assessment and to make that assessment authentic. In our large statistics service course we have used student experiments to achieve this. Their major project for the semester has simply read "Design and carry out a comparative experiment of interest to you". By choosing their project topic, students are being assessed on an area that they consider is important to them. They also appreciate from the outset that their opinions are valued in the course.

An interesting example of a student project considered whether caffeine, a stimulant, could counteract the effects of alcohol in slowing reaction times. The experiment that designed was as follows:

> Two groups of eight males were given five drinks of rum and coke over a two-hour period. One group had regular diet coke in their drinks while the other had decaffeinated diet coke. Reaction times were measured before and after the drinks.

This demonstrated a clever understanding of a blind experiment in action, and has been used as the standard lecture example in subsequent semesters. Unfortunately experiments involving alcohol are now vetoed, because of the obvious risks to the subjects and to the reputation of the university. However, students continue to create experiments to answer interesting questions, such as how wingspan affects paper plane flights, how different liquids and temperatures affect the times for tablets to dissolve, and a variety of classic agricultural questions tested with bean sprouts.

In addition to developing an ownership of their course, the projects also target some of the aims outlined by Chance [2]. These include encouraging the understanding of the statistical process, improving statistical and computer literacy and communication and collaboration skills, establishing a dynamic assessment process, and increasing student interest in statistics itself. Mackisack [5] notes that students also develop an appreciation of the practical issues involved in carrying out experiments and collection data, an outcome encouraged by Higgins [4].

The framework for the project has been designed to match as far as possible a complete scientific investigation. For example, students who are proposing projects that will involve human or animal subjects must first complete the standard University application form for ethical clearance. Guest lecturers from the library are invited to give overviews of how to research with online databases and electronic journals. At the end of the semester, the students present their project reports in the form of their own journal article.

## 2.3 Poetry

In 2002 we added a further opportunity for individual expression within a large class by running a poetry competition. Students were invited to write poetry about statistics or science in any style of their choosing.

A few students were somewhat perplexed by this competition, as reflected in the following haiku:

> Statistics is cruel
> Who has ever heard of this
> Poetry in math

However, the majority of the 350 entries showed wonderful creativity and it was difficult to judge the winner. Students put a lot of thought and effort into their poetry, many of which ran for several pages. As with the projects, they really shined when given the opportunity to be creative and to express themselves beyond the core mathematical content of the course. The poetry also stimulated discussion amongst students, helping to strengthen relationships which could then lead to increased collaborative learning groups as students began to take control of their own learning. This interaction also promotes the use of the technical language of the course in conversation.

Exploring students' creative sides was the initial aim of the poetry competition. This is an experience which is normally denied to students in a large first-year science program. However, it also provided very rich feedback about the course and student perceptions of its objectives, structure and content. Compared to traditional evaluations, the poetry vehicle gave students the opportunity for completely free feedback, concerning their emotional involvement with the course and unconstrained by the limits of prosaic writing. Some poetry was about the lecturer, some was about the course structure and assessment, some was about how statistics related to their discipline, and so on. The analysis of the poetry as qualitative feedback data for future use in curriculum design is continuing.

The winning entries and some honourable mentions can be viewed at www.maths.uq.edu.au/~mrb/stat1201/poetry.php. The following example illustrates some of the positive sentiment in students perceptions of the subject:

> Mirror, mirror on the wall,
> Who is the brightest of them all?
>
> Is it the lawyer, so skilled at moots?
> Is it the maths man, good at square roots?
> Perhaps the chemist with test tubes to burn,
> Or the vet, engineer, accountant, intern?
>
> No, the standout mind, who loves star plots and means,
> Who knows the results aren't what they first seem,
> Is the hero unsung, who loves numbers and data,
> Statisticians, you see, are the ones who are smarter.
>
> They're the ones who show others new ways one can think,
> Who move outside the square to look for a link,
> They make sense of surveys and summarise data,
> They analyse numbers - and that's just a starter.
>
> They're big on Greek letters and bar charts and plots,
> And spend half their time playing connect the dots,
> But the graphs that result from the things that they do,
> Apply cross the board what's been gleaned from a few.

So when it comes time for the next Nobel Prize,
And awards all go out to the just and the wise,
I believe one should go to the men who love maths
And apply it in practice, let's hear it for stats!

Three of the winning poems were read by students on ABC Radio National's *Science Show* in Australia, broadcast on 19 April 2003. This external recognition helps the students to be aware of the high value that is placed on their creative input to the course.

## 3 The Environment

The next step beyond the student is their immediate learning environment. The lectures serve a rather motivational role and while they cover all the content of the course we expect that deeper learning will occur outside of this setting.

Mathematics tutorials have traditionally focussed on weekly exercises with small groups of students working on problems, asking the tutor for assistance when necessary. The tutor may then also discuss common problems on the board. For the past three years, we have used a peer tutoring method to promote the engagement of students with the course, by replacing this model with peer-assisted study sessions (PASS). PASS are interactive, small group study sessions, facilitated by a pair of second year 'student leaders' who have been deemed course competent and have been trained in group dynamics and behaviour, and metacognitive and cognitive principles of learning [8, 11]. Rather than having a tutor who is typically a mathematics honours or postgraduate student, PASS is run by undergraduate students who have successfully completed the course during the previous year.

The aim of these sessions is primarily to create a collaborative learning environment where students can share and hopefully enhance their knowledge by group learning processes, guided by leader peers who have faced and overcome similar problems themselves in the not so distant past. Collaborative learning can promote 'active learning, critical thinking, conceptual understanding, long-term retention of material, and high levels of student satisfaction' [9]. Sessions are thus informal and students are encouraged to refer to their lecture notes and any text or resource based material provided by teaching staff or created by leaders in order to gain a deeper knowledge of course content. An important aspect of session structure is that the learning is student directed, so that issues important to students are covered to the level of detail that is required.

This environmentally supportive framework appears to 'scaffold' the enhancement of students' knowledge constructs based on the perspective of Vygotsky's social constructivism [12]. For optimal cognitive growth, students' learning is mediated by social interactions with more competent peers who are at a level of understanding just beyond that of the students themselves. This socially based learning environment can also extend learning beyond the purely cognitive dimension so that students can personally engage with the new language of the discipline. PASS is thus structured in a way that encourages students to explore the realms of their coursework through group discussion and problem solving rather than focus only on key issues or limit variability of worked examples to only potential examination questions. Therefore, by practising deep approaches to learning, students will hopefully attain a more solid understanding of the course than if they adopt a surface approach where 'the learner focuses on the words of the text... rather than on what the text means' [6]. Students are discouraged from rote learning to reproduce mis-understood knowledge and are given problem based tasks to help assimilate current information into existing knowledge constructs [1].

The importance of the level of learning that students reach can substantially affect their learning outcomes [1, p. 4]. PASS encourage students to adopt strategic approaches to study on a weekly basis, by monitoring the effectiveness of their study methods or help identify any ineffective, apathetic approaches to studying [3, 10]. PASS leaders do not work on prescribed exercises which are then assessed but are free to develop their own activities which they feel will best meet the learning needs of the students for a particular week, based on student feedback concerning difficulties in understanding the course work. There is no assessment tied to these activities so that the sessions are therefore stress-free, allowing students to sort out any misconceptions without fear of recrimination or jeopardising academic outcomes. In this respect, student and leader directed activities in the PASS learning environment support the constructive alignment of curriculum objectives with any on-going course assessment. With constructive alignment,

> '[curriculum] objectives are defined not just in terms of content, but in terms of the level of understanding applied to that content. The focus is not just on what students know,. but on how well they know it.' [1, p. 42].

Emphasis on the social interaction of groups of students in learning construction should not be underestimated. Being peers, PASS leaders are more aligned socially to the students; this makes it easier to develop a mutual relationship of trust. For a service course, leaders are also unlikely to be students who are enrolled in mathematics programs, so they also share similar career interests with students and are thus better equipped to appreciate the extent to which mathematics has been of use to them for their particular degree program. The fact that students can identify with leaders as individuals who have gone through a similar process that they themselves are currently experiencing, is another particular advantage of not having post-graduate mathematics students as tutors. PASS leaders, who have come from the service disciplines but who have also been successful in the maths course, are more suitable as role models for the current student population. Entwistle [3] remarked that explanation, enthusiasm and empathy in a teaching environment generally contributed to deep learning approaches in students: PASS leaders are also able to effectively, enthusiastically and empathetically facilitate student learning so that the students themselves can construct explanations for problematic areas in their understanding of the course work.

> 'Teachers mould experiences for their students with the aim of bringing about learning, and the essential feature is that the *teacher takes the part of the learner*, sees the experience through the learner's eyes, becomes aware of the experience through the learner's awareness.' [6, p. 179]

This approach can also be transposed to the context of PASS with leaders facilitating students' learning.

# 4 The Academic

So far we have described how the use of student-directed assessment and the encouragement of creativity can give students a greater sense of connection with the course. This is then built upon by developing a collaborative learning setting in the peer-assisted study sessions. Together these approaches help the students see themselves as part of a large and active learning community in which they can all participate and in which they can have their say and be heard.

We believe that building this community is the vital part of providing pastoral care for a large class. If individual members feel isolated or alienated, as can often happen with service courses in mathematics and statistics, then the academic can become overwhelmed by having to deal reactively with many shallow questions concerning the content, rather than proactively addressing the deeper challenges that students could face.

Once the community is established, we have augmented the provision of pastoral care to this "flock" through the use of technology. We have developed a software package, simply titled *Care*, which aims to give the academic a closer relationship with the students in the class.

For each student, *Care* displays their name and their student photo. In addition to the survey at the start of semester, students are also asked to write a one-paragraph autobiography concerning their origins and interests, and these are also available in *Care*. As the semester progresses, details about each student's assessment results become available, as well as the data they have collected for their project and the poetry they wrote.

This information can then be used in a number of ways. When the academic receives an email enquiry from a student they can view the face of the student and so the ensuing exchange becomes more personal. The academic also has a better chance of learning student names in this way, putting faces to names, and can ask "how the student is going" next time they meet in the hallway. The autobiography also allows email responses to be made in context. In particular the maths anxiety of some students can be noted through in their autobiography, and responses to them might be phrased differently to help alleviate their concerns.

*Care* also allows notes to be recorded with each student's data. There have been three main types of notes used in practice. The first is the recording of information about medical conditions and other personal issues that students have reported during the semester. These are used to keep track of students who may be at particular risk of falling behind in the coursework and thus ensure that they are aware of available support and possible extensions for submission deadlines. The second type of notes are brief positive notes about encounters with the students. These may include comments from students regarding deep questions about the material or pointing out subtle errors in the textbook. Finally, a range of other notes are recorded to augment the autobiography, building up a richer student profile as the relationship develops.

The data within *Care* is of course confidential. It represents a multitude of virtual personal relationships between the academic and the students for the purpose of providing care. As with the peer-assisted study sessions, it is not intended to be a tool used for assessment. While it includes assessment data, separate software is used for processing student marks. Instead, the aim is to allow the academic to manage relationships with each student with the aim of supporting the student's growth during the course, both academically and socially.

An unexpected and significant use of *Care* has been in writing references for students who have completed the course. Perhaps because of the personal nature of the course, a number of students, who may have found it difficult to find an academic who knows them in their other large classes, seek references for study abroad programs and vacation employment. Access to *Care* enables the academic to easily recall the student and any positive comments that have been made.

# 5 Conclusions

The notion of pastoral care also has significance in religious education. Many of the ideas from that literature have obvious connections with our interests here in undergraduate mathematics and science education. For example, Meehan [7] suggests that

> 'A curriculum that aims to promote spiritual development must be ... [one] where feelings, emotions, values, beliefs, aspirations and the intellect are catered for... [and which] engages the mind, body and spirit in a way where learning experiences are integrated in the sense that there is an attempt to unify the whole range of experiences that students encounter.' [7, p. 22]

While these thoughts are in the context of school education, we believe that the need for a holistic, integrative and aligned curriculum is a vital step in providing responsive and empathetic pastoral care for large tertiary classes. 'Good relationships help to cultivate a sense of self-worth and value' [7, p. 19], and this can be seen in the context of relationships among learning groups and relationships between the students and the academic.

In this paper we have described how an integrated curriculum involving authentic and creative assessment, a collaborative learning environment, and an empowered academic can lead to a unified community of students. In our experience this has benefitted the students and PASS leaders, both in their content learning and in their broader growth, but it has also benefitted academia by giving them a greater connection with their students.

## References


[1] J. Biggs, *Teaching for quality learning at university*, Open University Press, 1999.

[2] B.L. Chance, *Experiences with authentic assessment techniques in an introductory statistics course*, Journal of Statistics Education **5** (1997), www.amstat.org/publications/jse.

[3] N.J. Entwistle, *Promoting deep learning through teaching and assessment: conceptual frameworks and educational contexts*, TLRP Conference (Leicester), November 2000, www.ed.ac.uk/etl/publications.html.

[4] J.J. Higgins, *Nonmathematical statistics: A new direction for the undergraduate discipline*, The American Statistician **53** (1999), 1–6.

[5] M. Mackisack, *What is the use of experiments conducted by statistics students?*, Journal of Statistics Education **2** (1994), www.amstat.org/publications/jse.

[6] F. Marton and S.A. Booth, *Learning and awareness*, Lawrence Erlbaum, 1997.

[7] C. Meehan, *Promoting spiritual development in the curriculum*, Pastoral Care in Education **20** (2002), 16–24.

[8] V. Miller, E. Oldfield, and G. Gregg, *Peer assisted study sessions (PASS): Facilitating effective learning in a student directed environment*, Effective Teaching and Learning 2002 (Brisbane), November 2002, www.talss.qut.edu.au/etl2002/.

[9] K.M. Plank, *The process and product of collaborative activities or three men and an egg*, The Penn State Teacher II: Learning to teach; teaching to learn, 1998, www.psu.edu/celt/PST/KMPcollaborative.html.

[10] H. Tait and N.J. Entwistle, *Identifying students at risk through ineffective study strategies*, Higher Education **31** (1996), 97–116.

[11] K.J. Topping, *The effectiveness of peer tutoring in further and higher education: a typology and review of the literature*, Higher Education **32** (1996), 321–345.

[12] L.S. Vygotsky, *Mind in society*, Harvard University Press, 1978.



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# SOME EXPERIENCES WITH TEACHING LARGE CLASSES

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This paper addresses some problems faced when teaching large service courses such as 1,200 students in a stage one statistics course. Some of the issues discussed are the problems of finding sufficient space for students in spite of streaming, keeping the streams aligned, and, catering for the diversity of interest and capabilities of the students. The large numbers enrolled necessitates streaming. Previously, overcrowding during the first few lectures of the year was ignored as, overall, seating was sufficient; this is no longer true and videoing of lectures is now used to supplement the streams. The videos have proved popular and so we plan to keep CDs of these in the library. To keep the streams parallel, we use the same day and lecturer for the three repeated lectures and use pre-prepared overheads and power-point presentations. Since financial considerations make us accept students with insufficient background, we often have to cater for students with a wide level of preparedness. Regular tutorials and assignments are given but recently introduced "consultation" or "drop-in" sessions are proving to be immensely popular. Self-test quizzes on Web-CT have been added as a means for students to assess their grasp of the subject. Putting course notes on the Web before the lecture helps overcome the problem many have with copying notes. A work-book containing all the year's tutorials and assignments, and a CD of other course material including lessons and tests of the background material that students are expected to know is planned for next year. A small token mark (0.5% of final grade) is given for attendance at tutorials and this is making a difference to tutorial attendance. Excel has been chosen as the computer software for the course; special optional tutorial sessions for Excel are held at the beginning of the year.

When dealing with large service courses such as elementary stage 1 statistics (1,200 students at Canterbury), there are many problems peculiar to large numbers enrolled in the course rather than the subject material being taught. Some of these are:

- the need for several streams
- the large numbers in each stream
- keeping up the interest of diverse students
- large numbers of students with a limited amount of English
- organisation of tutorial and consultation sessions
- large number of queries - many of them routine and mundane
- speed of delivery – too fast for some; too slow for others
- coping with non-attendance at lectures, tests and exams
- varying levels of computer literacy

This talk presents ways in which we are trying to cope with some of these problems. Perhaps this talk will give you ideas but my main hope is that you will be able to contribute ideas on the teaching of large numbers of students.

The large numbers enrolled means that streaming is necessary; currently we have three streams. We have had three streams for many years. For several years now one of the streams at a more popular time had, for the first few lectures, more students than seats available in the lecture rooms. However, nothing was done about this till this year since

- overall there have been enough seats for all those enrolled;
- the over-crowding stops after the first few weeks and the shortage of seats soon ceases to be a problem;
- starting another stream would not have been possible because of both space and resource constraints.

This year we actually have more students than the seating capacity of all three streams. Although students attending lectures did decrease so that no one was turned away after the first few weeks, we felt that an alternate solution had to be found since we were really not providing enough seats for all enrolled. The solution we decided on was to video-tape the lectures. It was not possible to simulcast these as there were no suitable rooms available at the time of any of the lectures. What we are currently doing is converting the taped lecture into a format that can be re-played on a computer. Initially we made these available on the campus computers by converting the videoed lectures to streaming video and storing the resulting lecture records on one of the servers on campus. This way any student on the campus network (and this included the hostels) could view it at any time. The campus computers have the necessary software and sound cards to enable this to be seen. The students have to bring their own headphones to listen to the sound. The students log on to the Web-CT system and then get access to their chosen lecture. Feedback has been very positive – some using it because they missed a lecture, others for revision, or because they could not understand it the first time. However we felt that compressing the videos sufficiently to enable them to be streamed on the campus network resulted in rather poor quality of the final pictures. We are now also creating good quality videos using less compression and putting these on CDs that are available from the library on overnight loan.

Another major problem is keeping the streams parallel. We have for several years used the same day for the three repeated lectures so that holidays etc did not put the streams out of phase. We also use the same lecturer for all the three streams. The use of pre-prepared overheads and recently power-point presentations ensures that the material in each stream is essentially the same. The lecture notes are also put on the web-site before the lectures are actually presented – sometimes in its complete form and sometimes with some bits missing that the students add during the lectures. Most students preferred the complete version.

Since we are now funded according to the number of students we have had to start accepting students that are not up to the required standards for doing many courses. This often results in classes with students with a wide range of preparedness for the courses they take. To cater for varying abilities we have set up "consultation sessions" or "drop-in" sessions which have proved immensely popular, especially at times when assignments are due or when tests and examinations are near. We have also just started providing self-test quizzes on Web-CT. These tests do not count towards a final grade; they are just there for the students to test their knowledge of the subject. The help classes above would assist students if they have trouble answering the quizzes. The students are encouraged to do these quizzes by pointing out to them that if they can do well in the quizzes then they should be able to get a high grade in the finals. At the moment the quizzes stand alone. A student selects a chapter and the software randomly selects 20 questions from a bank of questions based on the chapter chosen. They have 50 mins to complete these questions though they are told that 30 minutes should be enough when they are sufficiently familiar with the material. After the quiz, the computer checks their answers and supplies the correct choice for each question. A score is kept of all their attempts and thus a student can keep track of his or her progress in the course. Students may repeat quizzes from a chapter as many times as they feel necessary. Currently the quizzes are not linked to any part of the course but we are contemplating putting links from quizzes to lessons for self teaching.

The placing of course notes on the Web before the lecture also helps overcome the speed of delivery problem. This certainly assists the slow ones who feel that everything has to be copied but can only copy at a very slow pace. There is a danger with this approach since, if the students kept busy copying they tend to stay involved; otherwise they could switch off.

Currently the students get one assignment and one tutorial a fortnight. These are placed on the Web site just before they are set. We are however wondering if we should be working towards making a work-book which contains all the year's tutorial problems and assignment problems. The students would then work through these over the year bringing their work-book to the tutorial classes to be marked and commented on. The workbook along with perhaps a CD of the entire course could be sold to the students before they begin the course. This CD would also have some lessons and tests of the background material that students are expected to know before beginning the course. Students would be advised to do these quizzes and if necessary go through these lessons before the course begins so that they can come better prepared for the course. Perhaps these pre-tests could be made freely available on the Web so that students can see if this is a course they are able to do, but on the other hand this could prove to be a deterrent for some, resulting in reduced enrolments.

Non-attendance at lectures and tutorials is a problem. It is difficult to know how to ensure that students attend lectures. We do not think that the old system of students signing a roll is really going to work in large classes. Especially now as the material is all the Web and Web-CT videos, it is perhaps a tribute to the lecturers involved that so many come to lectures. Tutorial attendance too has often been a problem although this year the numbers attending tutorials seem to be much better. The two marks (which count for 0.5% towards the final grade) they get for attending a tutorial may have a lot to do with this.

We think it is essential to have a teaching assistant to help the lecturers to deal with the paper work associated with the course as well as a number of minor but time consuming chores, such as the website and the Web-CT links, which invariably go hand in hand with a large class. Non-attendance at tests and exams also causes administrative problems.

In many subjects and certainly in Statistics, the use of computer packages is becoming increasingly important. The computer literacy of students tends to be very varied and this can be a major problem. We have chosen to use Microsoft Excel as it is readily available, easy to use and is a generally useful tool. It comes in the standard Microsoft Office package and is available freely or cheaply to most students. It is easy to use and many are already good at it, but even those who have a computer phobia can often be taught to use it in a very short time. Even if the students do not become statisticians, they go away with a useful tool that they can use in many other situations. We have a booklet that the students work through at the beginning of the year to ensure that they have the necessary Excel skills for the course. There are special optional tutorial sessions at the beginning of the year to help those who need help with Excel. Students book for these tutorials for a one hour session, but those who need extra help can go to sessions they have not booked for, if space permits. We have found this arrangement very satisfactory.

The department has now introduced a second semester version of this course whereby the full year course is taught at a faster rate in one semester. In spite of being put on at the last minute, there are nearly 200 students are taking it. The resources developed for the full year course are being used for this course with minor modifications where necessary.

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# SUPPORTING GOOD PRACTICE IN ASSESSMENT

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The authors have been commissioned by the Learning and Teaching Support Network for Mathematics, Statistics and Operational Research in the United Kingdom to write a series of booklets on Assessment in MSOR. This paper describes these booklets and gives a commentary on our writing, providing a rationale and indicating the pedagogical issues involved.

## Introduction

"What you assess is what you get" and so it is vital that we get assessment right! Hence the enormous current international interest in assessment. In November 2001, the Learning and Teaching Support Network Generic Centre (LTSN Generic) in the United Kingdom (UK) published *Assessment Series*, a boxed set of twelve booklets, also available for download [1]. These are intended as a resource to the whole academic community. The twelve titles are:

1. Assessment: A Guide for Senior Managers
2. Assessment: A Guide for Heads of Department
3. Assessment: A Guide for Lecturers
4. Assessment: A Guide for Students
5. A Briefing on Key Skills in Higher Education
6. A Briefing on Assessment of Portfolios
7. A Briefing on Key Concepts
8. A Briefing on Disabled Students
9. A Briefing on Self, Peer and Group Assessment
10. A Briefing on Plagiarism
11. A Briefing on Work-based Learning
12. A Briefing on Assessment of Large Groups

Subsequently, the LTSN for Mathematics, Statistics and Operational Research (LTSN MSOR) commissioned us to write a similar set of booklets specifically for the mathematical sciences community, primarily, but not exclusively, in the UK. In this paper we report on how we did this task. We describe our analysis of the generic *Assessment Series*, and the decisions we took concerning the structure and the content of the MSOR booklets. We outline the content of our booklets and discuss some of our writing in more detail. We include our reflections on the task, relating the ideas to our own research and practice and the published work of others. Thus we give insights, not only into the *product*, but also into the *process*. It is intended that our publication will be available for download from the LTSN MSOR website.

## Analysis of Assessment Series

Many of the authors of the twelve booklets are well known in the higher education scene in the UK as educationalists and educational developers. Most have published extensively. For example Chris Rust[1] (Booklet 12) is an Educational Development Consultant at Oxford Brookes University and Phil Race[2] is a free lance author and consultant. George Brown (Booklets 2 and 3) is a retired professor from Nottingham University and noted contributor to the national training agenda. Mantz Yorke (Booklet 1) is well known in the educational literature, a former Director of the Higher Education Quality Council, and is based at Liverpool John Moores University.

Each of the twelve booklets runs to about thirty A4 pages and about 10,000 words, except for No 10 which is about half that size. They are all truly *generic* in that the ideas apply to most subjects, but requiring specialist subject interpretation and modification. Our task was to interpret the ideas in terms of the assessment of mathematics, statistics and operational research (MSOR) and to write what are essentially modifications of the original, drawing on our own experience and knowledge.

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[1] http://www.brookes.ac.uk/services/ocsd/5_research/chris.html accessed June 16, 2003
[2] http://www.phil-race.net/default.htm accessed June 16, 2003

First, we isolated those booklets that would not lend themselves to MSOR interpretation. We identified booklet 1 – A Guide for Senior Managers and booklet 8 – A Briefing on Disabled Students, as falling into this category.

A "Senior Manager" is anyone above Head of Department level, and who would thus be interested in strategies for assessment that would be appropriate for the whole faculty or the whole institution. The author of booklet 1, Mantz Yorke, of Liverpool John Moores University, distinguishes between *leadership*, *management* and *administration* in relation to the assessment policies of an organisation. He writes on page 6 that leadership embodies "the capacity to envision or pick up new possibilities and to inspire others to join in." Of course, one would expect to see heads of department and programme leaders exercising this sort of visionary leadership, but quite often such visions flounder on the rock of institutional policies. Leadership for change must really come from the top, but may be inspired from below by lecturers who are visionary, charismatic and persuasive.

Booklet 1 also has an appendix on "Some technical considerations", which discusses different choices for grading scales, such as the common percentage scale and others with say 16 or 20 grade points. While this discussion is generic, it is included in our modification of the Guide for Lecturers, because it incorporates ideas, which are important for MSOR people when they are involved in assessing programmes which combine MSOR with another discipline.

We considered booklet 8 on assessing disabled students to be sufficient as it. The ideas presented are important and can easily be adopted for use in MSOR. Our modifications of the Guide for Heads of Department and the Guide for Lecturers draw attention to it. We have incorporated the remaining booklets into our writing, as outlined in the next section and as described in the remaining sections of this paper.

## The MSOR booklets - structure

We decided to write three booklets, which we shall refer to as booklets MSOR 1, 2 and 3. Booklet MSOR 1 was to be about 8,000 words and to include a Guide for Heads of Department and a Guide for Lecturers. Incorporating both of these Guides into one booklet means that both sets of people can easily read both Guides and thus obtain insights into each other's roles. Most Heads of Department will have come up "through the ranks", and so should be sympathetic to their subordinates, but sometimes they may forget what is required if they have been away from the "chalk face" for some time.

Booklet MSOR 2, with about 6,000 words, was to be the Guide for Students. By producing this in a separate volume, we would facilitate its reproduction for distribution to students. That left seven Briefings and we decided to incorporate our modifications of these into one volume – booklet MSOR 3 – as seven essays of about 2,000 words each. This packaging allowed us to commission other authors with particular expertise to contribute to the work. At the time of writing this paper (June 2003), not all of these essays have been written.

## The MSOR Guide for Heads of Department

The generic Guide for Heads of Department concentrates on explaining the need for a strategic approach to assessment, and presents a wealth of practical advice and experience on how to go about crafting a strategy. The authors, Alistair Mutch of Nottingham Trent University and George Brown, recently retired from the University of Nottingham, point out that "crafting is a subtle art", and go on to elucidate some of the subtleties. In particular these include an exploration of some of the managerial arts. For instance, in choosing an approach, one must be somewhere on a scale between completely accepting and copying an institutional strategy, and at the other end setting up a light structure and assuming a high degree of mutual trust and professionalism. There is also some considerable, and useful, discussion on how to manage the process of creating a strategy, with a view to ensuring that the eventual product is accepted and owned by the department members.
The generic issues raised in this guide are indeed relevant to and applicable in the area of MSOR, but will benefit in the MSOR guide from the addition of case studies of strategic approaches, showing how these can work within differing institutional cultures.

The booklet will be useful not only for Heads but also for others involved who, as they function as responsible professionals, need to understand the processes of which they are part.

## The MSOR Guide for Lecturers

Probably the most useful piece of advice a Guide for Lecturers could give is to say, "First of all write the *learning outcomes* of a module very carefully." Thoughtful learning outcomes (LOs) are the key to good assessment. But the next piece of advice would be, "Read the entire Guide before starting." This is the thrust of *Assessment Series* booklet 3, written by George Brown, recently retired from the University of Nottingham. He stresses the point that *alignment* between LOs, methods of learning, assessment methods and assessment tasks is crucial. Hence the lecturer, as teacher and assessor must know about all of these topics in order to perform well. Good MSOR lecturers with many years of experience will probably have had implicit LOs in mind, they will know how students learn and how to facilitate that learning. They will know how to set three hour, unseen, written, examination papers that are reliable and discriminating. But given the current demands from our customers, the students, for transparency in everything to do with assessment and relevance in everything to do with the curriculum, they, along with everyone else, must meet these demands. This is a good thing, because it requires the introduction of different methods of assessment, methods that are better than written examinations at assessing some LOs like, "On successful completion of this module students should be able to demonstrate effective teamwork skills."

There has been a recent development in the UK higher education scene in that Subject Benchmark Statements have been written and published by the Quality Assurance Agency (QAA) [2]. The MSOR Statement may be downloaded from the QAA website.[3] These statements each describe their subject and list, as LOs, the "benchmark" standards expected of graduating students at threshold honours level and at modal honours level. While the benchmark statement mainly relates to final year achievement, clearly the foundation for this must be laid in first year and built upon in succeeding years. The MSOR Guide includes a discussion of benchmarking.

The Guide starts with a reference to the MSOR briefing on *Key Concepts* and moves on to the writing of LOs. The reader is referred to Bloom's taxonomy [3] for help with constructing these. The Guide looks at the purposes and principles of assessment and catalogues many methods of assessment, commenting particularly on those that seem to be most appropriate for MSOR. There is also a comprehensive bibliography.

## The MSOR Guide for Students

The LTSN Generic Centre's *Assessment: A Guide for Students* by Phil Race, provides some excellent general guidance and advice to students. The *MSOR Guide* similarly seeks to reassure students and to make them aware how academics view assessment, addressing the issues that arise in MSOR. This requires introducing some of the standard ideas of assessment, and the current practice of providing information of what is being sought in assessment. There is a table, based on Race's [4] table in the generic guide (pages 11 to 15), listing the usual forms of assessment in MSOR.

One of the Guide's aims is to make students aware of what is valued in assessment and that they must engage with the material and the assessment to perform well. Rust, Price and O'Donovan [5] found that student learning can be improved significantly through students' knowledge of assessment criteria, although we need to note the limitations they found to this benefit. The corresponding advice for students is to be aware of what is actually assessed, and what contributes towards higher marks.

Many students have a false impression of what will gain them the outcome they seek. We all know of students who think that assessment consists of nothing more than regurgitating a few facts or solutions to standard problems. The guide contains advice on finding out what is assessed. The emphasis here is on readily available sources: the learning outcomes for modules and for entire programmes. Students' attention is drawn to the outcomes like "use skill in logical reasoning and problem solving", or the statement from the benchmark for MSOR that "A graduate who has reached the modal level should be able to demonstrate skill in abstracting the essentials of problems, formulating them mathematically and

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[3] http://www.qaa.ac.uk/crntwork/benchmark/phase2/mathematics.pdf accessed June 20, 2003

obtaining solutions by appropriate methods".[4] Both of these clearly seek more than simply acquiring knowledge. Equally, the criteria for the award of higher grades differ from those from lower ones by more than sheer quantity. This awareness, suitably reinforced by lecturers, has the potential to influence students' behaviour in ways all would approve of.

Apart from the emphasis on published learning outcomes, there is advice on types of assessment and how they work, on dealing with assessment, and on making use of feedback. The key point here is well put by Taras [6]: "for feedback to take place, the learner who is receiving it is required to be an active participant and use the information to alter the 'gap' " [between actual level and the level required].

The guide ends with some practical advice on tackling assessment in MSOR and some brief notes on plagiarism. MSOR has some particular problems with plagiarism in that we tend to quote the "common knowledge" of our subject without acknowledgement more freely than some others. Students need a clear statement of what this "common knowledge", for which they need not supply references, comprises. Your students need the protection of good advice!

## The MSOR Briefings

### *Key Concepts*

In the past, mathematicians have been slow to learn the language of education. Our community can no longer afford to be less than professional about this. Accordingly, this Briefing considers the Key Concepts of "Summative Assessment", "Formative Assessment", "Norm Referencing" and "Criterion Referencing". It takes the ideas from booklet 7 of *Assessment Series*, written by Peter Knight of Lancaster University, and explains them in the context of MSOR.

All MSOR lecturers, including recent graduates and others new to the profession, have experienced both summative and formative assessment, but may not have appreciated the distinct purposes and values of each. Quite often we use the same assessment tasks for both purposes. Thus homeworks and class tests are used in these ways – to test student learning for the purpose of reporting and for the purpose of giving feedback to enhance further learning. However experience is showing that this dual use may not always be the best thing to do. When students are given a grade or score for a piece of work, they are inclined merely to feel content or dissatisfied depending on the score, and then to file the work away without looking at it carefully to learn from any failure. On the other hand, if a grade or score is not given, but a careful commentary is provided, then the student is inclined to read the commentary and to learn from it. Indeed, by providing assessment tasks solely for formative assessment means that more challenging tasks, which might be considered unfair were they to be used for summative purposes, can be set.

Another aspect that our community might begin to consider is the idea that a particular assessment task be scored either 0 or 1, with a "1" being required to "pass" the module. In other words, the task must be completed satisfactorily (and "satisfactorily" could be defined), but a percentage score for inclusion in the overall module score is not assigned. Again the emphasis is on giving feedback to enhance learning.

The distinction between norm referencing and criterion referencing is often blurred in our assessment procedures. Thus we use a marking scheme (which may be hidden from students) to mark an examination paper – criterion referencing. But when it comes to awarding different classes of honours, we introduce an element of norm referencing, making final judgements to achieve the distribution of awards we think should be appropriate. Similarly, when it comes to awarding a score to a Project, we first award marks according to the criteria we have (usually) published, and then adjust these to achieve the rank order we consider appropriate.

Booklet 7 also deals with the Key Concepts of "Reliability", "Validity", "Affordability" and "Usability", and our Briefing explores these in the MSOR context. Again these words are from the language of education and may be foreign to many in our community.

[4] http://www.qaa.ac.uk/crntwork/benchmark/phase2/mathematics.pdf accessed June 20, 2003

### *Plagiarism*

This section aims mainly to raise the issues of plagiarism, with the starting point that students and staff need advice on what to avoid, and that some lecturers' behaviours may inadvertently lead to students adopting practice that is at risk of being plagiarism. The key point of the section is that lecturers should set appropriate tasks and advise students correctly in order to reduce students' temptation to plagiarise. Very poor work, which duly acknowledges sources, may be so derivative as to command few marks: if such work does not acknowledge the sources an accusation of plagiarism is possible. The borderline between plagiarism and poor academic practice may here be fuzzy, and making it clear to students that what they are trying to do is not what is valued may assist the larger problem. Students should be urged (repeatedly, as with several other messages) to give sources for and justifications of what they conclude. This reinforces the message that good MSOR work will in any case justify the steps taken in a clear argument, and would remove most of the plagiarism cases (but, sadly, not the poor work or the rare cases of cheating).

The guide contains a selection of sources of advice on avoiding plagiarism and on dealing with apparent cases as they arise. The action taken must, of course, be consistent with the institution's policy and the details will have to take this into account.

### *Key Skills*

The teaching of Key Skills is now, to a considerable extent, a requirement of all undergraduate programmes. It is widely recognised that the best way to do this is to embed their teaching in the subject curriculum. Those key skills, which are most important to the working mathematician, are identified and various ways of embedding them are discussed.

### *Self, Peer and Group Assessment*

These three forms of assessment tend to be treated, and to a considerable extent, they are indeed linked. The objective is not primarily to save the lecturer time and effort, although that may happen. Rather, the objective is to involve students in their own assessment so that they will increasingly "know themselves", and be better at discerning their own strengths and weaknesses. Engaging in peer assessment is useful because it helps develop students as self-assessors and, indeed, as diplomats. Furthermore, intra-group, confidential self and peer assessment is probably the best way of discovering how the group functioned. It is much to be preferred to observation by the lecturer, which will inevitable be invasive and therefore will distort the activity being observed.

### *Work Based learning*

In mainstream undergraduate mathematics teaching, the notion of work based learning of mathematics does not come into play. Work based learning in the part time job in the supermarket does of course contribute significantly to the development of a student's key skills, but this is uncontrolled, variable and not compulsory, and so cannot contribute to the university's assessment scheme. An exception is when a period of controlled sandwich placement in an appropriate environment forms part of a MSOR programme. This essay by Ian Taylor from the University of Ulster discusses the organisation and assessment of such a sandwich year.

### *Portfolios and Progress Files*

The idea of using a portfolio to assess a student's work is not widespread in the MSOR community. Nevertheless one of our universities (SHU) uses portfolios in its MSOR programme. The essential feature of a portfolio is that it comprises a collection of items rather than a single piece of work. Thus it is distinguished from a dissertation, although a dissertation may be included. Furthermore the portfolio contains those elements of their work that the student values and selects. It must include a written critical reflection on the work wherein the student presents evidence that they have achieved the LOs of their programme.

Progress Files are not considered in *Assessment Series* but their advent has been heralded and we considered it timely to include this topic. Universities will have to keep a progress file for each student, and, as part of this, students will have to keep an up to date "personal development plan"[5]. This is a new idea and it is good that we can share our limited experience in working with these.

***Assessing Large Groups***

This essay is written by JP Ward from Loughborough University and describes a computer assisted assessment (CAA) scheme in operation there. CAA certainly provides a mechanism for the weekly assessment of large numbers of students through on-line class tests. This technology is increasing in popularity. The essay describes the advantages and shortcomings of CAA and the improved results at Loughborough deriving from the overall system adopted there.

## Conclusions

The publication of *Assessment Series* by the LTSN Generic Centre was both timely and important. Assessment is currently a big issue in the higher education agenda, particularly in the domains of quality assurance and quality control, excellence in teaching and student learning. The arrangement of the *Series* in four *Guides* and eight *Briefings* gives it a distinguishing structure that immediately leads the reader to those topics of most immediate interest, and allows them to leave other topics for another day.

The experience has been an interesting one, and is not finished yet. Part of our brief is to disseminate the ideas and this paper and the Delta conference are part of the disseminating process. A draft of the work was available at the UK Undergraduate Mathematics Teaching Conference in September 2003, and the *Guide for Lecturers* was used as a resource by one of the working groups.

When we first read *Assessment Series*, we quickly became aware that a commentary on the generic material would be helpful to the MSOR community, through selecting material carefully and grounding it in the practice and understanding of academic mathematicians. As always, though, given the considerable freedom each university has to determine the curriculum, and each lecturer has to present it, our *Supporting Good Practice in Assessment in MSOR* will be nothing more that a guide containing well intentioned suggestions that we have found to work.

## References


1. LTSN Generic [Learning and Teaching Support Network – Generic Centre], *Assessment Series*, LTSN: York, 2001. Also available as papers ASS001 to ASS012 (30 November 2001) Retrieved June 16, 2003 from http://www.ltsn.ac.uk/genericcentre/index.asp?docid=17219.
2. QAA (Quality Assurance Agency), *Benchmarking Academic Standards*, 2003, Retrieved June 19, 2003 from http://www.qaa.ac.uk/crntwork/benchmark/index.htm.
3. B. S. Bloom, *Taxonomy of Educational Objectives: Cognitive Domain*, McKay, 1956.
4. P. Race, *Assessment: A Guide for Students*, LTSN Generic Centre, 2001.
5. C. Rust, M. Price & B. O'Donovan, Improving Students' Learning by Developing their Understanding of Assessment Criteria and Processes, *Assessment and Evaluation in Higher Education*, **28** (2), 2003, 148 - 164.
6. M. Taras, Using Assessment for Learning and Learning for Assessment, *Assessment and Evaluation in Higher Education*, **27** (6), 2002, 501 - 510.
7. D. Holton, editor, *Teaching and Learning Mathematics at Undergraduate Level*, Kluger, 2001
8. S. Griffiths, S. K. Houston & A. Lazenblatt, *Enhancing Student Learning through Peer Tutoring in Higher Education*, University of Ulster, 1995
9. P. Khan & J Kyle, editors, *Effective Learning and Teaching in Mathematics and its Applications*, Kogan Page, 2002



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[5] http://www.ltsn.ac.uk/genericcentre/index.asp?id=16906 accessed June 29, 2003

# SERIOUSLY TEACHING MATHEMATICS IS NOT TEACHING SERIOUSLY... FOR ALL THOSE SERIOUS UNIVERSITY STUDENTS

ANNE D'ARCY-WARMINGTON

Through action research, teaching first year service units in the university from Business to the Medical Sciences, many different learning strategies in the quest to communicate mathematical ideas to the "Mathematics Haters" have been found. Remember back to childhood when ideas just flowed into the brain and no resistance was given? The objective of this presentation is to show how strategies, often used in Kindergarten and Primary School, can be adapted for use in Tertiary Education. University educators can impart knowledge in a fun, enjoyable way for both students and lecturers. Seriously teaching Mathematics not teaching serious Mathematics.

The declining interest in mathematics behoves a rethink in traditional pedagogy, examining the results of an initial pilot study with a quantitative outcome, it will be demonstrated that effective teaching of mathematics rests in qualitative constructs. Analysing recent concepts of relational work and its place in institutions, this paper suggests that sensitive treatment of students' learning will foster an enthusiasm for mathematics. Socially mathematics is tolerated but not accepted as a viable discipline of study. Students usually ask, "Where will I use this type of mathematics?" In today's technical world, the scientific advances are transforming life at a pace never before experienced and economies are changing from national to global with knowledge needed to understand this evolution fragmented through different fields, both political and scholarly. The digital world's shift from literacy to numeracy, is one that cannot be neglected due to its pervasive application in various fields including Health Sciences, Business and Commerce, Anthropology, Information Technology, and Mining ([28]).

On a global scale, prices, interest rates and other economic variables have been shown to follow the Kondratiev cycle, which lasts between 50 to 60 years. These market forces drive the commercial world to adopt new technologies but social institutions lag behind, the market need people who can apply learning strategies to the work at hand, yet teaching the learning strategies behind mathematics are not the focus of educators ([4]). The teaching of mathematics has to keep pace with the changing tempo of the world and at the current mathematical crossroads; various innovative methods have been implemented to counteract the declining interest in mathematics ([7]; [18]; [24]; [25]).

Using the analogy of teaching kindergarten children to write a book, which progresses from the word → sentence → paragraph → essay eventually to the book. When a child first learns about words, the child is taught phonetically about the components of the word. They learn to utilise the knowledge of phonetics to pronounce new words and deal with the exceptions. This gives the child a basic tool and confidence to progress and unlock spelling, reading and comprehension of new words and ultimately the use of language because at university level, in basic mathematics course, it is often found that students are not confident with their ability to use mathematical language to progress to the solution ([14]). This paper suggests that the metaphor of the child should be incorporated to the non-mathematical student who is starting a journey similar to the long journey of language at kindergarten.

Studies at pre-primary level have shown that the teacher's body language and presence have a positive effect on the learning of the child. ([34]). Such tacit cues give the child guidance when combined with face-to-face teaching it provides learning with a competitive edge. Tacit knowledge is passed on mainly face-to-face although available to fewer people at a time it can give the competitive edge by reducing the risks of ambiguity and misconception of ideas, the teacher is thus the primary provider of tacit knowledge to the students. Codified knowledge, the written word obtained mainly by the Internet but available in other areas, has widened the scope of information available by offering quick and easy access to an increased audience. It has the ability to showcase new innovations and products to more people but since this knowledge is present everywhere offer the advantages of tacit knowledge ([11]). Tacit knowledge and experience are the principle components for innovation and is the springboard to enable individuals to utilise skills in other situations and disciplines ([29]). The teacher plays a very important part of the education process more than just teaching the subject matter. In the mathematics field, from pre-primary upwards, the teachers should impart to the students enthusiasm, passion and understanding for the topic of mathematics, these important qualities will foster maximum outcomes from students throughout school, university and eventually their chosen career. This enthusiasm, passion and understanding become the tacit knowledge to be passed from generation to generation. In the early stages of mathematics education, the teacher's role is pivotal to the child's approach and attitude to mathematics in later life. Research has shown that many pre-service primary teachers try to avoid mathematic electives, exhibit negative emotions and show anxiety towards mathematics. In addition parents can also contribute to the negative attitude towards mathematics by a blatant lack of enthusiasm and avoidance of help with homework, these emotions are often not hidden from the students who imbibe these feelings and so the cycle continues ([27]). Mathematics has gained a social stigmatic reputation as a difficult subject that should be disliked and avoided. In an effort to reduce this mathematical anxiety, the different learning styles must be appreciated and accommodated during the education of mathematics at every level. This issue has to be emphasised at an early stage so that confidence is built up in the child rather than destroyed ([6]; [15]; [17]) If this confidence is present then learning can really begin, as the mind is now open to ideas and concepts.

An interesting approach is to draw upon the similarities of schools and long established organisations; both are consumers of commercial products of training education, have instilled traditions, beliefs and practises in the framework and most importantly, need to please the consumer to be competitive ([23]). Commercial enterprises use consumer surveys and research to market a desirable successful product, so why shouldn't educators use a similar approach to learning? It is essential to find ways in which the consumer wants to learn, not the way the provider wants to deliver the product, in order to be productive and viable ([3]). In teaching mathematics, you are training the student to access more desirable ways of processing information and differing learning styles affect the development of ways to process information that can be utilised in various mathematical and non-mathematical situations. Previous studies describe the student's learning in terms of concrete and abstract concepts dividing them into four types: Divergers, Assimilators, Convergers and Accommodators ([20]; [21]).

Divergers use concrete experiences, reflection, and imagination. These students have the ability to create and share ideas in groups. They are emotive, people-orientated, and practical. Assimilators think abstractly, and then use reflective techniques. These students can follow procedures and directions well to assemble separate pieces of information to the puzzle. They are more at ease with abstract concepts than people and practical situations and like to work alone to form theories once the concepts have been established. Convergers use abstract means but learn by doing and thinking. They like to solve problems quickly and efficiently, although these students like the hands-on practical approach, and find it more appealing to work with objects rather than groups of people. Accommodators gain knowledge by concrete means then apply this knowledge actively. These students experiment with this knowledge in new situations in various ways including problem solving, taking risks, and trial and error methods. They love communication of ideas with others and as the name implies, accommodate and adapt to different circumstances with the greatest of ease.

The learning styles are combined with different approaches to learning and these styles will change with time as more understanding is gained. Three types of approaches to learning defined in previous research; Surface learners are as the name implies learners that do not go beyond the surface of the topic and tend to memorise just to pass the examination. Achieving learners would be interested in learning the topic with a variety of strategies including the understanding and procedures and when required memorisation. Deep learners look for the meaning and understanding of the subject and can easily relate the topic to other areas both in and outside the course ([2]). The student can use mixture of any three approaches to suit the situation. Armed with this knowledge of the different approaches and ways of learning, the student and educator can apply different strategies and increase the capacity of knowledge and understanding. There are the three modes of transferring this knowledge, oral, aural and visual and if done in the correct proportions it can suit all the different approaches to produce knowledge transfer.

It is important not only to know the learner type but also to understand the stages of cognitive development. Some students express the feeling that their mathematical ability is comparable to a pre-primary student's ability of language. Many schools use of Piaget's theory of development as follows:

- Sensori-motor stage - This is the infant period where the view of the world is limited and is acquired by physical interaction. Motor activity and mobility are the fundamental methods of gathering knowledge. Memory and language skills start to develop.
- Pre-operational stage - This is the toddler and early childhood period, the language and memory skills are maturing and the imagination is developing along with use of symbols. The thought processes are still concerned about the child and logic has not started yet.
- Concrete operational stage - This is primary and early adolescence period where concepts about number, length, area etc are developed with logical and thoughtful use of symbols relating to concrete objects. Mental perception of ideas increases.
- Formal operational stage - This is the adolescence and adulthood period where abstract concepts and symbols can be used logically together. Formal thinking begins but not all adults reach this stage ([19]).

The chronological age of the student is not always the best indicator of cognitive development stage reached and often the student's own perception of their mathematical stage is somewhere between pre-operational stage and concrete operational stage at the start of semester. If they perceive themselves to be at this stage then use the principles of teaching that will aid them to progress to the mathematically higher levels.

The teaching of mathematics at all levels involves many more aspects than just the subject matter. It is time now to put new life into the style of lecture given to non-mathematical students because it appears that negative emotion has one of the biggest impacts on the ability of the students to learn mathematics. Recent research has shown that there is a strong link between emotions and rationale theorising that positive emotion may facilitate an easier path to learning. Mathematics is often associated with deep negative feelings by a majority of students at all levels and this negativity can last a very long time. Pre-primary and kindergarten students have not yet been subjected to this emotion since the mathematics is still very much playing activities and hands-on approach. The environment is colourful, light-hearted and happy even though it involves the teaching of the fundamental concepts of mathematics but by high school and university this environment is stark and hostile and not conducive to learning any more ([35]). When and where did the fun leave mathematics teaching and how can we bring it back?

We have five senses with which to acquire knowledge so why not utilise as many of them as possible at the higher levels of education rather than leaving these tactics at pre-primary and primary level. Often, the practice of the whole class chanting the alphabet or reciting a story can be heard in early education classroom, here the information is being processed orally, aurally and visually. Adapting this activity to high school and beyond is easy and the knowledge transfer is processed in three ways, concentration to be able to read the words, thinking about the sound of the words as they are voiced and finally hearing the words. This method gives the student confidence in the mathematical process needed to solve a problem by vocalisation into smaller components. Mathematics is not the first choice of topic for discussion yet it is a subject that has potential to use creativity, imagination and personality ([22]). Unfortunately most people view it as a "clever subject" or "one that requires a lot of hard work". This vocalisation can be made as an informal, fun, personalised mathematical message for each student to lessen the anxiety associated with mathematics because identifying, acknowledging, and introducing strategies concerning mathematics anxiety for the student is often the first hurdle the lecturer has to overcome before teaching ([16]).

Every day people will discuss, debate and argue many topics from the weather to politics, during these exchanges knowledge will be acquired, reinvented, or reinforced. This medium may be used equally as well for the acquisition of mathematical knowledge because discussion is a little used strategy in mathematics but if utilised can build confidence in the student via teacher and peer support ([32]). Talking and listening about any subject is educational, as one can remain passive and just absorb information or one can participate without the threat of examination of one's opinions. Discussion of problems and concepts makes follow-up readings easier to do as we have acquired a certain amount of familiarisation with the topic. Mathematics is perceived to be a particularly hard topic to read if you have limited knowledge so discussion in class is of the utmost important to understand relate to, and personalise technical terms ([30]). The lecture becomes a forum for swapping ideas and knowledge, assessing strategies, evaluating skills such as note-taking, effective reading etc. These skills are often incorrectly assumed to be present and therefore not specifically taught. The students engage actively in the participative lecture even though often the teacher does most of the talking; the atmosphere is conducive for student's opinions and apprehensions to be aired. Students have given positive feedback for this style of lecture showing appreciation for the relaxed and comfortable atmosphere. Time has been allotted in the sessions for the practise of thought transition maps and self-correcting analysis ([31]). The days of the quiet passive lecture may soon be replaced by interactive tutorial sessions.

Thought transition maps are basically maps of emotion, reflection and strategy to solve a mathematics problem. After reading the question, the starting point is expressions of confidence or doubt about the solution; then what alternatives routes to the next line exist, explore all possibilities and reason out loud why some are rejected and others accepted and repeat until the final answer. The students and teacher will write the instructions on the paper next to the problem as hints and notes. Studies at primary level have used similar mind maps and the time allocated to this practice has be shown to be productive ([9; 8; 7]). Here the opportunity for the students to find a personalised pattern to memorise and through reflection on to a better understanding of the concept. It is important to let the students build up confidence in the subject even if this is through rote learning and memorisation by non-mathematical means. The students have the information needed to develop the deep meaning learning approach from surface and achieving learning approaches ([2]). There are many pathways to understanding, we do not all follow each other down the same road nor use the same mode of transport.

Self-correcting analysis involves notes, hints and corrections written at the side of the solution. Mistakes are to be learnt from and should not be crossed out, as often these same mistakes are repeated. Reinforcement of correct ideas can be viewed along side the previous incorrect attempt. This reflective behaviour eventually leads to more thorough understanding of the use of learning styles, which can transferred to other parts of the degree course ([26; 33]). Listening to a teacher's explanation then elaborating with your own words to fellow students provide more understanding for, the teaching student, pupil student and teacher. In these little discussion sessions, different strategies are revealed and adapted by students and the teacher has an insight into the students' perception of the lecture. In a non-intimidating environment the students can express, share and experiment with thought transition maps and self-correcting analysis and so build up confidence in talking about mathematics. Feedback from students showed all 33 participants agreeing to the usefulness of thought transition maps with almost 25% strongly in favour.

Mathematics is seen as a serious subject and teachers are often portrayed by the media as calculating machines, a lack of interest in people and any other non-mathematical activities. Students do not attend class with any enthusiasm, as it will be another hour of meaningless and incomprehensible instruction with little personal interaction socially with the teacher. Pre-primary and primary schools foster the social side of education whilst high school and university give greater priority to the subject matter. The emotional involvement with students during pre-primary and primary years does not hinder the learning process but rather enhances it. Studies have shown that young children value caring and support from teachers as this they feel creates a healthy learning atmosphere. This relational behaviour seems to be relegated and not viewed as important once the student progresses to high school and university, they are large institutions where one can feel isolated in a variety of ways and this can affect the learning capability of the individual. Relational behaviours are often deemed as unnecessary and out of the tune with the perception that the university is there to teach the chosen topic according to the syllabus, university is a life experience involving all aspects of social interaction not just learning ([10]; [13]).

Using learning strategies from pre-primary and primary at higher levels of education may be the key that will unlock the secret of learning for the non-mathematical students. It may help the popularity of mathematics at high school and beyond. Playing in pre-primary helps us start our journey of life experiences and hopefully the enjoyment and fun can remain for both student and teacher at all levels. Einstein used his imagination by pretending to be a charged particle to help with his theories, using our imaginations may well give us the competitive edge to seriously teaching mathematics not teaching serious mathematics.

## References


[1] Casey, L. (2001). Radical Algebra: Who figures in the equation? 148(4), 157-159. Dissent Publishing Company, Inc, Book Review of Moses, R.P., & Cobb Jr, C. E. (2001). Radical equations: Math literacy and civil rights. Beacon Press 192.

[2] Chapple, S.(1999). Understanding student learning, in K. Martin, N. Stanley & N. Davidson(Eds), Teaching in the disciplines/learning in Context, 67-71. Proceedings of the Eighth Annual Teaching Learning Forum, The University of Western Australia, February 1999. Perth: UWA. http://cea.curtin.edu.au/tlf/tlf1999/chapple.html

[3] Cohen, S., Dove, D. & Bachelder, E. (2001). Time to treat learners as consumers, Training & Development, 55(1), 54-57.

[4] Cone, D. (2001). Active learning: The key to our future, Journal of Family and Consumer Sciences, (93)4, 19-21.

[5] Daniels, D. & Perry, K. (2003). Learner-centered according to children, *Theory into Practice,* 42(2), 102-108.

[6] Deitte, J.M. & Howe, R..M. (2003). Motivating students to study Mathematics, *The Mathematics Teacher,* 96(4), 278-280. Retrieved 28th April 2003, from Pro Quest Social Science Plus.

[7] de Mestre, N. & Sugden, S. (1999). Designing a successful General elective mathematics course. In W. Spunde, R. Hubbard, & P. Cretchley (Eds), The challenge of diversity. Proceedings of the 2nd Delta Symposium on Undergraduate Mathematics, Queensland. 93-97.

[8] Doerr, H. & English, L. (2003). A modelling perspective on students' mathematical reasoning about data, *Journal for Research in Mathematics Education.*34(2),110-136.

[9] Ebdon, S., Coakley, M., & Legnard, D. (2003). Mathematical mind journeys: Awaking minds to computational fluency. *Teaching Children Mathematics,* 9(8), 486-493. Retrieved 28th April 2003, from Pro Quest Education Complete, Social Science Plus.

[10] Eveline, J., & Booth, M. (2003). Organising emotionality in the Ivory Basement: Fending off the Blues and the Blows, (forthcoming).

[11] Feldmann, M.P. (2000). Location and Innovation the new economic geography of innovation spillovers and agglomeration, in G.R. Clarke, M.P. Felmann & M.S. Gertler (Eds).*The Oxford Handbook of Economic Geography*, (Oxford: Oxford University Press), 371-394.

[12] Fennema, E., Sowder, J., & Carpenter, T.P. (1999). Creating classrooms that promote understanding, in E. Fennema & T.A. Romberg (Eds), *Mathematics Classrooms that Promote Understanding*, Mahwah, NJ: Erlbaum,185-199.

[13] Fletcher, J. (1999). Disappearing Acts: Gender, Power, and Relational Practice at Work, Cambridge, MA: MIT Press.

[14] Fung, P., & O'Shea, T. (1994). Using software Tools to learn formal reasoning, a first assessment CITE report 168 Open University.

[15] Furner, J., & Berman B. (2003). " Math anxiety: Overcoming a major obstacle to the improvement of student math performance", *Childhood Education,* 79(3), 170-174. Retrieved 28th April 2003, from Pro Quest Education Complete, Social Science Plus.

[16] Hackworth, R. (1992). Math anxiety reduction. Clearwater, FL:H&H Publishing.

[17] Hillel, J. (1999). A bridging Course on Mathematical Thinking. 109-114. in W Spunde, R Hubbard, & P Cretchley (Eds), The challenge of diversity. Proceedings of the 2nd Delta Symposium on Undergraduate Mathematics, 21-24 November 1999 Queensland.

[18] Hollingworth, R.W. & Mc Loughlin, C. (2001). Developing science students' metacognitive problem skills, *Australian Journal Of Educational Technology*, 17(1), 50-63. http://www.ascilite.org.au/ajet/ajet17/hollingworth.html

[19] Hummel J. (1998) *Cognitive Development,* Retrieved 18th June 2003 from: http://chiron.valdosta.edu/whuitt/col/cogsys/piaget.html.

[20] Kolb, D. (1985). LSI learning style inventory: Self-scoring inventory and interpretation booklet, (Boston:Mcber).

[21] Kolb, D. (1984). Experimental learning: Experience as the source of learning and development. Englewood Cliffs, NJ: Prentice –Hall.

[22] Lehrer, R., & Lesh, R. (2003). Mathematical learning in W.M.Reynolds & G.E. Miller (Eds) *Handbook of Psychology: Educational Psychology*, 7, 357-391.

[23] Lynch, J. (2000). Effective implementation of new technologies: legitimising change in strategies in schools. Conference of the Australian Association for Research in Education. Dec 4-7 2000 Melbourne: Australian Association for Research in Education 2000. Retrieved 7th June 2003, from Pro Quest Education Complete, AEI(Education).

[24] Mc Loughlin, C., & Luca, J. (2000). Cognitive engagement and higher order thinking through computer conferencing: We know why but we don't know how? in A. Herrimann & T. Kulski (Eds), Flexible Futures in Tertiary Learning, 219-234 Perth:Curtin http://cleo.murdoch.edu.au/confs/tlf/tlf2000/mcloughlin.html

[25] Mc Loughlin, C., & Oliver, R. (1998). Planning a tele-learning environment to foster higher order thinking, *Distance Education*, 242-264.

[26] Pape, S., & Smith, C. (2002). Self-regulating mathematical skills, *Theory Into Practice,* 41(2), 93. Retrieved 28th April 2003, from Pro Quest Education Complete, Social Science Plus.

[27] Peard, R. (1999). Encouraging Undergraduate Primary Teacher Education Students to Select Mathematics Content Electives. 160-165. in W. Spunde, R. Hubbard, & P. Cretchley (Eds), The challenge of diversity. Proceedings of the 2nd Delta Symposium on Undergraduate Mathematics, 21-24 November 1999 Queensland.

[28] Poindexter, S. (2003). The case for holistic learning, *Change,* 35(1), 24-30. Retrieved 28th April 2003, from Pro Quest Education Complete, Social Science Plus.

[29] Simmie, J. (2002.). Knowledge, Spillovers, and Reason for the concentration of innovative small-to-medium enterprises, *Urban Studies,* 39(5-6), 885-902.

[30] Stock, P.M. (2001). Towards a theory of genre in teacher research: Contributions from a reflective practitioner, *English Education*, 33(2), 100-108. Retrieved 28th April 2003, from Pro Quest Education Complete, Education.

[31] Swan, G. I. (2000). Reading, taking notes and problem solving in A.Herrmann & M.M. Kulski (Eds*), Flexible Futures in Tertiary Teaching.* Proceedings of 9th Annual Teaching Learning Forum, 2-4 February 2000. Perth: Curtin University of Technology. http://cea.curtin.edu.au/tlf/tlf2000/swan.html

[32] Swan, P. & Sparrow, L., (2001). "Strategies for going mental" in Mathematics: Shaping Australia: Proceedings of the 18th biennial conference of the Australian Association of Mathematics Teachers Inc 15-19 January 2001, Australian National University, Canberra, ACT. Adelaide: Australian Association of Mathematics Teachers 2001.

[33] Swedosh, P. (1999). "Dealing with Diverse Student Background" 206-212. in W. Spunde, R. Hubbard, & P. Cretchley (Eds), The challenge of diversity. Proceedings of the 2nd Delta Symposium on Undergraduate Mathematics, 21-24 November 1999 Languna Quays, Queensland.

[34] Valenzeno, L., Alibali, M.W., & Klatzky, R. (2003). "Teachers' gestures facilitate students' learning: A lesson in symmetry, *Contemporary Educational Psychology*, 28(2), 187-204.

[35] Weiss, R. (2000). Emotion and learning, *Training and Development,* 54(11), 44-48.

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# Experience of undergraduate teaching of mathematics in France

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**Abstract**

Over the last 20 years the volume of mathematics in secondary schools has been shrinking and the programs have been reducing the abstraction level. Therefore univer sity teachers are confronted with the decrease of the level of preparation and mathematical maturity. Moreover the number of students who choose science has been plummeting dramatically. Ways of dealing with the situation will be presented.

When 10 years ago I taught a course of functional analysis, compulsory for the students of Maîtrise (fourth year), there were more than 80 students in front of me. Now Maîtrise of mathematics counts about 30 students.[1] The tendency is similar in DEUG (diploma of general university studies).[2] Freshmen, who intend to study mathematics, enroll in DEUG of MIAS (mathematics, computer and applied science); after 2 days they will have to decide if indeed their major will be mathematics; if so, they will enroll in Licence of mathematics (third year). Students in France can stop their university education after either DEUG or Licence or Maîtrise or DEA (diploma of deeper studies)[3], and in each case they will receive a corresponding diploma. Many of them will pursue their instruction in special classes designed to prepare a competitive exams for teacher's jobs of various level (CAPES[4], Aggregation).[5]

Not only the number of mathematics students is plummeting. The number of hours consecrated to mathematics both in secondary schools and at DEUG have been diminishing over the last two decades, and the content of mathematical teaching has become less abstract and conceptual. The reformers apparently overreacted with respect to the previous situation in which over-theoretic mathematics were a fundament of most processes of academic selection.

As a result, the freshmen whom we meet now, have clearly smaller capacity of abstraction and of logical thinking than their peers, say, 15 years ago. But what are the reasons of their reluctance for mathematics as a major subject? One is probably a lesser intellectual attraction of what they are being taught at school as mathematics (cook book style). Another is a social phenomenon due to attractiveness of more practical, better paid professional activities like commerce and management. It follows from my personal polls that students perceive a mathematical curriculum as very demanding and more selective (also at the stage of hiring) than other studies.

Every four years universities elaborate their programs and submit them to the ministry. At Burgundy University, the first year the MIAS students follow a mathematics course of 4 (academic) hours (per week) completed with 4 hours of exercises. Mathematicians managed to convince the Faculty of Science that it was not enough, so there are 2 hours more of "techniques of calculus"

[1] By the way, the situation is even worse in physics. The trend concerns exact sciences all over in France.

[2] This term denotes the two first years of university studies in France.

[3] Diplôme d'Études Approfondies

[4] Certificat d'Aptitude pour Enseignement Secondaire

[5] These are last years of the existence of this specific French system, as the period of transition to the European "Masters" system has already begun.

during one semester that comprise such things like parametric curves, techniques of integration, multiple integrals and differential equations.

As for the main course of mathematics, it consists of basic analysis (limits of sequences, series, continuity of functions), arithmetic and algebra (divisibility, polynomials, matrices) and linear algebra. Two examples:

1. The determinant of a matrix is introduced as a recursive formula without any theoretic indication of the independence of the choice of a column or a row. I could not resign myself to leaving to the students this unappealing image of the concept, so I explored it, with my 2 exercise groups, as a multi-linear form and as a volume. My impression was that this was helpful to many of them.

2. Irreducible polynomials in $\mathbb{C}[X]$ and $\mathbb{R}[X]$ are discussed in detail (though, of course, without the fundamental theorem of

   algebra), but no reflection is carried in case of $\mathbb{Q}[X]$. So I completed the program by exercises on the Gauss lemma and the theorem of Eisenstein.

3. The program provides only basic tools for determining limits of

   sequences. I found that it was useful and reassuring for students to be able to compare the limits of the arithmetic and geometric means and to deduce $\lim_{n\to\infty} a_n^{\frac{1}{n}}$ from

   $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Most of the

   students could then calculate $\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n}$ for instance.

As I understand, some of my colleagues have a similar attitude. But the philosophy of some others is that the average level of the freshmen is such that teaching them anything deeper is useless. Moreover, sometimes their attitude conveys this message to the students. My conviction is different. I believe that one can show the beauty of rigorous mathematics even to very poorly prepared students. I taught a group of students who failed their first semester exams. They have right to follow a "catch up" course and then to repeat the first year in case they were successful with the "catch up" course. Sometimes, it seemed desperate to repeat and try them to repeat, many times, a simple important argument, that the level of their mathematical maturity could not motivate. However the outcome of the examination showed that about half of them achieved a good understanding of what was essential.

From a statistics that I made last year, it followed that the average note of the students of my 2 MIAS groups was 13% higher than the total mean. One can suspect that the adopted method was appropriate.

Last year I observed a new phenomenon: the students have a particular difficulty in linear algebra. A feeling is that the difficulty is due to deficiencies in logical thinking that becomes a more important component at a higher level of abstraction. From 70 examination sheets (some of them very
good) that I was marking, only one person answered correctly this simple
question: If $A$ and $B$ are linear subspaces, then $A \cup B$ is a linear
subspace if and only if either $A \subset B$ or $B \subset A$.

My past teaching experience in Poland, Italy and U.S.A. convinced me that many students are curious of philosophical aspects of mathematics. My few attempts to persuade my colleagues from Dijon to introduce some amount of logic and foundations to the program were not successful.

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# COMBINING ONLINE AND PAPER ASSESSMENT IN A WEB-COURSE IN MATHEMATICS

JOHANN ENGELBRECHT and ANSIE HARDING

Online assessment in mathematics is becoming more prominent as mathematics and the internet become more compatible. The immediate issues are why one should venture into online assessment and how best to do it. Other issues include whether traditional paper assessment can be totally replaced by online assessment, whether standards can be maintained while doing online assessment and to what extent an appropriate combination of paper and online assessment could provide for various learning styles. We address these issues from a background of having been involved in teaching online mathematics courses for the last four years.

## Introduction

Issues such as why one should consider online assessment, how viable online assessment is and what the role of online assessment is, in mathematics in particular, need to be addressed. Computer technology is establishing itself as an integral role player in teaching mathematics, yet many teachers of mathematics still shy away from granting technology the same significant role in the assessment process. Whether it is at all possible to sensibly assess mathematics online is an issue that is often regarded with scepticism. Concerns exist about whether valuable skills such as the development of a mathematical argument or the exposition of a problem solution, normally conducted on paper, will be forfeited. On the other hand, one can speculate whether it is possible to completely replace paper assessment in mathematics with online assessment, the precedence of which has been set in a number of other disciplines. If possible it will cut down on grading work for the teachers and fall in line with trends of increasing technology application. Steering midway offers the possibility of combining online assessment and paper assessment. If this option is considered, then one needs to investigate how to deploy such a combination. In other words, how does one maintain the best features of two worlds while venturing into a new way of doing? Hand in hand with this is another important issue, the question of whether it is possible to maintain standards while assessing mathematics online. In this paper we address the issues stated above from the perspective of having taught mathematics courses online for a number of years and making use of both online and paper assessment.

## Assessment

A danger amongst mathematics teachers is for assessment to be considered as an add-on to the course and not to form an integral part of the curriculum. This is not the case with students. For most students, assessment is the central driving force in the learning process. Teachers spend much time on developing detailed study guides containing carefully formulated learning objectives. The majority of students, however, have as their primary objective passing the course. For this purpose they are quick to consider whether their primary focus should be on preparing according to the study guide or being guided by the content of past papers. In most cases they prefer the latter. What we do not assess, students will not learn.

It has become customary to distinguish between four types of assessment. All assessment activities contain elements of one or more of the following components:

- *Diagnostic* assessment that enables the teacher and the student to detect weaknesses in individual or the group's progress continuously;
- *Formative* assessment where the primary purpose is to provide feedback to students on their progress;
- *Summative* assessment where the main objective is to generate a mark for grading purposes, and
- *Accountable* assessment certifying public accountability.

Diagnostic and accountable assessments are mainly of interest to the teacher whereas formative and summative assessments are more in the interest of the student. In general, students are mostly interested in the summative component. They are concerned about the accountability and realize the value of formative assessment. Yet, formative assessment is only effective if a student makes mistakes in a test, mistakes to learn from. Students with full marks for a test do not truly benefit in a formative sense. At most they benefit from peace of mind and increased confidence. So to benefit formatively from a test, a student has to lose marks in the test, not a popular occurrence in a student's life.

## Why online assessment?

The answer to the question why one should venture into online assessment probably lies in the fact that more and more situations are presently created where online assessment seems the obvious route to take. The most common of these stems from the idea that there is a core of knowledge that is essential in any particular course before problem solving could even be attempted. The process of assessing this core of knowledge, the "must knows", is also sometimes referred to as *gateway testing*. This type of testing is particularly suited to online assessment.

It is important to note, however, that online assessment is not only used to test the "must knows". The traditional perception is that Multiple Choice Questions (MCQs) can only be used for testing lower level cognitive skills. This is not true, according to Hibberd (9)

> ... they can be implemented to measure deeper understanding if questions are imaginatively constructed.

From the teacher's perspective the value of decreased grading time when assessing online should not be underestimated. When working with large groups of a hundred and more students the grading load impacts on valuable research time and affects staff budgets.

Online assessment has the further advantage of enabling the teacher to readily obtain question-by-question profiles. Subsequent refinement of questions and tests can be carried out. The empirical data that becomes available makes online testing a valuable diagnostic instrument. The objective of every MCQ should be clearly understood and a careful selection of the distracters can itself be utilised to provide diagnostic information.

The issue of assessment innovation is addressed widely such as by the United Inventors Association in the U.S.A. with their *Innovation Assessment Program* (18), the Freudenthal Institute in the Netherlands (7) with their programme on *Research on Assessment Practices*, the *Center for Innovation in Assessment* at the Indiana University (10) and the *MathSkills Discipline Network* (17) in the United Kingdom.

However, we still venture to say that in many instances much effort is expended on curricular innovation without the same effort being applied to assessment innovation. Many academics claim that they are "reformed teachers" without thinking seriously about new and innovative ways of assessing. Innovation in teaching methods requires innovation in assessment.

When teaching almost exclusively by means of technology, such as in a web-based environment, the foundations are laid for assessing online and it seems almost natural to assess online. This paper largely deals with such a case study. If technology is incorporated in the presentation of the course, it makes little sense to avoid technology in the assessment part of the course. According to Gretton and Chalis, (8),

> Assessment has various purposes. Is it for grading and sorting students? Is it for encouraging learning? The answer is yes to both, but when both technology and students' skills are evolving so rapidly, then assessment style must also evolve to ensure it continues to fulfil these objectives.
>
> Smith and Wood (15) believe that
>
> ...appropriate assessment methods are of major importance in encouraging students to adopt successful approaches to their learning. Changing teaching without due attention to assessment is not sufficient.

The National Council of Teachers of Mathematics 1995 Assessment Standards, (13) states that if students are to increase their mathematical power, a number of related shifts in assessment practice are warranted, including a shift toward using multiple and complex assessment tools such as performance tasks, projects, writing assignments, oral demonstrations, and portfolios, and away from sole reliance on answers to brief questions.

Critics view traditional paper-and-pencil tests as inadequate measures of student abilities because they fail to elicit complex thinking and deep subject matter understanding (6; 14). This need not be case but in traditional tests the emphasis is often on manipulative skills or rote learning.

## How should online assessment be applied?

Although it is not possible to lay down rigid rules as to how assessment should best be conducted, assessment should match the mode of presentation. One should provide for the fact that just as learning styles differ amongst students, so do assessment preferences. In other words, neither online assessment nor paper assessment nor oral assessment suffices on its own. Even within each of these categories the possibilities are numerous and none can be used exclusively. The answer lies in combining different modes of assessment to suit the course. We turn to our own experience in teaching online courses to illustrate such a combination.

## The teaching model

In 2000 the first web-based calculus course was introduced at the University of Pretoria. Experiences and findings are reported in Engelbrecht and Harding (1, 2, 3). Due to the success of the project it has been expanded to three successive web-based courses. So there are students now that have completed three semesters of calculus on the web. Large groups of students are involved, up to two hundred students per course.

All the web-based calculus courses run along the same model. A textbook (16) is prescribed and the student is guided through the course on a dynamical day-by-day basis. The course provides for one contact hour per week, a discussion session. The platform is WebCT, the reason being that the university subscribes to this software and so provides the necessary infrastructure and support. The study material is broken down, firstly into themes and secondly into units, each of which provides for more or less a daily portion. For each of the units detailed study objectives, short lecture notes and problems of the day are provided. None of these activities are monitored and therefore requires a fair amount of self-discipline from the student's side. Because of the importance of this matter all activities are geared towards cultivating self-discipline. Communication takes places via online discussion forums and e-mail.

## The assessment model

Online assessment forms an integral part of the assessment strategy. The most basic assessment activity is a weekly quiz done on the web with rapid feedback. Students do these quizzes individually. Although there is no security check, it does contribute to the semester aggregate and students soon get to use it as a formative tool and as a fair judge of their progress.

Also as a formative tool but in a totally different style of assessment is the assignment and project activity of the course. Students hand in four hard copy assignments and one project during the semester. Assignments mainly consist of selected problems and the project requires the use of mathematical software such as Matlab or Maple. The assignments and the project are graded after which the solutions are published on the website. These activities are done in group regard and the benefit of co-operative learning for this mode of learning is discussed in Engelbrecht and Harding (3).

The two formative tools discussed above differ in their role of contributing to the learning process but complement each other. Whereas the online quizzes assess almost continuously in an environment suited to the presentation style, the assignments cultivate discussion and the art of writing mathematics. Whereas the online quizzes have a time limit, simulating examination conditions, the assignments offer open-ended time. The quizzes offer no partial credit but the assignments do. Online quizzes consist of MCQs as well as Constructed Response Questions (CRQs) whereas assignments consist only of CRQs.

For the two semester tests and the final examination two modes of assessment are again combined. Each of these tests consists of an online as well as a paper section. The online section is done in a computer lab under supervision. The online and paper sections carry equal weight. We are convinced that each of these modes has its place in the assessment system and that the two modes have different advantages. Having been exposed to the weekly quizzes, students are familiar with this way of assessment and again it fits in with the mode of instruction. On the other hand, as is the case with the assignments, the paper section assesses skills such as formulation, exposition and sketching that are not possible to assess online. From the lecturer's side there is the added benefit of reduced grading and the diagnostic features of the online section.

## Maintaining the standard

Having given an example of how assessment styles can be combined, the question then is whether it is possible to set an online examination in which the performance standard is maintained. In addressing the issue of accountability, there is a concern that standards are sacrificed for the sake of accommodating online assessment.

Judging the level of difficulty of MCQs is not easy. Research by Lorge and Kruglov (11, 12) show that mathematics teachers can judge the relative difficulty of MCQs fairly well but cannot predict the pass rate. Finney et al (5) report that pooled judgment is better than individual judges' opinion. It stands to reason that judging the degree of difficulty and predicting performance of online questions is something one instinctively becomes more proficient at the longer one is involved with it, just as is the case with paper questions.

In an attempt to compare the performance in paper sections of tests with the online sections, data was collected on performance in both the online and the paper sections over a period of two years from eight semester tests. The average performance as a percentage for each group taking the particular test is given in Table 1. The Pearson correlation coefficients of the students' performance in the paper and online sections in each of the tests are also included in this table.

Table 1
*Comparison of performance in online and paper sections of semester tests*

| | Number of students | Paper section | Online section | Correlation coefficient |
|---|---|---|---|---|
| 2001 sem 1 test 1 | 87 | 46.29 | 58.30 | 0.10 |
| 2001 sem 1 test 2 | 87 | 40.86 | 53.45 | 0.13 |
| 2001 sem 2 test 1 | 180 | 51.96 | 47.49 | 0.49 |
| 2001 sem 2 test2 | 180 | 54.52 | 69.33 | 0.39 |
| 2002 sem 1 test 1 | 170 | 51.88 | 61.56 | 0.36 |
| 2002 sem 1 test 2 | 170 | 71.29 | 74.58 | 0.29 |
| 2002 sem 2 test 1 | 83 | 54.35 | 54.62 | 0.58 |
| 2002 sem 2 test 2 | 83 | 46.51 | 50.82 | 0.35 |
| **Averages** | | **52.21** | **58.77** | **0.27** |

The results in Table 1 on the average performance in the tests are represented graphically in Figure 1.

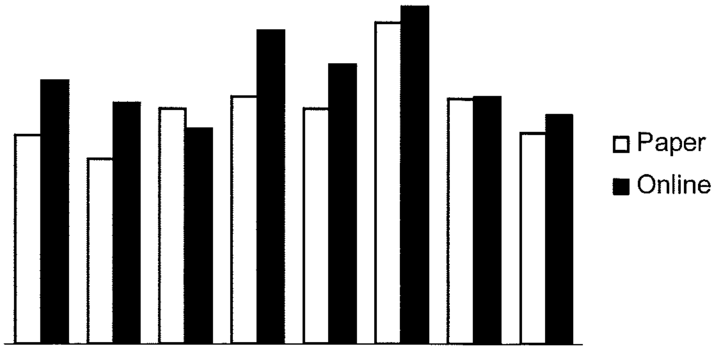


*Figure 1: Comparison of performance in online and paper sections of semester tests.*

To get an indication of the standards of the online and paper sections, three experienced colleagues independently rated the level of difficulty of each question in each of the tests on a four-point scale where 1 indicates an easy question and 4 a difficult question. A weighted average level of difficulty for each of the tests could then be calculated and these ratings are listed in Table 2.

Table 2

*Comparison of independent rating of difficulty level of online and paper sections of semester tests*

| | Paper section | Online section |
|---|---|---|
| 2001 sem 1 test 1 | 2.61 | 2.19 |
| 2001 sem 1 test 2 | 2.25 | 2.08 |
| 2001 sem 2 test 1 | 2.24 | 2.22 |
| 2001 sem 2 test2 | 2.41 | 2.37 |
| 2002 sem 1 test 1 | 2.67 | 2.38 |
| 2002 sem 1 test 2 | 2.37 | 2.32 |
| 2002 sem 2 test 1 | 2.23 | 2.43 |
| 2002 sem 2 test 2 | 2.04 | 2.25 |
| **Averages** | **2.35** | **2.28** |

Observations include:

- There has never been any "disturbing" difference between the online and paper sections. The biggest difference of 14.81% still is acceptable in light of the fact that there is a fluctuation of 30.43% in the paper tests and of 27.09% in the online tests.
- Students do seem to perform slightly better in the online section in general although this is marginal in most cases and not even always the case.
- The differences in performance in the online and paper sections seem to be getting smaller as time progresses. Setting online quizzes is a skill that has to be cultivated and although still far from having perfected it, it seems as if there is improvement.
- Although most of the correlations between the online and paper sections in Table 1 are significant on the 5% level, the values are still relatively low. The somewhat low values should not be viewed negatively. In fact this could be an indication that some students perform better in the online section and others better in the paper section and that the combination accommodates students with different learning styles.

- The overall Pearson correlation coefficient between online and papers sections of all testa taken over all tests is 0.27. Taken over a sample of 896 students, this is highly significant even on the 0.1% level of significance. This confirms the notion that although individuals could perform better in either one of the sections, overall there is little difference in performance.
- Table 2, which indicates no noteworthy difference in the level of difficulty between the questions in the paper and online sections, supports all the above observations.

When setting an online test it is important to use a combination of question types. Students tend to perform better in online MCQs than in online CRQs and better in paper CRQs than in online CRQs. This issue is discussed in detail in Engelbrecht and Harding (4).

Online CRQs requiring a single answer are often criticised both by teachers and students for the lack of partial credit. Perhaps there is merit in training students to work with care in order to deliver the completely correct product and not to rely on partial credit. Single answer CRQs without partial credit are fine as long as this forms part of the assessment process and is not the only way of assessing. This matter is discussed in more detail in Engelbrecht and Harding (4).

When comparing performances in the online and paper sections above, averages were considered. Unfortunately an average blocks out the distribution of marks within a certain group and valuable information could be lost. We have a closer look at the performance distribution for a particular test, the 2002 second semester test 1, where the average marks in the online and paper sections were almost identical. The distribution of marks are graphically represented in Figure 2. Both sets of marks are reasonably normally distributed, although the paper marks are more centred towards the average.

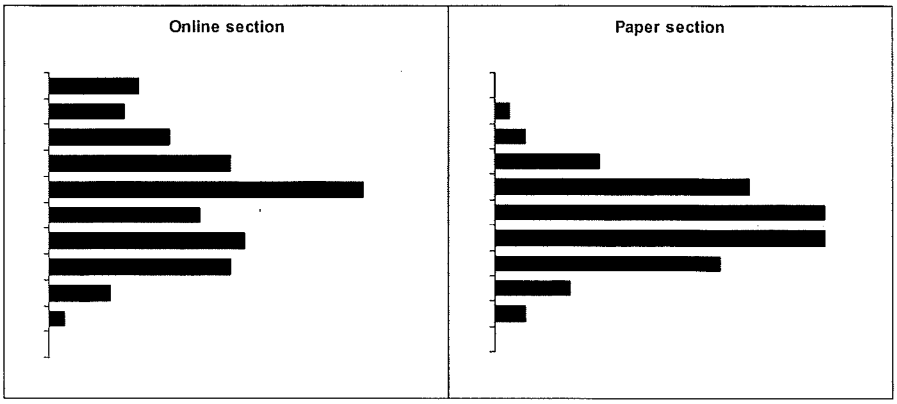


*Figure 2: Comparison of online and paper sections in Test 1, semester 2, 2002*

## Conclusion

Although no empirical information about student assessment preferences was collected, it seems as if the assessment model under discussion not only provides for different assessment preferences but also addresses each of the diagnostic, formative, summative and accountability components of assessment.

All assessment is ultimately subjective: there is no such thing as an "objective test". Even when there is a high degree of standardisation, the judgement of what concepts or techniques are assessed and what constitutes a criterion of satisfactory performance is in the hands of the assessor. However, despite subjectivity, the lecturer can still make every effort to ensure that assessment is valid and reliable.

Choosing between online assessment, paper assessment or a combination is ultimately also a subjective decision. The success of the assessment largely depends on the teacher's as well as the student's style, preferences and commitment. Indications are that a combination of paper and online assessment could provide better for student assessment preference and capability, but this should be investigated further.

Online assessment should be considered as a valuable additional component to the assessment spectrum. Replacing paper assessment totally by online assessment is not what we advocate. There is no doubt, however, that online assessment has a definite role to play and that it can be applied purposefully with definite advantages.

## References


1. J. Engelbrecht & A. Harding, WWW Mathematics at the University of Pretoria: The trial run, South African Journal of Science, 97(9/10) (2001), 368-370.
2. J. Engelbrecht & A. Harding, Internet Calculus: An option? Quaestiones Mathematicae, Supplement 1 (2001), 183-191.
3. J. Engelbrecht & A. Harding, Cooperative learning as a tool for enhancing a web-based Calculus course, Proceedings of the ICTM2, Crete, July 2002.
4. J. Engelbrecht & A. Harding, Online assessment in mathematics: Multiple assessment formats, New Zealand Journal of Mathematics ...........
5. S. J. Finney, R. W. Smith & S. L. Wise, The effects of judgment-based stratum classifications on the efficiency of stratum scored CATs, Paper presented at the annual meeting of the National Council on Measurement in Education, Quebec (1999), (23 pages).
6. N. Frederikson, The real test bias, American Psychologist, 39 (1984), 193-202.
7. Freudenthal Institute, Research on Assessment Practices, Retrieved September 2, 2003 from http://www.freudenthal.nl/en/projects/
8. H. Gretton & N. Chalis, Assessment: Does the punishment fit the crime? Proceedings of the International Conference on Technology in Education, San Francisco, (1999).
9. S. Hibberd, The mathematical assessment of students entering university engineering courses, Studies in Educational Evaluation, 22(4) (1996), 375-384.
10. Indiana University, Center for Innovation in Assessment, Retrieved September 2, 2003 from http://www.indiana.edu/~cia/
11. I. Lorge & L. Kruglov, A suggested technique for the improvement of difficulty prediction of test items, Educational and Psychological Measurement, 12 (1952), 554-561.
12. I. Lorge & L. Kruglov, The improvement of estimates of test difficulty, Educational and Psychological Measurement, 13 (1953), 34-46.
13. NCTM (National Council of Teachers of Mathematics) (1995). Assessment standards in Principles and Standards for School Mathematics, Retrieved June 10, 2003 from http://standards.nctm.org/Previous/AssStds/PurpMon.htm
14. L. B. Resnick & D. P. Resnick, Assessing the thinking curriculum: New Tools for Educational Reform, In B. R. Gifford & M. C. O'Connor (Editors), Changing assessments: Alternative views of aptitude, achievement and instruction, (1992) 33-75, Kluwer.
15. G. Smith & L. Wood, Assessment of learning in university mathematics, International Journal for Mathematics Education in Science and Technology, 31(1) (2000), 125-132.
16. J. Stewart, Calculus, Early Transcendentals, Brooks/Cole, (1999).
17. The MathSkills Discipline Network, Department of Education and Employment, U.K., Retrieved September 2, 2003 from http://www.hull.ac.uk/mathskills/themes/theme3/mathskill.html
18. United Inventors Association, Innovation Assessment Program, Retrieved September 2, 2003 from http://www.uiausa.com/UIAIAP.htm



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# ADOPTING A LEARNER-CENTRED FOCUS WITH ONLINE DELIVERY OF SERVICE MATHEMATICS

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AND RENAY BUCHANAN

With the advantage of hindsight, the comparison drawn by James Kaput in 1992, likening the task of describing the role of technology in mathematics education to that of attempting to describe a newly active volcano, appears an astute observation. His statement acknowledges the pressure that advances in technology exert as they move to drive education and the discipline of mathematics from outside and simultaneously act to support learning from within. This paper discusses a role for technology in mathematics education that few writers foresaw at the end of last century; that is, through the World Wide Web to offer online delivery of mathematics in environments that centre on the individual learner and their needs. The case for online delivery as a medium that adopts a learner-centred focus is put forward. This paper discusses an initiative to develop a learner-centred environment for a level-one business mathematics course at Central Queensland University. It will be maintained that the creation of such a learning environment offers individual mathematics-learners opportunities for understanding above and beyond those offered through traditional media.

## Introduction

In the same way that a "newly active volcano" [1] (p.515) changes forever the surrounding landscape, so technology has brought forth a multiplicity of seemingly permanent changes to the mathematical learning environment. The computer has not only changed the way that mathematics is applied in society but has also changed the face of the actual mathematics applications themselves, emphasising numerical methods and modelling and promoting the study of algorithms [2] (p.203). While thirty years of rapid technological change may have promoted this transformation in mathematics, it has also been responsible for increasing the power of calculators and computers, decreasing their size, heightening the levels of accessibility of these technologies among the populace and creating a global communication network to support enquiry through synchronous and asynchronous technologies. These changes have not been felt immediately in the classroom. Kaput indicates that, "while electronic computation has been in the hands of mathematicians for four decades, it has been in the hands of teachers and learners for at most two decades, mostly in the form of time-shared facilities" [1] (p.515). Very often the time spent by learners at such facilities has been devoted to the use of drill-and-practice software [3] (p.61). It would be difficult to gauge the separate contribution of these technological changes to improving access, flexibility and opportunity for mathematics learners, however, it appears that the creation and the rapid rise of the global communication network offer the most powerful challenges to date, for both learners and educators [4].

## Learning Technologies

It is only in recent years that some universities have taken up the challenge within a number of their courses to make full use of the Internet as a communication technology and to create individual e-learning environments for their students. Taylor's [4] classification system of the generations of distance education technology places the Internet as central to the implementation of fourth and fifth generation models; the Flexible Learning Model and the Intelligent Flexible Learning Model. What distinguishes these later generations of distance education technologies is their emphasis on the Internet as a medium that allows interactivity and communication and which through its use, promises universities attractive variable costing structures and benefits from economies of scale and value adding. Previous generations of models involve varying combinations of print, video, audio, computer-based learning and teleconferencing resources for student use. Because moving from earlier to later generations of models involves far more than merely converting material, some educators appear reluctant to make the move.

Fox and Herrmann [5] (p.73) in their discussion of the adoption of new educational technologies point to a reluctance on the part of educators to take up the challenge to design resources based around the communicative aspects of the Internet. They argue that this reluctance can be based on concerns about work practices and perceptions of the gap between the promises and the reality of using the media in educational settings. Issues such as access for students; the need for staff to receive appropriate funding and support for such activities; the perceived dehumanisation of complex teaching and learning processes by technology; and challenges to accepted, traditional models with the teacher as the locus of control are just some of the reasons put forward to explain this reluctance ([5]; Forsyth, cited in [6], p.112). Roberts [7] highlights the inadequacies of trying to use one technology or delivery medium as a basis for another when he notes the inappropriateness of using essentially sequential media such as face-to-face lectures and printed chapters from a book as content for online delivery. He states that "... materials will almost certainly need to be redesigned not only to make use of the specific strengths (hyperlinks, graphics, animations, etc) provided by the new medium, but also to take advantage of the fact that the Web enables simplified non-sequential access to the course materials" [7] (p.24). Concerns regarding the acceptability of traditional models are reflected in the comments made by one respondent in the Fox and Herrmann study. After transferring a print-based distance course to the Web they concluded; "...the Web's ability to hyper-link disparate ideas and themes made us rethink our learning strategy too...rather than adopt an objectives model, we decided to use a more constructivist approach to structuring the course" [5] (p.76). Given that these are but a few of the issues involved in moving to online delivery it is no small wonder that reluctance exists.

In 2000, Wattenberg and Zia [8], in their assessment of technology-enriched mathematics, concluded that the form of current learning environments in mathematics was under challenge from technological innovations. If improvements could be made in the areas of access and equity, and technical concerns such as bandwidth and transmission speeds could be addressed, then the possibilities for the future were very exciting. Such concerns may explain why some university mathematics educators persist in offering their students, distance or otherwise, resources and delivery characterizing earlier distance education models. What constitutes study material resources for students has basically remained unchanged over the last twenty years. Typically students may attend lectures and tutorials and be expected to obtain a printed study guide, a prescribed text, and audio and/or videotape material where necessary. Perhaps a level of satisfaction with the traditional on-campus presentation of the mathematics tutorial with its rich face-to-face learning environment [9], has led such educators to concentrate their energies in other areas and to not consider the possibilities available through the use of new learning technologies. As Taylor, cited in [10] states so succinctly:

> Such a state of affairs wherein teaching as a process is more-or-less taken for granted stems from the "tyranny of proximity", a frame of mind in which important issues are overlooked because they are so much an accepted part of day-to-day activities that they remain unchallenged and unquestioned. (p.2)

There is a strong contrast between the previously stated resources as examples of early learning technologies offered to students and the expectation that these same students will have access to the Web and be familiar with technologies such as programmable and graphics calculators and computer algebra systems within their chosen courses. While educators appear to presently give less emphasis to the communicative power of new technology, they seem to have grasped the tremendous interactive power of the Web with its library of rich resources available to both educators and students. Slavit and Yeidel [11] distinguish between two different uses of the Web - "transpositional" and "non-transpositional". The former involves shifting tools of learning from one medium onto the Web while the latter includes "downloading of data from existing web sites, the use of simulations to generate data, preprogrammed prompts or reactions to student responses, synchronous and asynchronous communication, and dynamic modes of displaying, processing, and analysing information" [11] (p.28).

It was the attractiveness of these very aspects of the power of the Web that led mathematics educators at Central Queensland University to investigate and attempt online delivery for one of its service mathematics courses. Unique resources for student use were created within the WebCT e-learning environment; the ultimate goal being, to offer mathematics learners a learner-centred environment where resources could supplement those provided under earlier distance education models. The argument of this paper is that such a learning environment offers individual mathematics-learners opportunities for understanding above and beyond those offered through earlier traditional learning technologies.

## Building a learner-centred environment through online delivery

A number of researchers have pointed to the power of later learning technologies in developing a learner-centred environment. According to the directors of the National Science Foundation [12] (p.205) a paradigm shift from teacher-centred to learner-centred undergraduate education should be one of the major anticipated achievements for the 21st century in science, engineering, mathematics and technology (SMET) research and education. In addition the directors also saw that various technologies needed to be exploited to allow students to: "explore theories and concepts without getting bogged down in tedious calculations or manipulations" and to "… learn outside the confines of a particular time and classroom or laboratory setting" [12] (p.205). Such expectations require a more thoughtful use of technology than just giving students access to the Internet and getting professors to use PowerPoint in their lectures [13]. The argument proposed by Brown and Duguid, cited in [13] (p.28) that learning technologies will be under-exploited unless more thought is given to students and their practical needs, appears to reinforce Taylor's previously cited comments regarding "the tyranny of proximity" [10]. Too often technology becomes an adjunct to teaching instead of being thought of as a way to promote learning and explore new learning environments.

Albon and Trinidad [14] in their paper to the 2nd International Lifelong Learning Conference at Central Queensland University note the importance of interactive technologies as instruments in learning for today's students. They point to the role of such technology in developing a learner-centred approach to education. Such an approach "acknowledges the roles of pace, repetition, learning styles, motivation, self-regulation, and responsibility to learn" and puts the "learner in control of constructing their own learning" [14] (p.51). Maier and Warren [15] (p.98), argue that communicative technologies can be used to facilitate online communities where learners can reflect on their own learning. Such approaches to technology in fostering learner-centredness contrast strongly with traditional teacher-centred approaches that utilize early learning technologies and which are so often characteristic of undergraduate mathematics courses. Here the teacher is seen to be the expert and decides how students will learn and on what they will concentrate while learning.

## Online delivery for Quantitative Methods A

The aspects of pace, reflection, repetition and learning styles were utmost in the mind of the developers when they set about to improve the learning environment for students enrolled in the Quantitative Methods A (QMA) course. The decision to develop a WebCT site for the course was in line with a general initiative to implement online delivery using WebCT in a number of courses across the university. Academic staff looked to the Multimedia Design Centre to assist in the development of the site. Involving design staff at this stage helped allay concerns from teaching staff regarding their confidence levels, competency and skill development in working with the new technologies. Prior to this initiative, the delivery of the course displayed a teacher (not learner) centred approach, with its emphasis on earlier learning technologies and the use of technology as a feature of teaching, not learning. While traditional technologies have been useful for some students in the past, feedback from unpublished student course evaluations reveals a level of dissatisfaction with these resources. This initiative not only sought to address the issue of resources but also to accommodate the significant issue of the mathematical background of the student.

Quantitative Methods A assumes no prerequisite mathematics knowledge and introduces a number of elementary mathematics topics such as simple functional models, inequalities, the mathematics of finance, simple linear programming techniques, matrix algebra and an introduction to single-variable calculus with applications to marginal analysis and optimisation. It is offered three times a year to students studying via distance education and to those enrolled internally at regional and international campuses in Australia and overseas. On average in any one term, there can be as many as 700 students and as few as 200 students enrolled in the course depending on the term of the year. Enrolled students exhibit varying backgrounds, abilities and attitudes towards mathematics, as well as differing levels of proficiency with the English language. The authors' experience in teaching the course over a number of years reveals that those students with successful recent experience in a formal mathematics course are less likely to find this course challenging. A background in elementary algebraic techniques and a level of competency in using the functions of a scientific calculator underpin many of the skills developed within the course. A problem solving approach to the topics is emphasised; hence the need for English language proficiency. Typically, distance education students make up a sizeable cohort and are frequently among those students who struggle with the content and with its former presentation using earlier technologies. Without ready access to tutors, lecturers and their peers for clarification using the technologies accessible through the now current website, these students would be left to individually read and decipher the printed word and perform the necessary calculations in the hope that their actions would lead to a successful outcome (a correct method and solution) and hence ultimately vindicate their own belief that they understand the material and that learning has taken place.

The QMA WebCT site is now linked to the faculty-default, current term course website and provides all enrolled students with an opportunity to work with a number of its learning features. Together, both sites provide the following features for students:

- *Information about staff and an email link to current teaching staff,*
- *Email contact to other enrolled students through a mailing list,*
- *Worked mathematical examples with accompanying audio-enhancement,*
- *Simulations that construct solutions to more complex mathematical problems,*
- *Lecture slides in PowerPoint form,*
- *The Course Profile and the Study Guide in electronic form,*
- *Non-assessable quiz questions and*
- *Chatroom facilities.*

Why concentrate on providing these online features or technologies and not develop others? The aim was to provide a course learning environment for all students that permitted access to all levels of learning technologies, especially the later models; the interactive and communicative features of which afford students different and additional opportunities for reflecting on their learning. Some of the technologies listed above have been in use for a number of terms and are almost generic to many courses across the university. Others, such as the worked examples, the simulations and the quizzes were supplementary to existing course resources and were developed in the WebCT environment. In particular the worked examples and the simulations reference individual mathematical topics and provide features that the authors believed would be extremely beneficial in assisting the students to learn. No theoretical approach to the development of online mathematical resources was accessed in their development, for as Slavit and Yeidel point out, "well-developed theory and supporting data on the benefits and limitations of web-based mathematics instruction remain scarce" [11] (p.29). Rather, the criteria for the formation of these resources emerged through the experiential and innovative development work of the staff involved. The current structure of these resources addresses criteria such as; the recreation of the face-to-face learning environment; a close connection between the learning activity and the unfolding solution; creation of a learning style familiar to the student and the creation of resources that are simple, robust and easy to replicate in other mathematical areas [9]. The chatroom facility was added to the resources this term in an effort to increase the opportunities for synchronous communication between students and between staff and students. Bradbury and Farrell [16] in their paper "Value adding in an online environment", specifically mention the creation and use of the chatroom as one resource that addresses the issue of meaningful communication for learners. Even though the QMA website resources include a mailing list for students, email facilities and audio-enhanced accompaniments to the worked examples and the simulations, it was considered necessary to provide some 'real time' communication opportunities for students. Bradbury and Farrell see the chatroom as a way to forge bonds between tutor and student and between students. "Specific questions and student pairings tend to engage the learner from the onset and ensure ownership and a sense of belonging" [16] (para. 22). They also caution that such a facility while offering many opportunities for learning may not be used by some students who lack the time to access such facilities, thereby diminishing the role of this aspect of their learning environment. An assessment of the students' use of this facility will be undertaken at the end of Winter Term, 2003.

The current website gives students additional opportunities to become immersed in the mathematics of the topics covered in the course; beyond those offered by traditional technologies. The worked examples and the simulations attempt to recreate for students the rich environment of the face-to-face mathematics tutorial. The tutorial in its most perfect form, with its thoughtful demonstration, expansive justification, timely cautions and sequenced solution, is recreated in the worked examples through a series of distinct learner regulated steps, each with "well-paced commentary and a head-and-shoulders photograph of the person providing the commentary. The user can then hear and see the person with whom they are working and learning" [9] (in press). The simulations provide a sequenced representation of some of the more complex mathematical topics such as matrix multiplication, equations of value and the development of the optimum point within a geometric approach to linear programming. In the tutorial, such solutions are multi-layered and information is incorporated as the solution is built. Likewise the simulations attempt to replicate the unfolding solution and even though they may not currently have any accompanying auditory comment they offer features superior to the space-constrained global representations of the solutions in printed texts and study guides. As these two resources were being developed it was also thought appropriate that little change take place in the format of the traditional mathematics worked example. Students are familiar with text based presentations, by courtesy of their early mathematics training, and regardless of the inadequacy of using one delivery medium as a basis for another, (as previous discussion points out), it was considered important not to represent to students anything too far from their previous experience. The addition of auditory commentary in the worked examples was not considered to be an extreme change to the familiar traditional format. Research has yet to propose the inadequacy of the presentation of the traditional mathematics solution in an online environment. With both resources, the learner is free to visit and revisit the topic and the steps not understood, and in the worked examples to listen to the auditory commentary as often as he or she might need in order to feel that a level of mastery and understanding of the material has been achieved. According to Bradbury and Farrell [16], such a level of interactivity allows the learner to personalise the experience and establish opportunities for engagement, competency and retention.

## Future directions

As universities move to cope with rapid technological change they have sought solutions, but not always at a speed commensurate to the changes themselves. Taylor [10] comments that many universities have struggled with the need for change and that those who appear to have coped the best, are those that have had a significant history in the provision of distance education. The pathway of change through the generations of learning technologies has been more clearly defined for such institutions. Decisions regarding the implementation of learning systems are usually made at management levels, within the university, above that of individual academic or teams of academics. Hence, it is more likely to be issues pertaining to cost that determine the speed and the direction of the initiatives, rather than the construction of appropriate learning environments for students. In an atmosphere of diminishing resources, more and more is expected of those university educators who are willing to take the initiative and create online environments for course delivery. The learning implications of this technology which are based on their teaching and research experience, are now becoming the topic of wide debate. A number of researchers have indicated the potential of interactive and communicative technologies for improving the learning outcomes in education, however, some concerns have been noted.

In the more specialized area of online delivery of mathematics, the debate is however a little quieter. This previous discussion has highlighted the need for mathematics educators to offer learners opportunities for learning that focus on the use of interactive and communicative technologies as opposed to traditional learning technologies. In fact, some educators have urged the mathematics community to tackle challenging issues such as creating the optimal environment for learning mathematics and identifying the effects of teaching and learning technologies on student learning [12]. The teaching staff from one undergraduate service mathematics course have been able to develop online resources using newer technologies by working within the powerful e-learning system, WebCT. Through the use of these resources, individual mathematics-learners now have opportunities for understanding above and beyond those offered through traditional media. The interactivity of the worked examples and the simulations on the website, and the opportunities for communication provided by the chatroom and the mailing list, address a number of issues which arise when creating a learner-centred environment for students.

It may be argued that at the present time the form of the resources and the structure of the site are not supported by any pre-existing theoretical model addressing effective online delivery of mathematics. It was not the intention of the staff to first find the theory and then begin the process of adapting and building the resources. Rather they responded to the online delivery initiative and attempted to incorporate outcomes from their teaching experience, and feedback from students that had been collected over a number of years as these students contended with traditional learning technologies. The quality criteria for the site that emerged as its development unfolded [9] assisted staff in maintaining a belief in the value of their work. Also there is limited feedback currently available from students that either supports or denies the benefit of the resources on the site, in contributing to the development of a learner-centred environment. The developers look forward to an extended period of association with the course and with the website so that substantial material can be collected to judge the impact that is anticipated.

While Central Queensland University moves from one e-learning system to another, the immediate future of the site appears assured. In an effort to bring online more mathematics courses offered within the faculty, the site will undergo several transformations to allow for expansion of its resources; the possibility of online assessment; and greater communicative capacity with the inclusion of an equation editor and an electronic whiteboard for student use. It is intended to be a frontrunner for online delivery of mathematics within the university.

As universities seriously consider the extent of their commitment to the mathematics discipline at the macro level, some individual mathematics educators are seeking to improve learning outcomes for individual students. Their optimism sees them aspire to a delivery environment for their students that initially involves considerable funding and support, but upon maturity promises to deliver so much for learning and teaching management.

## References


19. J. J. Kaput, 'Technology and mathematics education' in D. A. Grouws (ed), *Handbook of research on mathematics teaching and learning,* A project of the National Council of Teachers of Mathematics, MacMillan, New York, 1992, pp.515-556.
20. J. Kilpatrick & R. Davis, 'Computers and curriculum change in mathematics' in C. Keitel & K. Ruthven (eds), *Learning from computers: mathematics education and technology,* Springer-Verlag, Berlin, 1993, pp.203-221.
21. H. Povey & M. Ransom, 'Some undergraduate students' perceptions of using technology for mathematics: tales of resistance', *International Journal of Computers for Mathematical Learning,* vol. 5, 2000, pp.47-63.
22. J. C. Taylor, *Fifth generation distance education*, Keynote address presented at the 20th ICDE World Conference, Dusseldorf, Germany, 1-5 April 2001, pp.1-11, accessed 2 June, 2003, <http://www.usq.edu.au/users/taylorj/publications_presentations/5thGenerationDE.doc>.
23. R. Fox & A Herrmann, 'Changing media, changing times: coping with adopting new educational technologies' in T. Evans & D. Nation (eds), *Changing university teaching: reflections on creating educational technologies,* Kogan Page, London, 2000, pp.73-84.
24. J. Curtin, 'WebCT and online tutorials: new possibilities for student interaction', *Australian Journal of Educational Technology*, vol. 18, no. 1, 2002, pp.110-126.
25. T. S. Roberts, 'From distance education to flexible learning', *Proceedings of the 16th Annual Conference of the International Academy for Information Management,* 2001, pp.21-27, accessed 30 May, 2003, <http://www.aisnet.org/iaim/ICIER2001/ICIER 2001.htm>.
26. F. Wattenberg & L. Zia, 'Technology-enriched learning of mathematics: opportunities and challenges', in M. Burke & F. Curcio (eds), *Learning mathematics for a new century,* 2000 Yearbook, National Council of Teachers of Mathematics, Reston, Virginia, 2000, pp.67-81.
27. R. McDougall, M. Flanders, R. Buchanan & S. Lindsay, 'Developing a local framework for quality in an online learning environment: a case study', in *Quality Education @ A Distance: Proceedings of the International Federation for Information Processing (IFIP) Working Group 3.6 Conference,* Geelong, Victoria, 3-6 February, 2003, in press.
28. J. C. Taylor, 'Distance education technologies: the fourth generation', *Australian Journal of*

*Educational Technology*, vol. 11, no. 2, 1995, pp.1-7.
29. D. Slavit & J. Yeidel, 'Using web-based materials in large-scale precalculus instruction' *International Journal of Computers for Mathematical Learning,* vol. 4, 1999, pp.27-50.
30. E. Teles, Major issues and future directions of undergraduate mathematics: instructional techniques in freshman and sophomore mathematics courses, 1999, pp.205-208, accessed 30 May, 2003, <http://www.dean.usma.edu/math/ activities/ilap/workshops/1999/files/teles.pdf>.
31. R. Garrison & T. Anderson, 'Transforming and enhancing university teaching: stronger and weaker technological influences' in T. Evans & D. Nation (eds), *Changing university teaching: reflections on creating educational technologies,* Kogan Page, London, 2000, pp.24-33.
32. R. Albon & S. Trinidad, 'Building learning communities through technology' in K. Appleton C. Macpherson & D. Orr (eds), *Building learning communities through education: refeered papers from the 2nd. International Lifelong Learning conference,* Yeppoon, Queensland, 16-19 June, 2002, Central Queensland University Press, Rockhampton, 2002, pp.50-56.
33. P. Maier & A. Ransom, Integr@ting technology in learning & teaching: a practical guide for educators, Kogan Page, London, 2000.
34. E. Bradbury & P. Farrell, *Value adding in an online environment,* accessed 30 June, 2003, <http://online.nmit.vic.edu.au/resources/valueadding.pdf>.

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# HELPING UNDERGRADUATE ENGINEERS TO LEARN MATHEMATICS

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This report outlines the background, aims and objectives of a major three-year curriculum development project – HELM: Helping Engineers Learn Mathematics – and provides details of the progress made in its first year.

## Introduction

The importance of mathematics as a tool for the description and analysis of engineering systems and processes has long been acknowledged and the UK's Engineering Council rightly demands a high level of mathematical knowledge and skill in its accredited engineers. As a consequence, the design and delivery of an appropriate mathematical curriculum for engineering undergraduate students must be of central importance to engineering educators. That mathematics is a fundamental constituent of the education of an engineer is substantiated by the facts that it is often the only subject specified as a prerequisite, and it is a common thread in almost all engineering courses.

This paper first outlines some of the difficulties experienced in teaching mathematics to engineers. It then argues the case for introducing an 'open learning environment' and describes a possible solution to the mathematics problem, the HELM learning environment, which is being sponsored by UK Government funding. The HELM learning resources and assessment regime are described in some detail and future plans outlined.

## The 'Mathematics Problem' in the engineering context

In recent years the mathematical preparedness of students embarking upon science and engineering degree programmes has been the subject of close scrutiny in numerous research reports [for example 1, 2, 3, 4], and the topic has been debated at many conferences [for example, 5, 6, 7], with disturbing conclusions. A common theme running through all this work is that these students are, on the whole, mathematically weaker than students coming to university a decade ago. Traditionally, student embarking upon engineering degree programmes had to demonstrate a very respectable competence in mathematics and physics through the achievement of good A level grades. However, the recent past has seen a widening of access: due in part to financial encouragement from the Government, in part from proactive legislation, but mainly the result of individual institutions endeavouring to counter substantial shortfalls in recruitment to engineering degree courses.

Grade inflation, syllabus weaknesses, high failure rates, proliferation of qualifications, increasing attractiveness of other school subjects, undergraduate courses and careers, have all contributed to the problems faced by those whose task it is to recruit and educate engineering undergraduates. Furthermore, students and their parents are becoming increasingly demanding, especially so since the introduction in the UK in 1998 of tuition fees for university education. They are no longer content to put up with anything less than a first-rate experience. It is therefore of vital importance to have in place learning mechanisms to deal with the well-known and widely reported 'mathematics problem'.

It is now generally the case that the range of mathematical ability and knowledge base of any one group of engineering students is extremely wide. Many of the weaker students find it particularly difficult to learn from the traditional lecture, which becomes an exercise in note taking with little or no learning. Most students find it difficult to learn from textbooks. Many textbooks which were in use some ten or so years ago are too advanced for the present day student. Their range of material is too wide and they make too many assumptions about prior knowledge, particularly, the lack of experience many students now have in performing basic mathematical operations.

These problems affect the whole of the UK higher education system and are equally apparent in other parts of the world. They come at a time when academic staff are under increasing pressure to retain students rather than have them fail and leave, whilst at the same time delivering more research and publications, and when institutions have seen the per student unit of resource halved. Academic staff face the conundrum of needing to give less time (or at least less energy) to teaching and administration whilst tailoring courses to the needs of the wide range of individuals.

In teaching mathematics to engineers, the primary vehicle of transmission still remains the lecture. It is particularly efficient with large groups, although interaction is then especially difficult. Syllabus requirements may, however, encourage lecturers to try to cover too much material thus hindering student understanding. In any case, lectures are not the best place for transmitting a great deal of technical information, especially when students are trying to copy it from an OHP or writing board. Understanding becomes more difficult and many errors of transcription are made. The lecture notes of an average student often bear only passing resemblance to what was actually delivered, which can be crucial to understanding in mathematics.

Much material is now available in electronic form, on CD or on the web, and computer aided learning (CAL) has tremendous potential to assist the learning process. Information can be presented in written format (on screen) which can be supplemented with animation, with commentary and with video. Interaction adds more interest. It can be delivered locally on an intranet network or over the web. It can be recorded on a CD and used by students not wishing (or able) to use an internet connection. However, the development and start-up costs are extremely high, both in time and in resources. There is much anecdotal evidence to suggest that even though there is quite 'good' CAL material available, students prefer not to learn primarily using this approach. What CAL material is available is usually in stand-alone form without accompanying written material, which is a considerable drawback. This has been the Achilles heel of many CAL projects – failure to adequately link with the curriculum.

## The Loughborough experience

At Loughborough we have put in place an environment for learning mathematics that we believe will be attractive to the vast majority of undergraduate students, whatever their level and whatever their previous experience.

In 1997, funding was made available by the University for the 'Open Learning Project' in Mathematics for Engineers which has provided high-quality student-centered workbooks, computer aided learning material closely allied to the workbooks, and a strategy for computer aided assessment which can be used for self-assessment and for module assessment. For students following this open learning regime, lectures are now optional as they can now choose to study, with guidance, the mathematics independently.

The success of the Open Learning Project encouraged staff to seek funding to develop further this work resulting in the HELM project (Helping Engineers Learn Mathematics) which is supported by a £250,000 HEFCE- FDTL4 grant for the period Oct 2002-Sept 2005.

## The HELM project

The HELM team consists of staff at Loughborough and four consortium partners in other English universities: Hull, Reading, Sunderland and the University of Manchester Institute of Science and Technology (UMIST). The project aims to considerably enhance, extend and test Loughborough's original Open Learning materials, in particular by the writing of additional Workbooks and incorporating engineering exercises and case studies closely related to the mathematics presented, extending the question databanks, and promoting widespread trialling. The HELM project's output will consist of Workbooks, CAL segments and a Computer Aided Assessment Regime which is used to help 'drive the student learning'. To view sample materials see http://helm@lboro.ac.uk.

The Workbooks may be integrated into existing engineering degree programmes either by selecting isolated stand-alone units to complement other materials or by creating a complete scheme of work for a semester or year or two years by selecting from the large set of Workbooks available. These may be used to support lectures or for independent learning.

Nothing on this scale has been attempted before for free dissemination across the HE sector in England and Northern Ireland. The emphasis is on flexibility – the work can be undertaken as private study, distance learning or can be teacher-led, or a combination, according to the learning style and competence of the student and the approach of the particular lecturer.

## HELM project Workbooks

40 Workbooks are planned of which 33 will be available by Autumn 2003. These comprise:

- 37 Student Workbooks (listed in the Appendix) written specifically with the typical engineering student in mind containing mathematical topics, worked examples, mathematical exercises and related engineering exercises.
- 2 sets of engineering Case Studies ranging over many engineering disciplines provided by an experienced group at Hull University and others.
- 1 Tutor's Guide relating success stories and challenges and encapsulating good practice derived from trialling in a variety of institutions with their individual contexts and cultures.

The main project materials are the Workbooks which are subdivided into manageable Sections. As far as possible, each Section is designed to be a self-contained piece of work that can be attempted by the student in a few hours. In general, a whole Workbook represents about 2 to 3 weeks' work. Each Section begins with statements of pre-requisites and the desired learning outcomes.

The exercises include space for students to attempt the questions, and guide them through problems in stages, where appropriate. In some cases it is possible for the lecturer to select certain Sections from a Workbook and omit other Sections, reducing the reproduction costs and better tailoring the materials to the needs of a specific group.

## HELM project CAL Materials

The project has 67 CAL segments which link to about half of the Workbooks (those inherited from the original Open Learning Project). These enable web–based versions of the Workbooks to contain animations which generate student interest. Revision exercises with randomly generated questions are provided for the benefit of students working independently. Web-delivered CAL segments complement the Workbooks and include audio, animation and self-assessment aspects. These have been found to be especially useful for supporting students of moderate mathematical ability, and for revision. They are also useful for illustrating lectures. However, following discussions with HELM triallists, the project team has decided to give low priority to the generation of further CAL segments (for the newly written Workbooks) preferring to put its resources into developing the assessment regime described below.

## HELM project Assessment regime

In formal educational environments assessment is normally an integral part of learning, and this is recognised by the HELM project. Students need encouragement and confirmation that progress is being made. The HELM assessment strategy is based on using Computer-Aided Assessment (CAA) to encourage self-assessment, which many students neglect, to verify that the appropriate skills have been learned. The project's philosophy is that assessment should be at the heart of any learning and teaching strategy and Loughborough University's own implementation of HELM makes extensive use of CAA to *drive* the students' learning.

HELM provides an integrated web-delivered CAA regime for both self-testing and formal assessment. Students following the project's regime at Loughborough are typically tested four or five times each semester with questions delivered over the web. Currently there are over 3000 questions in 25 question banks; most have a page of feedback. These are being further enhanced and extended. Students are encouraged to engage in their own learning by allowing them unlimited trial tests before taking a one-attempt summative test. As each summative test is worth 6% of the module mark, students are motivated to keep up with their studies, thereby improving achievement and progression.

CAA is an essential part of the project and this raises potential difficulties over transferability as each HEI would need to support this on completion of the project. The adoption of Question Mark Perception (QMP) at Loughborough has allowed us to deliver tests to large numbers of students over the web since October 2000. Although not without its weaknesses, QMP has the significant benefit of being IMS QTI compliant, which is the accepted standard for question mark up. Other HEIs using any QTI compliant assessment engine will be able to import the Loughborough question banks for use in their own institutions. Such HEIs would need to put in place an appropriate system in order to properly administer student test taking and associated information through CAA. It is envisaged that within the time-span of the project (2002-2005) most UK institutions will be in a position to be able to exploit the HELM Assessment regime one way or another.

Web-delivered CAA is convenient but not essential and alternative implementation based on CDs is being trialled. All of the HELM tests (that is, all the questions and all the linked feedback) together with all the Workbooks easily fit onto one CD. Students provided with such a CD can then operate in *distance learning mode* doing the required work and completing tests on the CD. This is easy to implement if only self-testing is required; formal testing is more challenging, requiring a server so that the students can submit their completed tests for processing. This way the test can be done off-line and students need not be *live* on the web for long periods; only the few seconds it takes to upload test results. This scheme has already been successfully incorporated in another Loughborough University based project, undertaken on behalf of the Royal Academy of Engineering, entitled Best Maths [see 8].

## Loughborough's implementation of the Assessment regime

In a typical testing regime students are given a Workbook for a new topic in Week 1 (for self-study or lecture support). Week 3 is then a Test week (during which lectures and tutorials run as normal, but on a new Workbook). The Test week is organised as follows:

- From Monday to Wednesday a trial test is available on the web. Students may access this test *at any time* within this period and, as it is web-delivered, anywhere in the world. It can be trialled as often as the student desires. No record is kept by staff on student performance on these trial tests (although it could be – for research and development purposes, for example). Some students simply access the trial test to get information on question types and level of difficulty without attempting to answer any questions. Most will make a serious attempt at the trial test at least once; many up to five times. Many will work in small groups sorting out difficulties with the trial test. A good number seek help with the trial test from staff in our Mathematics Learning Support Centre. Others will access the test, input spurious answers just in order to get the feedback or possibly to try to discover all the possible questions! We find 95% of students engage in some way. This as a valuable learning mechanism and it is clear that students now engage with the learning process at some level *throughout the semester*.

- On Thursday and Friday the actual coursework test is available. Again, students may access this test at anytime within this period and from anywhere. However, they are only allowed to take this test once.

Both tests have an identical form, selecting questions randomly from previously created question banks covering aspects of the topic just covered in lectures. If a student gets a question wrong on the trial test a single page of feedback is available. The feedback may be exemplary (addressing the solution of similar problems to the one presented) or specific (in which the solution to the given problem is detailed). The only feedback available on taking a formal coursework test is the overall score and an indication of which questions were answered correctly and which incorrectly.

Although there are many possible question types we generally use just three: numeric input (the majority), multiple choice, and hot-spot in which the student is usually required to use the cursor to identify a 'deliberate' error in the solution to a problem. Multiple response questions may be used in the future.

Following extensive feedback exercises we find this testing regime to be generally popular with both staff and students. Students particularly like the flexibility this method of assessment offers. They like the facility to trial tests and the possibility of doing tests when *they* are ready. A recurring complaint about this approach is its *unforgiving* nature: if the final step is incorrect then no marks are awarded even if every other step was accomplished correctly. However CAA questions are relatively straightforward and so we would expect students to get them right each time. It also encourages careful working, which is no bad thing.

An occasional and valid concern raised by academics is the current practice at Loughborough of using the same data bank for both trial testing and formal testing. It is intended to have separate banks in the future (and tell the students) to discourage a rote-learning approach. Another legitimate concern is over allowing Loughborough students to undertake the formal tests unsupervised. There are great benefits to the students to allow them this freedom but at least some supervised tests would seem wise. This is very much up to the individual academic or institution to decide upon.

## Trialling and evaluation

The HELM learning resources will be on trial at each of the five consortium members. In addition, we have made provisional arrangements or will be negotiating to trial them in 2003-2004 at the following universities and colleges: Aston, Birmingham, Bournemouth, Bournemouth & Poole, City, Derby, Glamorgan, Glasgow, Glasgow Caledonian, Glenrothes, Harper Adams, Hertfordshire, Kingston, Lancaster, Leeds, Leeds Metropolitan, Moray, Leicester, Liverpool, Nottingham, Nottingham Trent, North Devon, Newcastle, Northumbria, Oxford Brookes, Portsmouth, Plymouth, Queen's Belfast, Robert Gordon, Salford, Southampton, Surrey, Ulster, UWIC and Westminster Kingsway.

Strictly speaking, this project is limited to HEIs in England and Northern Ireland but there has been considerable interest from Scotland, and some from Wales and the funding body (HEFCE) have agreed that institutions in those countries can be included in the trialling. Interest abroad has been shown but the project's intention is to wait until it enters its final year (2004-5) before considering linking outside the UK. Trialling will determine whether these resources can be successfully used across the HE sector and in particular will enable us to assess the viability of the assessment regime elsewhere.

## References


1. Sutherland R. & Pozzi, S (1995) The Changing Mathematical Background of Undergraduate Engineers – A review of the issues. London: The Engineering Council
2. Institute of Mathematics and its Applications, (1995), *Mathematics Matters in Engineering* – Report. London: (The Institution of Chemical Engineers, the Institution of Civil Engineers, the Institution of Electrical Engineers, the London Mathematical Society, the Institute of Mathematics and its applications.)
3. Working Group of London Mathematical Society et al (1995) *Tackling the Mathematics Problem*, Report – London: The London Mathematical Society, the Institute of Mathematics and its Applications, and the Royal Statistical Society.
4. Institute of Mathematics and its Applications, (1999), *Engineering Mathematics Matters*, Report, IMA, London.
5. Proceedings of the 2nd IMA Conference on the Mathematical Education of Engineers (1997), Loughborough University, Eds. Hibberd, S & Mustoe L.R. IMA ISBN 0 905 091 05 01
6. Proceedings of the 4th IMA Conference on the Mathematical Education of Engineers (2003), Loughborough University, Eds. Hibberd, S & Mustoe L.R. IMA
7. Proceedings of the 9th SEFI European Seminar on Mathematics in Engineering Education, Helsinki, 1998.
8. http://bestmaths.lboro.ac.uk/

## Appendix Proposed HELM Workbooks

01 Algebra
02 Functions
03 Polynomials, Inequalities & Partial Fractions
04 Sets, Probability & Descriptive Statistics
05 Modelling
06 Logarithms & Exponentials
07 Matrices
08 Matrix Solution of Equations
09 Vectors
10 Complex Numbers
11 Differentiation
12 Techniques of Differentiation
13 Applications of Differentiation
14 Integration
15 Applications of Integration
16 Sequences & Series
17 Conics & Polar Coordinates
18 Functions of Several Variables
19 Differential Equations
20 The Laplace Transform
21 Discrete Probability Distributions
22 Continuous Probability Distributions
23 The Normal Distribution
24 Numerical Methods 1
25 Numerical Methods 2
26 Functions of a Complex Variable 1
27 Functions of a Complex Variable 2
28 Eigenvalues & Eigenvectors
29 Fourier Series
30 The Fourier Transform
31 Partial Differential Equations
32 Numerical Methods 3
33 Numerical Methods 4
34 Statistics 3
35 Statistics 4
36 Multiple Integration
37 Vector Calculus
38 Case Studies 1
39 Case Studies 2
40 Tutor's Guide

Those Workbooks or parts underlined are newly written materials for the HELM project. The remainder are enhanced and extended Workbooks based upon the Loughborough Open Learning Project materials.

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# AN INTERESTING DIFFUSION PROBLEM FROM FOUR DIFFERENT ANGLES

GLADYS GUINEO COBS & VICTOR MARTINEZ LUACES

## Introduction

Chives (Allium schoenoprasum) are a plant whose long, tube-shaped leaves are used in typical dishes of several countries. For example in France, is called "Ciboulette" and is highly valued, being used instead of onions, because of their similarity.

This plant has an increasing importance from economical viewpoint, but in order to pack it for supermarkets and restaurants, it is necessary to dry it first. This process of chives drying is a source of very interesting mathematical problems that will be analyzed here.

The long tubes of chives are approximately cylindrical; so one possibility consists in solving the diffusion equation in Cylindrical coordinates [1]. The other option is cover a plane surface with the plants and dry them from the upper side using air of low humidity. Obviously, in this last case, Cartesian coordinates are more useful [2],

For each geometrical situation (plane or cylindrical) we can propose two different approaches: analytical and numerical. So, there are four different angles to be considered, for the same problem.

In this paper we will analyze these four problems, their mathematical richness and their educative possibilities. Moreover, we will comment students' opinion about these problems, which were widely used in both undergraduate and postgraduate courses, for Chemical Engineering and Food Technology Engineering.

All these experiences were carried out in the last six years with very positive reactions from the students. Taking into account these experiences, we will present here several conclusions and recommendations.

## The problem in cylindrical coordinates

When chives are dried, a water molecule diffuses from the long tube to the cylindrical surface. This diffusion corresponds with a PDE problem where the Laplacian is written in cylindrical coordinates as follows: $\frac{\partial c}{\partial t} = D \cdot \left( \frac{\partial^2 C}{\partial t^2} + \frac{1}{r} \cdot \frac{\partial C}{\partial r} \right)$

where t is time, r is radius, $C$ is humidity (grams of water divided by grams of dried plant) and $D$ is diffusivity (in this case $D$ is a function of temperature)

As we can see in the figure 1, we can assume that $R << L \ and \ 0 \leq r \leq R$. For symmetry reasons at $r = 0$ we do not have diffusion of water, so the net flux is zero in the axis of the cylinder. Then one of the boundary conditions is

$$\frac{\partial C}{\partial t}(0,t) = 0 \quad \forall\, t > 0 .$$

The other boundary condition is $C(R,t) = C_{eq.} \quad \forall\, t > 0$, that is, humidity at the surface reaches instantaneously the equilibrium humidity ($C_{eq}$).

Finally, the initial humidity in chives is $C_0$, with $C_0 > C_{eq}$. Taking into account all these facts, the problem is:

$$\begin{cases} \dfrac{\partial C}{\partial t} = D \cdot \left( \dfrac{\partial^2 C}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial C}{\partial r} \right) \\ \dfrac{\partial C}{\partial r}(0,t) = 0 \quad \forall t > 0 \\ C(R,t) = C_{eq} \quad \forall t > 0 \\ C(r,0) = C_0 \quad 0 \le r \le R \end{cases}$$

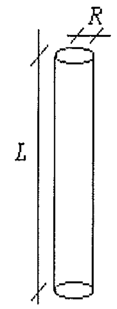


*FIGURE 1.* Geometry of ciboulette.

As it was mentioned before, diffusivity D is an exponential function of temperature more precisely we have:

$$D = a_1 \cdot e^{-\frac{a_2}{T}}$$

In our country all the drying process is done in a sun-convective-dryer, so temperature is like air temperature, a sinusoidal function of time:

$$T(t) = a_3 + a_4 \cdot \sin(\omega \cdot t + a_5) \qquad with \quad a_4 < a_3$$

Then, we can write $D = a_1 \cdot e^{\frac{-a_2}{a_3 + a_4 \cdot sin(\omega \cdot t + a_5)}}$

Now, we will solve this problem both by analytical and numerical methods.

## The analytical approach

Using the method of separation of variables we can put $C(r,t) = X(r) \cdot G(t)$. Then, substituting in the equation, we obtain two O.D.E. One with variable r: $\dfrac{d^2 X}{dr^2} + \dfrac{1}{r} \cdot \dfrac{dX}{dr} + \beta^2 X = 0$ and the other with variable t: $\dfrac{1}{D(t) \cdot G(t)} \cdot \dfrac{d\,G(t)}{d\,t} = -\beta^2$

The first one can be converted in a Bessel equation by a linear change of variable [1]. Solutions of the Bessel equation can be written as linear combinations of Bessel functions of order zero, but solutions must be bounded when r→0 for chemical reasons, so all the possible solutions are multiples of $J_0(\beta \cdot r)$.

Using boundary conditions it is easy to prove that X(r) is always a coefficient multiplied by $J_0\left(\dfrac{\lambda_n}{R} \cdot r\right)$ where $\lambda_n$ are the positive roots of $J_0$, the Bessel function of the first type, of order zero.

The second O.D.E. (that is, the time-equation) can be solved with elementary methods, and the solution is: $G(t) = G(0) \cdot e^{-\beta^2 I(t)}$ where $I(t)$ is the integral function $\int_0^t D(t)dt$

Then, we have a transcendent function to integrate, because $D = a_1 \cdot e^{-\frac{a_2}{T}}$ and $T(t) = a_3 + a_4 \cdot sin(\omega \cdot t + a_5) \qquad a_4 < a_3$. For this purpose we can expand the exponential function in power series, that is: $e^{-\frac{a_2}{T}} = \sum_{k=0}^{\infty} \frac{\left(-\frac{a_2}{T}\right)^k}{k!}$

Then, we will have to solve the integrals: $\int_0^t \left(1+\frac{a_4}{a_5}\cdot\sin(\omega\cdot t+a_5)\right)^{-k} dt$ for k: 1, 2, ... and finally use convergence theorems to arrive at the desired function $I(t)$.

In this case, the change of variable $u = 1+\frac{a_4}{a_5}\cdot sin(\omega\cdot t+a_5)$ is recommended and putting $U = -u^2 + 2u + \frac{a_4^{\,2}}{a_3^{\,2}} - 1$. Then, the above integrals become $\int_{u_0}^{u} \frac{du}{u^k U^{1/2}}$ which can be solved by recurrence methods [6]

Taking into account all these facts, the final result is: $C = C_{eq} + \sum_{n=1}^{\infty} c_n . e^{-\frac{\lambda_n^2}{R^2} . I(t)} . J_0\left(\frac{\lambda_n}{R}.r\right)$ where

$c_n$ can be obtained using the initial condition as follows: $C_0 - C_{eq} = \sum_{n=1}^{\infty}\left(c_n \; J_0\left(\frac{\lambda_n}{R} r\right)\right)$

This is a Fourier-Bessel series, where coefficients $c_n$ can be obtained using derivation and integration formulas for Bessel functions and ortogonality properties of: $\sqrt{r}.J_0(\lambda_n r) \quad n = 1,2,3,\ldots$

Applying all these techniques, we finally arrive at the analytical solution:

$$C(r,t) = C_{eq} + (C_0 - C_{eq})\sum_{n=1}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)} J_0\left(\frac{\lambda_n}{R}\cdot r\right)\cdot e^{-\frac{\lambda_n^2}{R^2} I(t)}$$

## The numerical approach

A first option to be considered for this kind of parabolic P.D.E., consists in using the explicit numerical method. In this case the recurrence formula is:

$$\frac{w[i,j+1]-w[i,j]}{k} = \frac{D_j}{h^2}\left(\left(1+\frac{1}{i}\right).w[i+1,j]+w[i-1,j]-\left(2+\frac{1}{i}\right).w[i,j]\right) \qquad \begin{matrix} h \approx \Delta r \\ k \approx \Delta t \end{matrix}$$

This method is unstable for moderate and large values of "$t$", but is useful to introduce this technique in the classroom, due to didactical reasons [3].

If we are interested in the solution itself and in particular, in the stability properties of this solution, then Crank-Nicholson method is the most recommendable [4]. In this case, the recurrence equation is:

$$\frac{w[i,j+1]-w[i,j]}{k} =$$

$$\frac{D_j}{2h^2}\left(\left(1+\frac{1}{i}\right).(w[i+1,j]+w[i,j+1]) - \left(2+\frac{1}{i}\right).(w[i-1,j+1]+w[i,j]) + w[i-1,j] + w[i+1,j+1]\right)$$

There exist important differences between both methods. Explicit method provides the following system:

$\mathbf{W}^{j+1} = \mathbf{A} \cdot \mathbf{W}^{j}$ where **A** is a tri-diagonal matrix. Crank-Nicholson method, after several transformations led us to a different equations system, like this: $\mathbf{A} \cdot \mathbf{W}^{j+1} = \mathbf{B} \cdot \mathbf{W}^{j} + \mathbf{C}$ where **A** and **B** are tri-diagonal matrix and **C** is a column vector. This represents an extra difficulty to solve the algebraic system, and then, to solve the PDE problem
Finally, a graphical representation of the solution is shown in Figure 2.

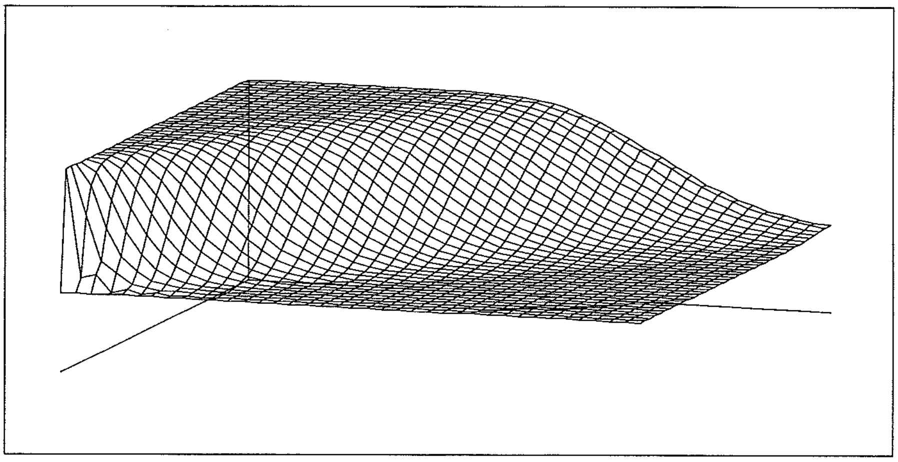

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*FIGURE 2.* Numerical solution of Ciboulette problem in Cylindrical coordinates.

## The problem in Cartesian coordinates

As we can see in the figure, in this case diffusion takes place only in z-axis, so the problem is [5]

$$\begin{cases} \dfrac{\partial C}{\partial t} = D \dfrac{\partial^2 C}{\partial z^2} & \\ C(L,t) = 0 & C(x,0) = C_0 - C_{eq} \\ D = a_1 e^{-a_2/T} & T = a_3 + a_4(\sin wt + a_5) \end{cases}$$

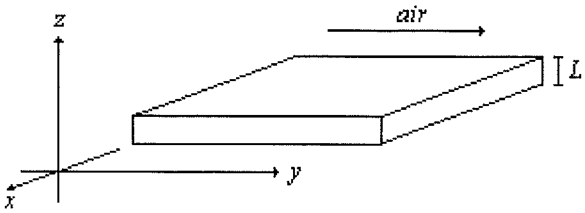


*FIGURE 3.* Drying of Ciboulette. Plane geometry.

where boundary and initial conditions can be easily explained with the same arguments already mentioned in the cylindrical coordinates problem.

## The analytic approach

Again, we can put *C(z,t)=F(z)G(t)* (Separation of variables) to obtain an ODE system. Time equation of the ODE system is the same than in the other method (the cylindrical coordinates problem), so we will consider only the positional equation (that is, the ODE with variable z).

Equation in z is in this case: $$\frac{1}{F(z)} \frac{\partial^2 F}{\partial z^2} = -\lambda^2$$

And the solution is $F(z) = A\sin(\lambda z) + B\cos(\lambda z)$

Using boundary conditions is easy to prove that A=0 and $\lambda = \dfrac{2n-1}{L} \dfrac{\pi}{2}$ where n= 1, 2, ...

Then, the solution of the problem is given by the series:

$$C(z,t)=\sum_{n=1}^{\infty} b_n \cdot \cos\left(\frac{2n-1}{L}\frac{\pi}{2}z\right)\cdot e^{\left(-\frac{(2n-1)^2}{L^2}\frac{\pi^2}{4}I(t)\right)}$$ where I(t) is $\int_0^t D(t)dt$ already considered and

$b_n$ can be computed from the initial condition: $C_0 - C_{eq} = \sum_{n=1}^{\infty} b_n \cos\left(\frac{2n-1}{L}\frac{\pi}{2}z\right)$

This is a cosine series, but not the traditional one (i.e. $\sum c_n \cos\left(\frac{n\pi}{L}z\right)$ or similar), so, it is convenient to proof the ortogonality and completeness of the basis formed by those functions.

Ortognality fallows straightforward from elementary integral formulas of trigonometric functions, while completeness can be demonstrated showing Parseval formula. For this last purpose, several numeric series related with Bernoulli numbers must be used [1].

Using all these techniques, the final result is:

$$C(z,t)=\sum_{n=1}^{\infty}\left[\frac{4\left(C_0-C_{eq}\right)(-1)^{n-1}}{(2n-1)\pi}\cdot\cos\left(\frac{2n-1}{2L}\pi z\right)\right]\cdot e^{-\frac{(2n-1)^2\pi^2}{4L^2}I(t)}$$

## The numerical approach

As in the other case, explicit and semi-implicit methods can be used. The first one is more didactical but unstable, and the recurrence formula [5] is in this case:

$$\frac{w[i,j+1]-w[i,j]}{k}=\frac{D_j}{h^2}\left(w[i+1,j]+w[i-1,j]-2\cdot w[i,j]\right)$$

The second one (Crank-Nicholson) is more accurate because of its stability properties. In this case the formula is: $\frac{w[i,j+1]-w[i,j]}{k}=$

$$\frac{D_j}{2h^2}\left(w[i+1,j]+w[i+1,j+1]-2\left(w[i,j]+w[i-1,j+1]\right)+w[i-1,j]+w[i,j+1]\right)$$

It is important to note that in this case it is possible to obtain a tri-diagonal matrix system [5], so the solution takes significatively less time of computation. Graphically, this solution can be represented as in Figure 4.

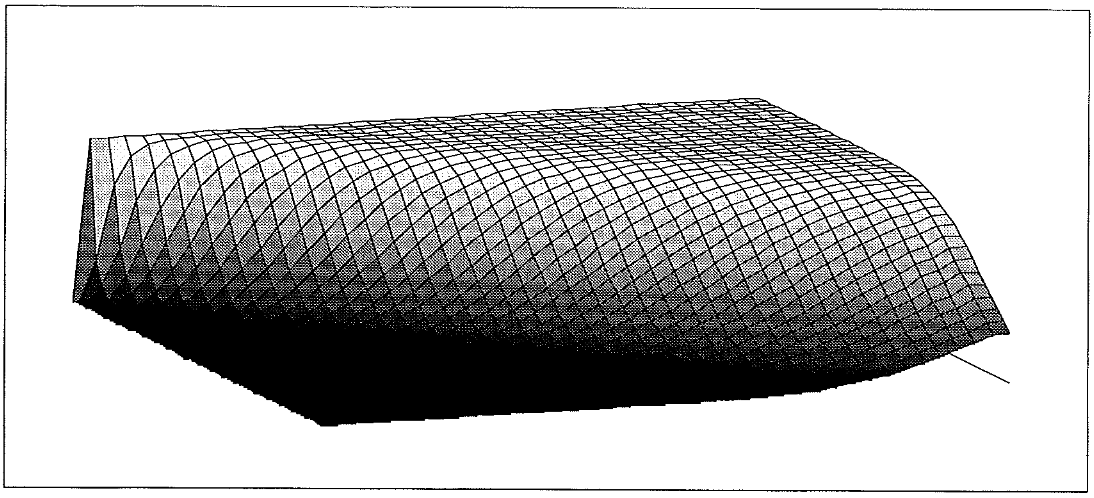

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*FIGURE 4.* Numerical solution of Ciboulette problem. Cartesian coordinates.

## The mathematical education point of view

This problem, and its different versions from four different angles provides an interesting form to present several mathematical concepts, theorems, methods, etc., in a motivating and innovative way. In fact, these different versions were used in postgraduate courses [6] and in a couple of undergraduate courses: Differential Equations and Numerical Calculus. These last courses are two of the three Mathematical second-year courses for Chemical Engineering and Food technology Engineering students (the other one is "Probability and Statistics").

In a first stage, these three courses (Numerical Calculus, Statistics and Differential Equations), were based on real problems, strongly related with other disciplines. The other three courses offered by the Mathematics Department (Calculus I, Calculus II and Linear Algebra), remained traditional, at least in this first experiment. So, at present time, all second year courses of our department are in connection with other disciplines, while first year ones will be changed probably next year (this will be the second part of this experiment). There are other important differences between first and second year courses. For example, in second year courses, real problems represent more than fifty percent of final examinations. Moreover, in several cases, these final examinations can be substituted by project-work, where students try to solve this kind of problems with help of computers or electronic calculators (and, of course, with orientation of teachers).

## Results and conclusions

In a previous paper, an expert group was consulted, and almost all the experts remarked the importance of teaching significant concepts and procedures in service courses ([7] and [8]). From a different point of view, Chemistry students showed an important preference for teachers who make the effort of presenting real-life problems, related with their own careers.

Finally, Cluster Analysis and other Multivariate Statistical methods showed a very similar situation [9]. More precisely, in our group of mathematical teachers (that is, twelve teachers of the Mathematics Department at the Chemistry Faculty in Montevideo), the Cluster Analysis of "Applications", separate a group of them as the better ones (this variable "Applications" consists of an 12 vector with the average results of two questions: one of them related with real-life problems and the other one about the connection with other disciplines). This group of five teachers was integrated almost exclusively with teachers of second year courses (Numerical Calculus, Statistics and Differential Equations) and almost all of them participated in interdisciplinary work with teachers and researchers of other departments and laboratories. Moreover, two teachers of this group are researchers in Applied Mathematics. From these comments and results, it is obvious that real applications produce positive reactions in Chemistry students, in concordance with experts´ opinion [10].

In an important paper of ICMI [11], this style of teaching, where Mathematics is applied to other disciplines, was considered as "the ideal situation" for mathematical service courses. Other aspect, very important to be considered is assessment. The evaluative process must not be dissociated from the style of teaching. So, if we try to teach through problem-solving of real life situations, in context with other subjects, assessment must be carried out in the same way. This purpose can be put into practice through project-work, where students (with orientation of an interdisciplinary team of teachers) try to solve real problems of their careers, in order to approve their mathematical courses.

It is important to remark that Mass Transfer is an excellent source for this kind of problems. In several cases, they remain almost unexplored in their mathematical richness. Also, this subject provides a good opportunity for interdisciplinary work in research and teaching. Finally, as it was mentioned before, these problems represent an interesting challenge for applied mathematicians and Mathematical Education researchers.

## References


1. Martínez Luaces, V., Guineo Cobs, G. *Las EDP en problemas industriales de secado de alimentos: su resolución analítica y su transferencia al aula*, III Seminario Internacional de Matemática, Física e Informática Educativa. Camagüey, Cuba, 2002.
2. Martínez Luaces, V. & Guineo Cobs, G., to appear *Resolución analítica del problema de secado de Ciboulette en coordenadas cartesianas.* Submitted to InMat-2003. Buenos Aires, Argentina. 2003
3. Guinēo Cobs, G., Martínez Luaces, V. *Resolución numérica de Ecuaciones de difusión en coordenadas cilíndricas: su transposición didáctica a un curso de cálculo numérico*, III Seminario Internacional de Matemática, Física e Informática Educativa. Camagüey, Cuba, 2002.
4. Dahlquist, G. Bjorck, A., Anderson, N., 1974, *Numerical Methods*. New Jersey: Prentice – Hall.
5. Guineo Cobs, G., & Martínez Luaces, V., to appear. *Resolución por métodos numéricos del problema de secado de Ciboulette en coordenadas cartesianas y análisis de estabilidad de las soluciones.* Submitted to InMat-2003. Buenos Aires, Argentina. 2003
6. Martínez Luaces, V. *Sobre el rol de las Ecuaciones en Derivadas Parciales en los cursos para Ingeniería*, CD: Memorias del IV Taller sobre la Enseñanza de Matemática para Ingeniería y Arquitectura. CUJAE. La Habana, Cuba, 2000.
7. Martínez Luaces, V.; Casella, S. *La Educación Matemática en las diferentes ramas de la Ingeniería en el Uruguay hoy* in Memorias del II Taller sobre la Enseñanza de la Matemática para Ingeniería y Arquitectura, Cuba: ISPJAE, 1996. pp. 386-391.
8. Martínez Luaces, V. *Matemática como asignatura de servicio: algunas conclusiones basadas en una evaluación docente.* Números. Revista de didáctica de matemáticas ***36***. (1998). 65 – 67.
9. Gómez, A., Martínez Luaces, V. *Evaluación docente utilizando Análisis Multivariado,* in Acta Latinoamericana de Matemática Educativa **15.2,** (2002). R.M. Farfan (ed.), México: CLAME. 1016-1021
10. Martínez Luaces, V. *Considerations about teachers for Mathematics as a service subject at the university* in Pre-proceedings of the ICMI Study Conference, Singapore: Nanyang Techonological University, (1998). 196-199.
11. ICMI, Mathematics as a service subject, L´Enseignement Mathématique 32, (1986) 159-172.



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# MAPLE LABS: CALCULUS FROM ALL ANGLES

JOHN HANNAH

One of the guiding principles of the calculus reform movement has been the idea of multiple representations, especially as captured in the Rule of Four: where appropriate, topics should be presented geometrically, numerically, analytically and verbally. In this paper I shall describe an attempt to apply this idea in the context of laboratory sessions where a computer algebra system (Maple) is used to encourage learning about advanced calculus. As an example I shall discuss two lab sessions about solutions to second order linear differential equations with constant coefficients. In this context, geometric and analytic representations are supplied by Maple's plotting and solving routines, physical representations come from the students' experience of damped spring systems and simple RLC electrical circuits, and verbal representations come from the natural dialogue occurring between lab partners working on the same computer. The availability of different representations means that students can generate expectations (from a familiar representation) which can then be used to check conclusions (derived from an unfamiliar representation). The interplay of ideas deepens the students' understanding, not just of the mathematics of the differential equations, but also of the physical systems which these equations have been used to model.

## Introduction

It seems likely that, right from the beginning, multiple representations have been important to the communication and practice of mathematics. Old Babylonians used geometry and tables to supplement their basic calculating tools [**3**]. Similarly, ancient Greeks used verbal argument and labelled diagrams to support one another in their development of deductive mathematics [**6**], and used an appeal to a mental image of motion to solve geometric problems such as squaring the circle ([**5**], page 82).

More recently, research into how our brains function mathematically tells a similar story. There is no mathematical region in the brain, but rather a dispersed collection of centres responsible for different tasks, such as representation of number words, visual recognition of numerals, or choice of strategy and so on, all cooperating when we solve mathematical problems (see for example [**4**], pages 52-53).

So when proponents of calculus reform suggested the Rule of Four mentioned above, they were reminding us both of how our subject developed historically, and of how our brains do mathematics. One response to the plea for multiple representations has been to teach mathematics as a laboratory subject, as in the Connected Curriculum Project (CCP) materials authored by Lang Moore, David Smith and others [**2**]. Here students augment the traditional mathematical toolbox with a computer algebra system like Maple, along with experimentally derived data and access to the World-Wide Web.

In conjunction with a visit by David Smith in 2001, the author experimented with some of the CCP modules in an advanced calculus course, focusing mainly on the use of Maple in a lab environment. Apart from abandoning CCP materials in favour of worksheets written specifically for the course, the lab approach was retained in each of the next two versions of the course. This paper discusses the extent to which students used the various representations available to them in these sessions.

## Background

This study is part of a program of action research carried out by the author during three successive versions (2001 to 2003) of a one semester second year calculus course (MATH 264 / EMTH 204) at the University of Canterbury. The first half of the course is about multivariate calculus, and the second about differential equations. In 2003 there were 97 students in the class, all of above average ability (getting at least B+ in their first year mathematics course, or else gaining direct entry from high school). About 70% of the class were engineering students, with most of the remainder being science majors.

The class met for 4 one-hour lectures and 1 one-hour 'tutorial' each week. The students were divided into six groups which met every fortnight for traditional tutorials: discussing previously assigned problems from the text book, working through additional problems set during the tutorial itself, and getting help from one another or the tutor if they got stuck. In the intervening weeks, the same six groups met for the Maple labs. Here students were organized into pairs, although occasionally they worked solo or in threes. In the lab sessions they followed Maple worksheets which were similar in style to those of the Connected Curriculum Project [**2**]. Consultancy hours were offered each week to help students with any problems they might have with their homework or with the lab assignments.

In this paper I shall focus mainly on two successive labs dealing with second order linear equations with constant coefficients, and with the use of Laplace Transforms. The first of these labs looked at the solutions in the homogeneous case, exploring the effect of different initial conditions, and of different levels of damping (see the Appendix). The second lab set similar tasks for various forced systems, using Laplace Transforms to construct solutions.

## Method

My own observations come from a diary of incidents which occurred while I was tutoring in the laboratory sessions, recorded during the sessions or straight afterwards if I was too busy during the session. I was the tutor for all six weekly sessions for the first lab, and for three sessions of the second lab. As I generally responded only to student requests for help, this probably biases the observations towards students' difficulties, but it is still possible to extract positive information too. Further information came from grading the students' final submissions.

Students' attitudes were collected via an open-ended survey at the end of the course. The survey was completed by 85 of the 97 students taking part in the course.

## Observations

*Example 1.* All the lab sessions featured a very high level of animated conversation among the students, particularly compared with the relatively intermittent nature of conversation among the same students in the pen-and-paper tutorials which alternated with the labs. As I walked around the labs, looking discreetly at the screens over people's shoulders, the discussion was all relevant, usually focusing on which commands were needed next, or on how to interpret the results of the latest command.

Typically, one student talked to the other while gesticulating towards the screen. Usually I did not have time during the labs to observe whether there were dominant partners, but during one consultancy session I did notice a group of three students who, during the course of the 50 minute session, each took a turn at expounding their point of view to their partners. Some time later one of this same group told me that she felt their group was disadvantaged because 'it took us longer to agree because there were three different opinions instead of two.'

Students' survey comments about working in pairs give a similar view of what was happening in the groups:

> Much better than individual work since both people could bring different ideas and there was often discussion about areas of difficulty.
>
> Working in pairs was a good idea and was especially helpful if you were paired up with someone with a different background in maths than you had.
>
> It helped to get 2 perspectives for questions and you had to learn to express your self better, to enable the other partner to know what you meant.
>
> Its good for me, forces me to follow other peoples lines of thought, in lectures its easy to just take the notes down.
>
> Worked well for me, able to discuss different interpretations.

Definitely helpful as it encouraged communication within the pairs and allowed for another perspective on the lab material.

Much better than individual work since both people could bring different ideas and there was often discussion about areas of difficulty.

*Example 2.* A simple example of interplay between analytic and graphical representations occurred at the start of the first lab session, where students were asked to plot the solution to an initial value problem: $y = \sin(50\sqrt{2}\, t)/10$. Using a standard Maple plotting range (0 to 10) produces a plot as in Figure 1.

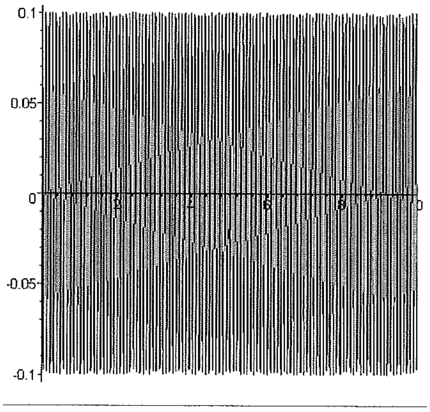


*FIGURE 1.* Plot of solution using default settings.

Students who used this plotting range often stared at the screen for some time. When I asked them what sort of behaviour they were expecting, what the period was, and (if that didn't work) how many periods fitted into the chosen plotting interval, they always resolved the problem by going back to the analytic expression for the solution, or to the (invariably symbolic form of the) period found earlier.

*Example 3.* Later in the same session, where the effect of increased damping was examined, there was another opportunity to use the analytic representation to help understand the graphical representation. Despite being encouraged to use the auxiliary equation, some pairs just tried a sequence of values (like 0, 20, 50, 100) and then stopped. A typical comment was

> By trialling increasing values of p it seems like when p=100, there's no oscillation, since the graphical trace of the motion ... does not even complete a full cycle.

even though the accompanying formula for the solution involved trigonometric functions.

*Example 4.* In the second lab session on differential equations, students had to solve one differential equation with a series of different forcing terms, including one with a Dirac delta impulse, and another (defined the using Heaviside function) consisting of an initial segment of periodic forcing which was then switched off. As in the first lab session, they had to interpret the solutions in terms of a corresponding spring system.

After one pair plotted the solution to the impulse case, one partner exclaimed 'Hey! It did what I predicted.' He then explained how he expected the spring system to respond to an impulse, and what the resulting graph should look like.

Other students developed similar expectations, either from the (imagined) physical representation of the spring system, or from solutions to simpler systems they had already solved. Many of these related to the example involving the Heaviside function, since the students had already solved cases similar to the individual segments of the forcing term. The expectations usually surfaced when they were disappointed (as mentioned earlier, I usually just responded to requests for help). For example, one pair wanted to know why they couldn't see the oscillations seen in an earlier example, even though the example started in exactly the same way as an earlier example – it transpired that they were looking at a graph of the forcing term itself, and not the solution. Two other pairs expected that the part of their solution corresponding to the 'turned off' segment should be identical to an earlier example which had no forcing term – it turned out that they were ignoring the effect of different initial conditions.

*Example 5.* In the first lab session many students developed expectations in the reverse direction. For each equation they solved they were asked to interpret the solution in terms of the behaviour of a simple electrical circuit. Although 32% of the students were electrical engineering students simultaneously doing a course on circuits, almost all students used the graphical representation to work out how a circuit would respond to given initial conditions, rather than using their experience of the physical model. One student explained to me that they had never had to interpret their solutions in the circuits course – 'that's the physics people's job.'

## Discussion

As we have seen, the students were at least exposed to multiple representations of these differential equations and their solutions. Did the students link these representations? Did they learn from the experience?

The evidence from Example 1 offers quite an optimistic view, at least as far as the students sharing their own perspectives goes. Bookman and Malone [1] report similar collaborative discourse whenever the interactive computer lab approach is used. Examples 2 and 3 show that the graphical representation may be preferred to the analytic one, at least by the students who needed help, although even these students showed an ability to link the analytic and graphical representations once the possibility was pointed out to them. It has to be admitted that software like Maple makes it much easier to guess and check, for a suitable range of parameter values (say), than to do a side calculation which might home in on the exact values required. Bookman and Malone [1], who also studied student pairs using CCP materials, give similar examples of students not always choosing what we might think is the best tool for a specific task (see their Vignettes 1, 3 and 5).

Examples 4 and 5 show that linking the graphical representation with an imagined physical representation seems to occur fairly readily, at least as long as the physical model is reasonably familiar or easy to imagine. Example 4 also gives some evidence that students had learned, between the first and second of these lab sessions, to use the links between the graphical and physical representations to help predict the behaviour of a solution.

Given the method of data collection, my evidence is necessarily fairly flimsy compared with that gathered in an intensive study of individual pairs like Bookman and Malone's [1]. On the other hand, the evidence presented here was collected in a real classroom, and so presents a complementary, yet supporting, picture to the one offered by their study.

## References


1. Bookman, J. & Malone, D. (to appear), "The nature of learning in interactive technological environments: a proposal for a research agenda based on grounded theory," Research in Collegiate Mathematics Education V.
2. Connected Curriculum Project, Duke University, web materials (available at www.math.duke.edu/education/ccp/).
3. J. Hoyrup, Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and Its Kin, Springer-Verlag, 2002.
4. P. D. Klein, Rethinking the multiplicities of cognitive resources and curricular representations: alternatives to 'learning styles' and 'multiple intelligences', J. Curriculum Studies, **35(1)** (2003) 45-81.
5. W. R. Knorr, The ancient tradition of geometric problems, Dover, 1993.
6. R. Netz, The shaping of deduction in Greek mathematics: a study in cognitive history, Cambridge University Press, 1999.

## Appendix: Maple worksheet (edited slightly)

This worksheet looks at the solutions to homogeneous second order linear differential equations with constant coefficients

$$y'' + p\,y' + q\,y = 0$$

and at what these solutions mean for

a simple spring system, or
a simple AC electrical circuit.

Maple will be used to solve the various differential equations, and to plot and compare their solutions.

### Part 1. Undamped free vibrations

In this section you will look at the solutions of a typical undamped equation with no forcing term:

$$y'' + 5000\,y = 0$$

focusing particularly on what happens when you change the initial conditions.

The following command stores the differential equation as a variable for use in later commands.

```
>   DE := diff(y(t), t, t) + 5000*y(t) = 0;
```

**General solutions**

Maple's dsolve command can be used to find the general solution to this differential equation.

```
>   dsolve(DE,y(t));
```

What kind of behaviour would you expect of such solutions? (If there are oscillations, what can you say about their period or amplitude? If there is exponential behaviour, what can you say about the growth or decay of the solution?)

Interpret your answer for the situation where the differential equation comes from a simple spring system:
what kind of spring system does the differential equation represent?
what does the general solution tell you about the motion of the mass in such a spring system?

Now interpret your answer for the situation where the differential equation comes from a simple AC electric circuit:
what kind of circuit does the differential equation represent?
what does the general solution tell you about the current in such a circuit?

**Initial value problems**

To get Maple to solve an initial value problem you can again use the dsolve command, this time putting the differential equation and the initial values into a list.

For example, to find and plot the solution to the above differential equation which satisfies the initial conditions

$$y(0) = 0.1 \quad \text{and}$$
$$y'(0) = 0$$

you could use the commands

```
>   dsolve({DE, y(0) = 0.1, D(y)(0) = 0}, y(t));
>   z := rhs(%);
>   plot(z, t = 0..?);
```

The following command saves this plot for later use.

```
>   plot1 := plot(z, t = 0..?, colour = magenta):
```

**Holding y'(0) fixed and increasing y(0)**

To see the effect of holding y'(0) fixed and increasing y(0) , you can construct a series of plots and superimpose them on a single diagram.

Suppose we fix $y'(0) = 0$ .

```
>   with(plots):
```

Copy, paste and modify the following group of commands to produce plots corresponding to increasing initial values for y(0) :

```
>  dsolve({DE, y(0) = ?, D(y)(0) = 0}, y(t));
>  z := rhs(%):
>  plot2 := plot(z, t = 0..?, colour = ?):
```

Displaying all these solutions in one diagram:

```
>  display({plot1, plot2, ?, ?});
```

Describe the effect of holding y'(0) fixed and increasing y(0).

Interpret your answer for the situation where the differential equation comes from a simple spring system:
what do these initial conditions mean for such a system?
what do the solutions tell you about the resulting motion of the mass?

Now interpret your answer for the situation where the differential equation comes from a simple AC electric circuit:
what do these initial conditions mean for such a circuit?
what do the solutions tell you about the resulting current in the circuit?

**Holding y(0) fixed and increasing y'(0)**
Use the same method to explore the effect of holding y(0) fixed and increasing y'(0).
Suppose we fix y(0) = 0.1 .
Copy, paste and modify the following group of commands to produce plots corresponding to increasing initial values for y'(0) :

```
>  dsolve({DE, y(0) = 0.1, D(y)(0) = ?}, y(t));
>  z := rhs(%):
>  plot1 := plot(z, t = 0..?, colour = ?):
>  display({plot1, plot2, ?, ?});
```

Describe the effect of holding y(0) fixed and increasing y'(0).

Interpret your answer for the situation where the differential equation comes from a simple spring system:
what do these initial conditions mean for such a system?
what do the solutions tell you about the resulting motion of the mass?

Now interpret your answer for the situation where the differential equation comes from a simple AC electric circuit:
what do these initial conditions mean for such a circuit?
what do the solutions tell you about the resulting current in the circuit?

**Part 2. Damped free vibrations**

In this section you will look at the effect of introducing a damping term in a second order linear differential equation with constant coefficients and no forcing term:

$$y'' + p\,y + 5000\,y = 0$$

```
>  DE := ?;
```

Fix both your initial conditions and determine the effect of increasing the value of p (from 0).

Copy, paste and modify the following group of commands to produce plots corresponding to several different values for p:
Hint: Use the auxiliary equation to decide on a suitable range of values for p.

```
>  p := ?;
>  dsolve({DE, y(0) = ?, D(y)(0) = ?}, y(t));
>  z := rhs(%):
>  plot(z, t = 0..?);
```

What is the effect on the solutions of increasing the value of p?
At which value of p does the solution cease to oscillate?

Interpret your answer for the situation where the differential equation comes from a simple spring system:
what does an increasing p value mean for such a system?
what do the solutions tell you about the resulting motion of the mass?

Now interpret your answer for the situation where the differential equation comes from a simple AC electric circuit:
what does an increasing p value mean for such a circuit?
what do the solutions tell you about the resulting current in the circuit?

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# TECHNOLOGY IN THE MATHEMATICS PREPARATION OF TEACHERS: A WORKSHOP

GARY ALVIN HARRIS

The advent of graphing calculators and computer algebra systems continues to have, tremendous impact on mathematics education at all levels. Mathematics educators at all levels not only have to re-evaluate how they teach mathematics, but what mathematics they teach and why. The following paper describes our efforts to incorporate such mathematics-specific technologies into the mathematics curriculum of students preparing to teach middle school and high school mathematics.

## Introduction

Mathematics education at the post secondary level in the US was influenced during the 1990's by two major reform efforts: the Calculus reform movement as described in two Mathematical Association of America publications (MAA, 1994; MAA, 1996), and reform in the undergraduate mathematics preparation of teachers as called for by the National Council of Teachers of Mathematics (NCTM, 1991) and the National Research Council (NRC, 1996). These and other professional organizations in the US observed the obvious fact: anyone teaching mathematics in the future will be expected to use mathematics-specific technologies such as graphing calculators and computer algebra systems (CAS) in their classrooms. Our efforts to address this need have focused on incorporating the use of such technologies into the mathematics courses required of students preparing at our university to teach children ages 4 through 18 (pre-college). These efforts are culminated in the creation of two lab-based, capstone mathematics content courses, one for students preparing to teach mathematics in middle school (ages 11 through 14) and one for students preparing to teach high school mathematics (ages 15 through 18). Details about these two courses can be found in our two previous articles, Harris (2001) and Harris (2000) respectively. Herein we discuss the underlying philosophy that has driven our reform efforts in general and the development and implementation of these two courses in particular. Next we describe the resulting methodology employed in the two courses and conclude with sample exercises. The workshop consists of a hands-on exploration of materials developed for, and used in, these two courses.

## Philosophy

The continued development and delivery of the two technology-based mathematics courses we offer for prospective teachers is driven by four basic assumptions:

- Teachers need to have an in-depth understanding of the mathematics they will be teaching, both at the skill level and at the conceptual level.
- Mathematics-specific technologies exist and are increasing in power and accessibility.
- Mathematics-specific technologies are becoming an integral part of mathematics.
- Teachers need to have an in-depth understanding of the mathematics-specific technologies they will be teaching, both at the skill level and at the conceptual level.

By an in-depth understanding of mathematics-specific technologies at the "conceptual level" we do not mean that teachers need to be software, or electrical, engineers; rather, we mean that they need to understand the interrelationships between the capabilities of the technology and the underling mathematical concepts. For example, when discussing large prime numbers it makes no sense to use a machine that approximates large integers using scientific notation. In a similar vein, the teacher should understand that any finite decimal expression is a rational number even if the technology cannot convert it to an integer fraction form, while some long decimal strings are converted to an integer fraction form that can not possibly be equal to the given decimal. Examples of such phenomenon are included in the Examples section below.

## Methodology

In addition to incorporating the use of math-specific technologies into the mathematics content courses required of prospective teachers, we created two lab-based capstone mathematics courses; one for the middle school teachers (Harris,2001) and one for the high school teachers (Harris, 2000). "Lab-based" means the courses are taught in a computer lab with access to a CAS (MAPLE in our case) and to graphing calculators (TI-86 or 83 in our case). By "Capstone Course" we mean a course that revisits and attempts to tie together the topics the students studied in their previous mathematics courses. For the middle school teachers these include topics from arithmetic, algebra, finite mathematics, number theory, and introductory calculus. For the high school teachers the topics are drawn from arithmetic, algebra, number theory, calculus, linear algebra, and differential equations. In theory, our two courses review mathematical concepts the students have already mastered, but do so in the context of modern mathematics-specific technologies.

In keeping with the above philosophy, we want these students be become confident that they can successfully deal with the evolving technologies they will encounter in their teaching careers. Thus, in both courses we attempt to create an interactive, cooperative, and discovery-learning environment. Students work in groups of two or three on series of exercises designed to reinforce their conceptual understanding of the mathematics, while at the same time forcing them to consider the implications and issues related to the technology. There are no lectures and minimum technical instruction is provided, mostly via examples. Students are expected to use their familiarity with standard mathematics terminology and notation, supplemented with online help and menus, to deal with syntax and idiosyncrasies related to particular technology. Of course the instructor is always available to help when necessary. This is also the primary methodology we expect to employ in the workshop presentation.

## Examples

All the exercises used in the two lab-based courses were originally developed by the presenter in the early 1990's and have been revised at least once per year since. The examples included herein are intended to be indicative of the types of exercises used in both classes, but will cover only topics suitable for both courses, that is to say, suitable for students preparing to teach middle school mathematics. The workshop participants will have the option of exploring topics from both courses, and in greater depth than presented here. The exercise outcomes are all obtained using the TI-86 calculator and/or MAPLE, but they are intended to focus on the interrelation between mathematics and mathematics-specific technologies in general, not just these particular platforms.

**Exact and Approximate Arithmetic**. Initially students are guided through a series of simple exercises in which they learn basic ways of communicating with the technology. For example, upon entering 2(3+5) into the calculator the output is 16; however, upon entering the execution command "**2(2+5);**" into MAPLE the output is 2. The students quickly realize that MAPLE does not interpret juxtaposition as multiplication and the correct MAPLE command should have been "**2*(2+5);**". After several experiments, with some directed prompting by the instructor, the students discover that MAPLE appears to interpret 2(whatever) as the constant function 2 acting on whatever.

Students observe that MAPLE defaults to exact arithmetic whereas the calculator is a numerical approximation tool. Students compare the results of various operations on very large and very small numbers using the calculator and MAPLE. For example students are asked to compare and comment on the results of computing 50! and 50!+1 using both. Then they are asked to compute $1/50!$ and $1/50! + 1$ and comment on the results. The students are also guided through several exercises involving greatest common divisors, gcd, and least common multiples, lcm, again seeing difficulties when trying to use the calculator to compute with large numbers. After segments on the Fundamental Theorem of

Arithmetic and lots of examples involving factoring integers, the students are asked to provide an argument for why gcd(m,n)lcm(m,n)=mn for all positive integers m and n.

In a subsequent lesson students are given the following instructions:

> Recall that for two positive integers m & n with n>m we say the remainder of n divided by m is r if n = qm + r, for integers q &r with $0 \leq r < m$. In MAPLE this remainder is denoted by "n mod m". In the TI-86 CATALOG it is denoted by "mod(n,m)".

Students work through a series of examples using both technologies to compute lots of remainders. They are asked to explain why an integer n is odd if and only if n mod 2 is 1, and explain why any power of an odd integer is also an odd integer. They are then asked to compute the following numbers using both the calculator and MAPLE:

$17^9 \bmod 2$, $17^{11} \bmod 2$, $17^{12} \bmod 2$.

Of course the MAPLE response is 1 in each case; however, the TI-86 responses are 1, 1, and 0 respectively. Students are asked to explain the calculator results. This is a bit more subtle than it looks at first glance given the following TI-86 results:

$17^9 = 118587876497$, $17^{11} = 3.42718963076E13$, $17^{12} = 5.8262223723E14$.

**Rational and Irrational Numbers**. The students are asked to use both the calculator and the CAS to compute $\sqrt{2}$. The calculator returns 1.41421356237, while MAPLE simply returns $\sqrt{2}$. The students are then asked to use MAPLE to get a decimal approximation to $\sqrt{2}$, resulting in 1.414213562. Next the students are asked to use the corresponding machine to convert the resulting decimal approximation of $\sqrt{2}$ into fraction form ($\sqrt{2}$ ">Frac" on the TI-86 and "convert($\sqrt{2}$ ,fraction);" with MAPLE). The calculator returns 1.414213562, whereas MAPLE returns $\frac{47321}{33461}$. The students are asked if this means the calculator "knows" that $\sqrt{2}$ is irrational and MAPLE "thinks" it is rational, and to explain their reasoning. It is suggested that they conduct an experiment comparing the results gotten from both machines when converting longer and longer decimal strings of 3's. Results from this experiment are included in Table 1.

TABLE 1
*Converting Decimals to Fractions with TI-86 and MAPLE*

| #3's | Decimal | TI-86 Conversion | MAPLE Conversion |
|---|---|---|---|
| 1 | 0.3 | $\frac{3}{10}$ | $\frac{3}{10}$ |
| 3 | 0.333 | $\frac{333}{1000}$ | $\frac{333}{1000}$ |
| 4 | 0.3333 | 0.3333 | $\frac{3333}{10000}$ |
| 9 | 0.333333333 | 0.333333333 | $\frac{333333333}{1000000000}$ |
| 10 | 0.3333333333 | 0.3333333333 | $\frac{1}{3}$ |
| 11 | 0.33333333333 | 0.33333333333 | $\frac{1}{3}$ |
| 12 | 0.333333333333 | $\frac{1}{3}$ | $\frac{1}{3}$ |

The results of an experiment changing the last digit in the strings of lengths 9 through 13 are included in Table 2. We see that MAPLE ignores all digits after the 9th while the TI-86 starts ignoring them after the 12th. The students are asked to provide a plausible explanation for what is going on. Of course the answer has to do with the degrees of accuracy. Here it should be mentioned that this data was obtained using the MAPLE's default degree of accuracy, Digits:=10, which could be set to any level. It should also be noted that the MAPLE conversion to fraction routine can be made exact, thus never yielding $\frac{1}{3}$ for any finite decimal string of 3's. The students are asked to explain why no finite decimal string of 3's can be equal to $\frac{1}{3}$.

TABLE 2
*Effect of Changing the Last Digit*

| Digit | Decimal | TI-86 Conversion | MAPLE Conversion |
|---|---|---|---|
| 9th | 0.333333336 | 0.333333336 | $\frac{41666667}{125000000}$ |
| 10th | 0.3333333336 | 0.3333333336 | $\frac{1}{3}$ |
| 12th | 0.333333333336 | 0.333333333336 | $\frac{1}{3}$ |
| $13^{th}$ | 0.3333333333336 | $\frac{1}{3}$ | $\frac{1}{3}$ |

**Roots and Graphs.** Students are guided through a series of exercises in which they use the calculator and the CAS to compute the roots of polynomials, first for degrees 1 and 2, then for higher degrees. They are also asked to plot graphs of the polynomials and asked questions like "How do the root of a polynomial relate to its graph?" Quickly they encounter issues involving exact versus approximate roots, and with representation of complex roots. For example when asked to solve for the roots of $x^2+x+1$ the calculator returns (0.5,0.8660...) and (0.5,-0.8660...), while MAPLE outputs $-\frac{1}{2}+\frac{1}{2}I\sqrt{3}$ and $-\frac{1}{2}-\frac{1}{2}I\sqrt{3}$. The students are asked to explain what is going on. It is suggested that they compute the roots using the quadratic formula, as well as look carefully at the graph of the polynomial.
Students are shown how to plot multiple graphs on the same axis and are asked to explore the effects of vertical shifts on the graphs and the roots of various polynomials. Figure 1 shows the effects of changing the parameter m on the graphs of the polynomials $p(x,m)=x^2-x+m$ and $q(x,m)=x^3-x+m$.

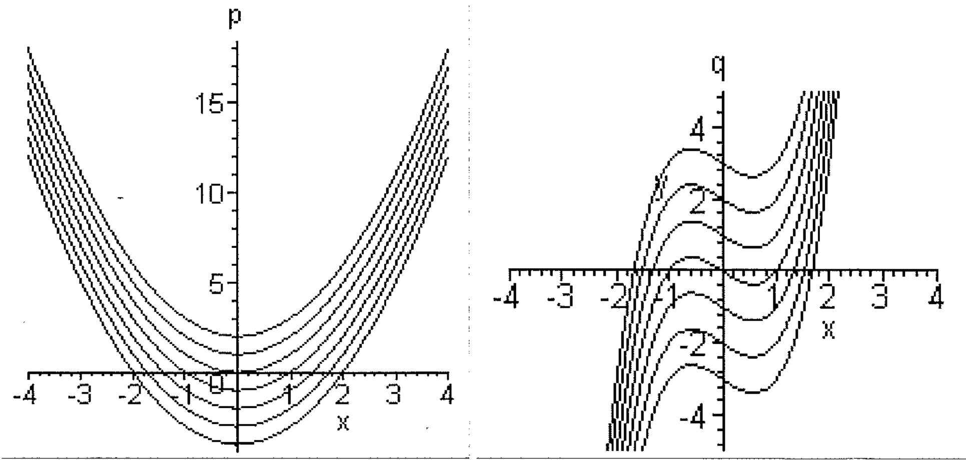


*FIGURE 1.* Graphs of p(x,m) and q(x,m) for m=-3 to 3.

They are asked to find a real number M so that the polynomial $p(x,M)$ has no real roots. They are then asked to do the same for the polynomial q. After several such exercises for polynomials of various degrees, the students are asked to explain why they can always find such M for a polynomial of even degree but never for one of odd degree.

Here the students are also introduced to the symbolic capabilities of the CAS. For example they are asked to see if MAPLE "knows" the quadratic formula. It is expected that the students will first try the MAPLE command "solve(a*x^2+b*x+c=0);" which returns the curious expression "{$c=-ax^2-bx, z=a, x=x, b=b$}." The students are prompted to try the command "solve(a*x^2+b*x+c=0,x);" which returns the usual form of the quadratic formula. The students are asked why they think ",x" needs to be inserted into the solve command, and then asked to see what happens if they ask MAPLE to solve for either a, b, c, or d.

The students are asked to determine if MAPLE "knows" the corresponding general formula for the roots a cubic polynomial. The answer is "yes" but we don't have nearly the space here to display the MAPLE output. The students are asked to speculate has to why high school students are required to memorize the quadratic formula but never asked to memorize its cubic counterpart.

MAPLE appears to use the cubic formula when asked to find the roots of the polynomial $f = 5x^3 - 5x + 1$, returning a lengthy list of expressions involving radicals. When the students use the MAPLE command "evalf(f);" to compute an approximation of these roots MAPLE returns 0.878...63-0.2 $10^{-9}I$, -1.088...15-0.259...12 $10^{-9}I$, 0.209...84+0.259...12 $10^{-9}I$.

The students previously observed that all odd degree polynomials must have at least one real root, and this appears to have all three roots imaginary. They are asked to explain what is going on. The answer involves the propagation of error in estimating the complicated radical expressions. Here they are shown the MAPLE command "**fsolve(f);**" returns the roots 0.878...62, -1.088...15, 0.209...84, which agree to 9 decimal places with the roots obtained from the TI-86. But then the students discover that "**fsolve(g);**" returns only one real root of the polynomial $g = x^5 + x^4 + x^3 + x^2 + x - 2$, which they know to have five roots. This motivates a discussion about real approximation techniques like Newton's Method and the existence of more sophisticated techniques for approximating complex zeros. Of course the students are encouraged to look at the graphs of both functions as given in Figure 2.

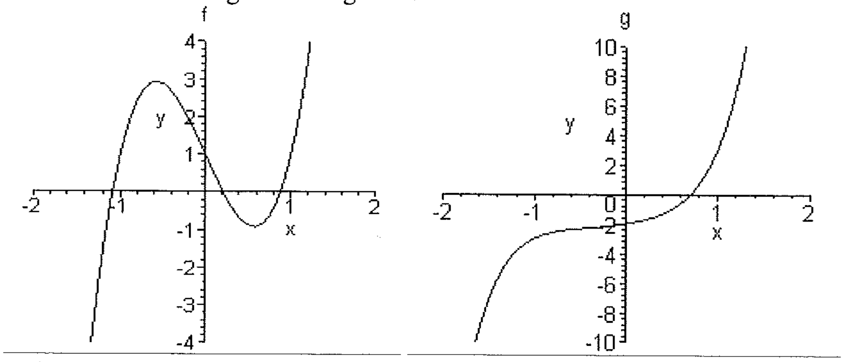


*FIGURE 2.*Graphs of f and g.

**Slopes and Areas.** At this point the course for the future middle school teachers begins to differ significantly from that for the future high school teachers. The workshop participants can opt to consider the more sophisticated and broader high school version. In both cases derivatives and integrals are introduced as measures of steepness and areas. Here we present the initial approach to area.

After being asked to compute the areas of given rectangles and assorted polygons the students are asked to find the area of the region in Figure 3. They are guided through the

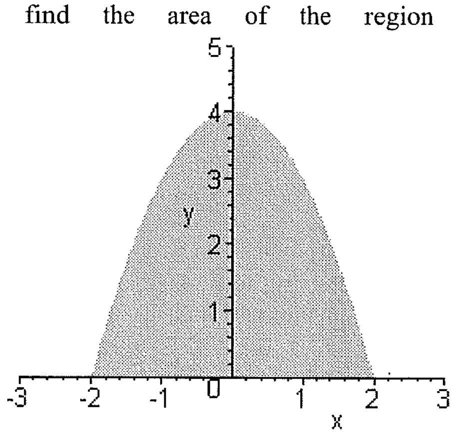


*FIGURE 3.* Region under the graph of some unknown function.

development of area as a limit of sums of areas of rectangles, with extensive use being made of MAPLE Student Package. The primary commands used from this package are "leftbox," "rightbox," "middlebox," "leftsum," "rightsum," and "middlesum." Figure 4 shows the MAPLE output using the command "middlebox(h,x=-2..2,8);" for the function $h = 4 - x^2$.

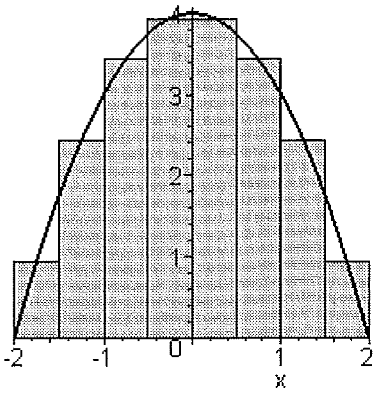


*FIGURE 4.* Region under the graph of h=4-$x^2$.

The MAPLE command **"middlesum(h,x=-2..2,8)=evalf(middlesum(h,x=-2..2,8);"** returns the MAPLE output: $\frac{1}{2}\left(\sum_{i=0}^{7}\left(4-\left(-\frac{7}{4}+\frac{i}{2}\right)^2\right)\right) = 10.750000000$.

The students try the same command sequence using more and more boxes, (using 200 boxes yields 10.66680000) and finish the exercise with the command

**"Int(h,x=-2..2)=int(h,x=-2..2);"** which returns $\int_{-2}^{2} 4 - x^2 dx = \frac{32}{3}$.

The course for middle school teachers continues with basic operations on matrices and finishes with a discussion of programming with LOGO. The course for future high school teachers goes more deeply into calculus, then covers vector spaces (eigenspaces, etc.) and finishes with topics from differential equations.

## Conclusion

We believe that mathematics teachers must be competent and confident users of mathematics-specific technologies and must be able to adjust quickly to evolving technologies. They must have an in-depth understanding of how these technologies relate to the fundamental mathematical concepts they will be teaching. Additionally they need some familiarity with topics from such areas as numerical and complex analysis not traditionally taught at the pre-college level, and often not in college level pre-service mathematics courses. In summary, we believe that providing the future teachers of mathematics with these traits is the primary challenge facing mathematics education at the beginning of the 21st century.

## References

1. G. A. Harris, *The Use of Computer Algebra Systems and Commercial Web Technology in the Undergraduate Mathematics Preparation of School Teachers,* Quaestiones Matheamticae, **Supplement number 1** (2001), 217-226.
2. G. A. Harris, *The Use of Computer Algebra system in Capstone Mathematics Courses for Undergraduate Mathematics Majors,* The International Journal of computer Algebra in Mathematics Education, **7(1)** (2000), 33-62.
3. Mathematical Association of America, *Assessing Calculus Reform Efforts*, MAA, 1994.
4. Mathematical Association of America, *Calculus the Dynamics of Change,* MAA, 1996.
5. National Council of Teachers of Mathematics, *Professional Standards for Teaching Mathematics,* NCTM, 1991.
6. National Research Council, Mathematical Sciences Education Board, *The preparation of Teachers of Mathematics: Considerations and challenges,* NRC, 1996.

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# MATHEMATICS AS AN APPLICATION TOOL – STUDYING MATHEMATICS AND STATISTICS IN CONTEXT

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This is a report into an action research study in a Bridging Mathematics course in which students learn to use mathematics and statistics as a tool. The students begin with inadequate mathematical and statistical understandings to successfully complete their other mainstream degree papers, which they are studying concurrently. The aim is to empower the students to be able to use and apply their mathematical knowledge in other subjects and circumstances. To succeed in this they need the ability to connect and link mathematical ideas and topics, and to build up a coherent mathematical structure, which makes sense to them. They also need the confidence to use mathematics and the ability to recognise appropriate mathematical or statistical tools to use. Methods used involve questioning them so as to assess their level of mathematical understanding and then, with them, together build on the foundation of what they already know to a position where they can understand and succeed at the course. Frequently the first task is to open doors for students by removing fear of mathematics. Often students with similar needs and starting points can be grouped. The collaborative discussion possible when working together in groups on a problem has frequently been found to be a source of permanent learning as well as positive feelings of success. The tutor's input is that of a facilitator, probing and questioning when the group are stuck or digressing from the problem; challenging misconceptions and thinking; and guiding direction with links to other knowledge of the student. The course is taught mainly through contextual problems and has been successful for a large number of students.

## Introduction

*Tertiary Numerical literacy*

The need for adult numerical literacy is increasingly being advocated especially in the US and Canada. (1, 2) The International Literacy Study (IALS, 1995) included quantitative literacy and was the first large-scale (19 countries) comparative assessment of adult literacy. The International Adult Literacy Survey (IALS) in 2002 includes a survey on the numeracy abilities of adults with a focus on numerate behaviour, defined as the way a person responds to mathematical situations. In England the Department for Education and Employment has introduced the first compulsory Adult Numeracy Core Curriculum with stated aims to develop learners ability to formulate problems as well as to solve them and communicate the result. It is even more important that tertiary graduates should be numerically literate.

Numerical literacy has come to be seen as a fundamental generic skill as much as written and oral literacy. In the modern technological age almost everyone let alone a University graduate requires some understanding of mathematical processes especially the use of calculators and computer software programmes. Statistics are in common use in the media, advertising, academic papers and reports. A graduate needs an understanding of the use and misuse of statistics and the ability to interpret the same. The ability to use mathematical concepts to aid in the solution of numerical problems, to connect and link mathematical ideas and topics, and to build up a coherent mathematical structure which makes sense, gives a student the power to use and apply their mathematical knowledge in other subjects and circumstances. When reading University texts, frequently tables, graphs, mathematical symbols and numerical formulas may be included. Sometimes, as well, the mathematical or statistical ideas are expressed in words such as inflation, exponential growth, rate or proportion. Mathematical literacy is assumed when reading many scientific or financial texts. To fully understand the text the information contained in these quantitative representations need to be considered and understood, and inferences and conclusions need to be drawn from them. In some cases the use of numerical or statistical tools can advance a reasoned argument or help in the solution of a problem. Therefore confidence in using mathematics and the ability to recognise appropriate mathematical or statistical tools and knowledge of how to use them is essential for most graduates.

*The affective domain*

Learning Mathematics appears to be closely linked to the affective domain (3). Student's confidence and feelings about mathematics seem to be crucial to their success (4). Often past lack of success has made it harder for them to learn from the traditional instruction through worked examples. Research has shown that student's attitudes, beliefs and confidence influence their ultimate success or failure at learning mathematics.( 5,6,7).

One of the problems for mathematical literacy is the inability of many adults to transfer or apply their textbook learning of mathematics. They can frequently only complete problems identical to ones they have done before. When they meet a problem where mathematical tools would be useful, they often ignore these tools and instead use less powerful methods. There appears to be a chasm between 'textbook mathematics' useful for tests and exams, and 'real mathematics' that we actually use in the real world. A student may leave school or University successful at mathematics but still unable to 'use' mathematics. In that case the mathematics learned is useless for that person and like any unused appendage, it is likely to wither and die.

Why does this problem occur? Perhaps it is a lack of understanding of the linkages of the subject and an inability to make sense of the material that they learn frequently by rote.

## Action Research

I used action research to assess the usefulness of some strategies to help tertiary students completing a bridging mathematics course improve their mathematical competencies. The strategies were designed to scaffold learning and were: stressing mathematical applications, making links to the students' prior mathematics learning, and using questioning as a means of guiding students through the process of problem-solving.

One problem in teaching is that the construction of knowledge takes place in the head and is invisible to others including the teacher/tutor. Though learning is invisible, evidence of its existence can be seen and can be revealed in oral or written communication. Like the wind, which cannot be seen, the effects it has on the learner and the learning environment can gauge the evidence of its presence and its strength.

Socio-constructivism considers that learning takes place through language and culture in learning interactions. In constructing new knowledge the student moves from one zone of proximal development to another (8). The teacher has a role in scaffolding the movement between zones. Wood, Bruner and Ross (9) coined the term *scaffolding* to describe the "process that enables a child or novice to solve a problem, carry out a task or achieve a goal that would be beyond his unassisted efforts" (p. 90). The teacher/tutor and student together build a mental scaffold that assists the student in achieving the task. Gradually the scaffold becomes redundant, as the learner becomes increasingly an independent learner. Brown calls this process fading (10) but Mason prefers directed–prompted–spontaneous (11). Language is vital in assimilation of a concept. Vygotsky says that concept attainment is complete only when a student can frame the concept in appropriate words (12).

## Background

This research was undertaken while teaching a bridging mathematics course and tutoring mainly adult students who needed extra support in mathematics for other courses. The bridging course is compulsory for many of the mainly adult students without University entrance level mathematics who needed mathematics to handle the mathematical components of some of their other courses. The course has been running since 1990 but there were concerns by 1997 about the low pass rate (25% to 40%) and high drop out rate (44–59%) in the course and that the course was failing to supply what many students needed. Those who dropped out early appeared to be the very ones for whom the course was designed. There were doubts as to the long-term value that even those who succeeded in passing the course gained from it. The course was self-taught using a programmed textbook, with workshops being an opportunity for students to sit tests and ask for help from tutors when needed. Many rarely attended except to sit tests and a large number were not seen after the first two or three weeks. The tests were 80% numerical calculations.

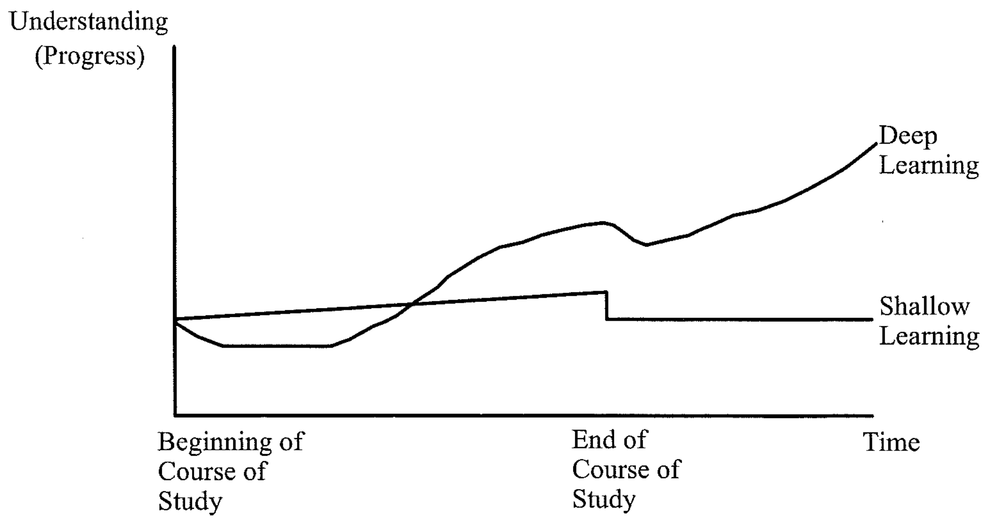
## 1998 trial

A trial was undertaken in 1998 when 133 students took the course. It was decided to have compulsory lecture/workshops during which topics were introduced through problems in context, where methods to solve the new problem were linked to earlier topics or assumed knowledge. It was explicitly stated that solving the problem was the focus of the topic, not developing isolated procedures. Questions were used to find a starting point of well-understood mathematical concepts, and these were built on to extend thinking. Counter-examples were provided that disturb the misconceptions that some students have and can lead to new ways of thinking about the topic. More than one way was often used to approach a particular problem because adult learners have often already developed successful methods. Linking with the previous ideas of the group and encouraging flexible thinking was seen as important for learning.

The problems were completed on the over-head projector by the researcher, who was also the Lecturer/Convener for the course, but with control of what was written given to the students. Questions were used as cues or prompts to scaffold the learning of problem solving, rather than telling or explaining. Questions asked in probing were often higher-level questions directing the student as to the ultimate purpose of the task or links with other topics. However these were often followed by lower level questions giving clues to the process needed.

After completing one or two problems as a large group, the students were given problems usually framed in words and in a context to work on either in groups, which was actively encouraged, or on their own. Tutors were available at all times to assist by questioning rather than telling and to encourage justifying their methods and answers to the rest of the group. The assessment tests were changed to include over 50% of contextual problems to ensure that the thinking skills needed to accomplish these were seen as valued. The aims for the course, which were made explicit on many occasions, were that they learnt not just how to do mathematical manipulations, but how to use and apply them.

They were taught in workshops of 20 to 40 students depending on their other timetabled courses. In addition there were opportunities for small group tutorials taken by the researcher or other tutors. Early tests were designed to ensure that students achieved success so that they could develop confidence and continue with the course. I have developed a diagram to illustrate the difference between traditional individualised instruction and a discussion-dominated problem solving method.




Individualised learning style compared to learning through everyday problems

The text book/worksheet method usually used for individualised learning creates some progress in getting correct answers to carefully programmed questions but the result is often only shallow learning, much of which is lost soon after completion of the course. This method tends to "discourage intuitive approaches and promote a mathematics that comes from an outside authority rather than a personal mathematics that can be applied in many situations" (13). The learning has not been assimilated into the student's previous schema and therefore is not retained in long-term memory. It is rarely transferable to problems encountered elsewhere. The problem solving method by contrast leads at first to some frustration and negativity as previous misconceptions are deliberately targeted to cause cognitive conflict. Progress seems to be small or even going backwards. However in the long term the students who persevere are more likely to develop a deeper understanding, some of which will be transferable to other studies.

## Results

- Better retention rate.
- Only 25% dropped out of the course compared to 44–59% in the previous years.
- Higher success rate.
- 65% passed the course compared to 25% to 40% in previous years.
- Positive feedback especially from those re-doing the course.
- Improved confidence, attitudes to mathematics and perceived understanding.

## 1999 Experience

In 1999 the results of the trial were used to judge the effectiveness of the changes. It was decided that the improvements were such that the new mode of delivery should be continued. Some slight changes were made, the main one being the decision to give a pre-test at the beginning of the course. On the results of these some students would be placed in a fast track stream. Some would be advised to take the full year version to give them extra time to successfully complete the course. 153 students enrolled in 1999.

*Results*

Similar results were found to those of 1998.

- Retention rate improved still further with only 11% not completing the course.
- The success rate also improved further going up to 78%.
- Feedback was again mainly positive.
- Confidence levels improved between the start of the course and the end, as did also feelings about mathematics and perceived levels of understanding.

## 2000 -2003

Each year the results of the previous year were used to refine the course. In 2000 for example the number of assessments was reduced from 10 to 5 and a new module was added on calculus. The results over the next few years have been similar to that of 1998.

## Discussion

From my research I suggest that there are 5 stages in the thinking/learning process and that each is characterised by a change in language.

1. The "can't do" stage where there is little response to questions asked. Answers are usually "don't know" or a shrug or a random guess.
2. The "taking risks" stage begins to develop as the tutor works with the student and the student increases in confidence. The tutor gains his or her trust and the student loses the fear of giving a wrong answer.
3. In the "making connections" stage students can remember related problems and methods and the answers given begin to show signs of learning. They relate what they are considering to other things learnt in earlier sessions. At first they only do this as a result of the tutor prompting them with a question but gradually they begin to do this naturally as the tutor fades the scaffolding (10)

4. In the "reflective" stage the students begin to question themselves as they begin to internalise the thinking process. They answer the metacognitive question (14) posed by John Mason as "what question am I going to ask you?" They need only occasional probes or prompts to initiate questioning of themselves. Interestingly they often cannot see the progress they have made. As the self-questioning becomes natural to them, they cannot remember a time when it was not and frequently do not see the part it plays in their success at solving problems.
5. Finally in the "independent learning" stage the students are able to use an internal monitor (15) to ask themselves questions, leading them to ways to tackle new problems. At this stage they sometimes start to extend a problem into new creative areas by asking their own questions on the topic or extending the data given.

*Interview results*

The interviews produced rich data about the student's previous experiences with mathematics, their feelings about maths and their growing knowledge of maths. Some themes emerged in the data that may partially explain the success or otherwise of the students in learning mathematics. Some of these are:-

- Negative feelings about maths and ability to learn maths.
- Past feelings of failure due to their experiences at school.
- Beliefs about the irrelevance of mathematics to their real world experience, while at the same time paradoxically believing that maths is important.
- Enjoyment of group work and finding it valuable to their learning
- Realising the gains that they achieve from helping others, as well as the gains they get from others helping them.
- Seeing the relevance of the maths they are learning when it is in context but at the same time sometimes finding those problems more difficult.
- Being appreciative of starting back with arithmetic and then building on to that to develop the understanding of algebra.

*Differences between successful and unsuccessful students*

The difference between the two groups (successful or unsuccessful) does not appear to be a difference in mathematical prior knowledge. Many of the students who started out in the faster group failed or dropped out of the course. Those who started with a very weak grasps of topics such as decimals and basic arithmetic and no idea of algebra, as often gained success as those who started able to cope with these topics fairly competently.

It does not appear to be caused predominantly by cultural differences either as in this research group there appeared to be an equal number of Maori, pacific islanders and Asian in the successful and unsuccessful groups. The European/pakeha group however had a larger proportion of their numbers in the successful group, which does imply some cultural bias to their advantage. The language problem may affect the international students more, especially where English is their second language but English speakers often also have difficulty in interpreting problems couched in language.

*Confidence and success*

There appears to be a positive relationship between initial confidence and initial level of competence as judged by the pre-test, but this is not a strong relationship (correlation coefficient of 0.308). It is weaker when comparing initial confidence with the final result (correlation coefficient of 0.151). From the interview data and the questionnaires it appears that the course has, for some students particularly older women, improved their confidence so that they do much better than their pre-test results would have predicted. Beliefs that maths is beyond them have been removed and they now succeed. At the same time some (often males) seem to have allowed over-confidence to have dictated a pattern of a minimum of participation in the course which ultimately has resulted in poorer results than the pre-test had predicted. The relationship is stronger between pre-test and final results (correlation coefficient of 0.526) that is not surprising as one expects better prepared students to make more progress than those with major gaps in their prior knowledge.

The ten most initially confident students were five males and five females, ranging in age from 18 - 39 with a mean age of 26 years. All except one had completed up to at least Sixth Form mathematics. The only one who'd not gone beyond 5th form got 21% in the pre-test. The pre-test results of the rest ranged from 32 - 77% with a mean of 54%. One student dropped out of the course and two failed. The other seven all passed with five gaining marks between 72% and 86%, one getting 97% and one 59%. The most confident student of all, giving 26 confident answers to the 26 questions failed the course, both on internal assessment and examination. It appears that over confidence, especially when misplaced, can be damaging to chances of success. Confidence however helped two students who scored only 32% on the pre-test but gained 79 and 81% on the final results.

The ten least confident of success (again five of each sex) with only between two and four confident statements out of a possible 26 each, had a wide range of ages from 17 to 48 with a mean age of 27.5 years. Four had not gone beyond fourth form mathematics, four had stopped at fifth form and two had done sixth form certificate. Eight had pre-test marks between 13 and 43%, and one had 58%. The least confident of all (a 48 year old who had not gone beyond fourth form) did not sit the pre-test. Their final results saw five passing, four of them with marks between 63 and 71%. Three failed and two did not complete the course). Generally these were the ones with the poorest pre-test scores but here again there were anomalies with one who gained only 16% on the pre-test having a final pass mark of 65%. One who failed the course had a pre-test score of 43%. It appears that previous failure at mathematics has an affect especially on confidence but, if this can be overcome, students can be very successful and appreciative of their success.

## Summary

Learning is a complex process and there is no easy answer as to why some students are successful and some not. The changes made to the course seemed to improve the student's confidence, feelings about mathematics and understanding, and increased the pass rate for the course. Fewer students dropped out of the course. Some of this may have been a consequence of the change in assessments and the focus on applications; some may be related to the teaching style. The analysis of the questionnaires and interview data indicates that much of the improvements were related to the teaching strategy and the use of questioning rather than showing/telling.

## References.

1. Curry,D, Schmitt,Mj,& Waldron,S. (1996). A Framework For Adult Numeracy Standards: The Mathematical Skills And Abilities Adults Need To Be Equipped For The Future. Boston, Ma: World Education.
2. Gal, I., Van Groenestijn, M., Manly, M., Schmitt, M.J., & Tout, D. (1999). Adult Literacy And Lifeskills Survey Numeracy Framework Working Draft. Ottawa: Statistics Canada
3. Tobias, S (1978) Overcoming Math Anxiety. New York. W.W.Norton & Co.
4. Colwell, D. (1998) The Role Of Feelings And Logic And Their Interaction In The Solution Of Everyday Problems. In M. Van Groenstijn & D. Cohen (Eds), Mathematics As Part Of Lifelong Learning (Proceedings Of The Adults Learning Mathematics Conference. Utrecht, Netherlands. ALM
5. Garafolo.Joe, And Frank K Lester. "Metacognition, Cognitive Monitoring, And Mathematical Performance." Journal For Research In Mathematics Education 16 (May 1985): 163-176
6. Schoenfeld, Alan H. Mathematical Problem Solving. Orlando, Fla: Academic Press, 1985.
7. Mcleod, Douglas B, "Affective Issues In Mathematical Problem Solving: Some Theoretical Considerations." Journal For Research In Mathematics Education 19 (March 1988): 134-141.
8. Vygotsky, L. (1978). Mind In Society: The Development Of Higher Psychological Processes.Cambridge, Ma, Harvard University Press.
9. Wood, P., Bruner, J., & Ross, G. (1976), The Role Of Tutoring In Problem Solving, Journal Of Child Psychology And Psychiatry, 17, P89-100.
10. Brown, S., Collins A., & Duguid P. (1989), Situated Cognition And The Culture Of Learning, Educational Researcher,18, (1), 32-4
11. Mason, J. (1998), Asking Mathematical Questions Mathematically. In Teaching And Learning Of Mathematics At University Level (Proceedings Of The International Commission On Mathematical Instruction Study Conference). Singapore.
12. Vygotsky, L. (1986). Thought And Language. Cambridge, Ma, Mit Press.
13. Tout, D. & Schmitt, M.J. (2002) The Inclusion Of Numeracy In Adult Basic Education. In J. Comings, B. Garner, & C. Smith (Eds), Annual Review Of Adult Learning And Literacy (Vol 3, P. 162). San Francisco: Jossey-Bass
14. Lester, F. (1982), Mathematical Problem Solving: Issues In Research, Franklin Institute, Philadelphia
15. Mason, J. & Burton L. & Stacey K. (1982), Thinking Mathematically, Addison Wesley, London

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# CONCEPTESTS: ACTIVE LEARNING IN CALCULUS

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ConcepTests—a powerful tool for improving student learning—were originally developed by Eric Mazur at Harvard to teach introductory physics. They were subsequently adopted in chemistry and biology. ConcepTests are now available for calculus, where they have shown the same impressive results. The pedagogy underlying Conceptests is active learning and peer-instruction. These have proved effective in many contexts, particularly for non-traditional students who do not learn as well in a passive lecture format.

Over the past two decades, many faculty have found ways to improve student learning by making better, more active, use of class time. One of the most effective methods—the use of ConcepTests and peer instruction---was developed by Eric Mazur of Harvard. His pioneering work in physics has been successfully replicated in other departments, and its efficacy is clearly supported by data. Information is available under "Peer Instruction" at http://galileo.harvard.edu.

ConcepTests have now been written for calculus. [1] This is the first time the method has been made widely available for mathematics. Evaluation suggests that ConcepTests are as effective in calculus as they are in physics.

## What are ConcepTests and how are they used?

ConcepTests are questions designed to promote the discussion and learning of mathematical concepts during a calculus class. The questions are usually conceptual, often multiple choice or true/false, with some free response questions as well. They are used as an aid in promoting student involvement in discussing mathematical concepts rather than as a method of testing students for a grade. (Some questions have more than one correct answer.)

Because of the variety of forms of these questions, instructors can use them in several different ways. Three of the possibilities are:

- As an introduction to a topic. This works especially well if the topic is closely tied to previous lessons or is something with which most students have some familiarity.
- After presentation of a specific topic by the teacher. Here the ConcepTest may be used to see if the students have grasped the concept, or if the topic needs more discussion or examples.
- As a review of material that has been thoroughly discussed.

Instructors usually display the ConcepTest using an overhead projector, or distribute a copy to each student. Students are then given short time (one to four minutes, depending on the question) to think about the question and then vote for the answer they think is correct. Providing almost all do not vote for the correct answer, the students are then given a few minutes to discuss the ConcepTest with adjacent students and then are given another chance to vote on the correct answer. Students are also asked to develop reasons to support their answer. The instructor then discusses the correct answer (or answers) and has students present their reasons.

## Why do ConcepTests Work?

Calculus instructors often strive to involve students in classroom discussions. However, it is not uncommon for their efforts to be met by silence as students wait for the instructor to answer his or her own questions. ConcepTests work because they remove the barriers to student response. At the outset, instructor and students both know that there will be several minutes to grapple with the question, so there is no pressure on the instructor to answer too quickly. The responses are essentially anonymous, so students can comfortably put forth tentative ideas. If the students are lost or unsure, they have the opportunity to talk things over with their classmates and to revise their first impressions. The give-and-take helps both weak and strong students and, even if it does not generate a correct answer, often generates excellent questions from the students to the teacher.

Regular use of ConcepTests ensures that students first grapple with difficult mathematical ideas in class, where help is available, rather than on homework, when help is not available. Having wrestled with ideas in class, students are much more likely to be able to tackle problems actively on their own.

## What Kinds of Classes Benefit from ConcepTests?

While originally developed for use in a large classroom, we find ConcepTests equally effective in small or moderate sized classes, where they promote "active'" or "discovery" student learning. The ensuing discussion greatly increases the students' familiarity with the subject and helps them formulate mathematical ideas in their own words. This increases the students' base of knowledge and enthusiasm for mathematics.

## Evaluation of ConcepTests

Scott Pilzer describes [2] Eric Mazur's development of ConcepTests and Mazur's evaluation data. Scott Pilzer, who taught with ConcepTests in physics before writing them for calculus, gathered similar evaluation data for mathematics. He found that at the start of the subsequent semester, students taught Calculus I using ConcepTests and peer instruction performed much better on conceptual questions and somewhat better on standard computational problems than those who had been taught standard lectures. See Table 1.

TABLE 1
*Comparison*

| | Conceptual Questions | Computational Problems |
|---|---|---|
| With ConcepTests | 73% | 63% |
| Standard Lecture | 17% | 54% |

From Scott Pilzer.

These dramatic results mirror the results found in physics: a large increase in conceptual understanding in addition to an increase in standard computational problem solving ability. In his article, Scott Pilzer describes how this enthusiasm translated into more mathematics majors.

We hope that many departments consider teaching with ConcepTests and taking part in developing expertise with a pedagogy that promises to be as productive for mathematics as it has been for physics.

## References

1. S. Pilzer, M. Robinson, D. Lomen, D. Flath, D. Hughes Hallett, B. Lahme, J. Morris, W. McCallum, J. Thrash, *ConcepTests,* John Wiley, 2003.
2. Scott Pilzer, *Peer Instruction in Physics and Mathematics,* Primus, **XI(2)** (June 2001), 185-192.

## Appendix A: Examples of ConcepTests

1. Using the graph in Figure 1, arrange in ascending order:

   a) Slope where $x = 0.2$

   b) Slope where $x = 1.5$

   c) Slope where $x = 1.9$

   d) Slope of line connecting the points where $x = 1.5$ and $x = 1.9$

   e) The number 1

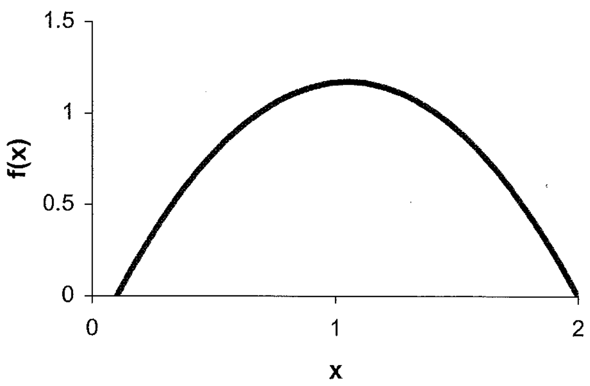


FIGURE 1.*Graph of function*

2. Let $N = f(t)$ be the total number of cans that Sean has consumed by age $t$ in years. Interpret the following in practical terms, paying attention to units.

   a) $f(14) = 400$

   b) $f^{-1}(50) = 6$

   c) $f'(12) = 50$

   d) $(f^{-1})'(450) = 1/70$


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# AN AFFORDABLE, REALISTIC STUDENT MODEL OF A MOTOR-VEHICLE SUSPENSION SYSTEM

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**Abstract**

A realistic model of the suspension system of a motor vehicle should incorporate the tyre as part of the system. Indeed, the tyre can be regarded as a "damped spring" forced by the road and coupled to the damped suspension spring. For both of the "damped springs" involved here, it is easy to determine the spring constant $k$ by means of an experiment. It is also easy to determine the coefficient of damping $\phi$ in the suspension spring system. Indeed, take a shock absorber and pull it out at a constant speed $v$ and measure the force $F_d$ needed to do this. Because $F_d = \phi v$, one can solve for $\phi$. However, in order to measure the damping force of the "tyre spring", much more needs to be done. The solution to this problem turns out to be non-trivial, but well within the grasp of undergraduate students who have studied a first course in ODEs and have access to a computer algebra system (CAS). Gathering the data on the coefficients involved requires the following apparatus: a ruler, a vernier calliper, a motor vehicle with its jack, a bathroom scale and a stopwatch. Consequently, the data-gathering part of the experiment is almost cost free, a factor that many cash-strapped schools will appreciate!

## 1 Introduction

We describe below a simple, low-cost, realistic student experiment to model and simulate the vibrations felt by a motor vehicle as it enters a series of road corrugations. Before the advent of computer algebra systems (CAS), such a simulation would almost certainly have been beyond the capability of most undergraduate students at technikons (universities of technology) in South Africa (as well as the author), merely because we would have run out of stamina writing down the computations involved. In this article we do not merely outline the mathematical ideas needed to derive the model, but also illustrate how we measure the coefficients involved using low-cost apparatus and simple experiments as well as how we use a CAS to effortlessly do the "book-keeping" necessary to write down the solutions to the model. To the best of the author's knowledge this non-homogeneous model with sinusoidal forcing, as well as methods for finding the coefficients involved does not appear in any undergraduate ODE textbook. However, we emphasise that this is by no means a new model nor the true model (see the specialist text [5]). It is merely a first attempt at an application of coupled IVPs to approximate what happens in a real situation. The technology students involved are not mathematics majors and have two semesters of calculus to their credit. They are typically a mixed bag of analytical chemists, chemical engineers, metallurgical engineers and computer systems engineers. Towards the end of a semester class on ODEs, after the bouncing soccer ball model discussed in [?] has been mastered, this model is introduced. This is not window dressing - it is part of the mission of a technikon to teach appropriate mathematics, using technology. Indeed, my students **expect** to be able to apply what they learn to actual situations even if the results are "ball-park" quantities. The author has postgraduate students working on more complex models with an eye to designing suspension systems to prevent "dirt road corrugations" (see [3]).

The author worked on a 1948 Willys Jeep with a leaf spring system, liquid-filled shock-absorbers and 265/75 R15L radial tubeless tyres pumped to a pressure of 150 kPa. For simplicity, we take the symbol $M$ to represent a quarter of the licensing mass of the Jeep (ie we will use a "quarter-vehicle model" [5]). On the other hand, $m$ represents the mass of a wheel of the Jeep (and for simplicity we assume half of the axel mass to be small compared to $m$). We found

$$M = 300\,\text{kg} \quad \& \quad m = 27\,\text{kg}$$

Figure 1 (see [5] page 256) is a schematic representation of a motor-vehicle suspension system which includes the tyre as part of the system.

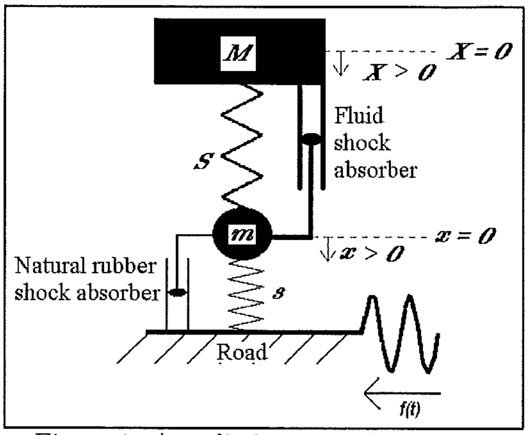

Figure 1: A realistic suspension system

$S$ represents the compressed spring attaching the wheel to the vehicle (the suspension spring), while $s$ represents the spring-like property of the compressed tyre, attaching the wheel to the road. If the vehicle is at rest, $X$ represents vertical displacement from rest of that part of the body of the vehicle directly above the wheel, while $x$ represents vertical displacement from rest of the centre of the wheel. We assume that the vehicle is travelling along a level, smooth road at a constant velocity $u$ and that there are no vertical velocities or displacements until the vehicle hits a series of corrugations, which exerts a time-dependent force $f(t)$ on the wheel.

## 2 Setting up the model

If we ignore damping in both the "tyre spring $s$" as well as the "suspension spring $S$" and remove external forcing from the system, we can set up the equations for coupled springs found in [4] Section 7.6. This system can be converted to a fourth-order ODE with constant coefficients that has a characteristic equation which can be reduced to a quadratic form. Hence the solution is easy to obtain by hand in terms of the coefficients and initial conditions, without using a CAS (see Fay and Graham [1] section 2.1). Use of this model is necessary to give students practice in mathematical manipulation, and they should attempt the exercise to gain insight into what we describe below. However, such practice and insight should not be considered sufficient knowledge, the reason being that this model is far from what actually takes place on a road. Taking the powerful computing tools available today, if we, as mathematics educators, intend to present applicable mathematics examples to our students, then we have failed hopelessly in our task if we stop here. The situation described by Figure 1 is modelled by the following coupled IVPs:

$$f(t) - kx - \phi\dot{x} + K(X - x) = m\ddot{x} \tag{1}$$

$$-\Phi\dot{X} - K(X - x) = M\ddot{X} \tag{2}$$

$$x(0) = X(0) = 0 \text{ and } \dot{x}(0) = \dot{X}(0) = 0 \tag{3}$$

Here $k$ and $K$ are the spring constants of $s$ and $S$ respectively; $\phi$ and $\Phi$ are the respective viscous damping constants and $f(t)$ is some impressed force from the road. This model is well within the grasp of undergraduate students who have studied a first course in ODEs. However there is a price to pay, as Fay and Graham point out, in [1] section 2.3. Introducing damping and forcing is a far more complex problem and this is where a CAS comes in handy, as we shall presently see.

## 3 Gathering data

We can obtain rough experimental values for $k$ and $K$ from Hooke's law as follows:

- Jack the vehicle up under a hub so that the spring extends to its maximum and the wheel is off the floor.
- Place an adjustable stand under the chassis and lower the jack so that the chassis is supported completely by the stand and the wheel is still off the floor (this is a safety precaution as well as a necessary step).
- Remove the jack and the wheel in question.
- Place the jack under the hub in question.
- Now, using a metre ruler, measure the distance $D$ through which the spring is compressed when the jack is raised to the point where the chassis starts to lift off the stand.
- Re-attach the wheel, remove the stand and lower the jack so that the wheel just touches the floor.
- Now, using a vernier caliper, measure the distance $d$ through which the tyre is compressed when the jack is lowered so that the wheel takes the full load of the vehicle.

We found

$$D = 0.075\,\mathrm{m} \quad \& \quad d = 0.0191\,\mathrm{m}$$

Taking $g = 9.8\,\mathrm{m\,s^{-2}}$, according to Hooke's law,

$$K = \frac{Mg}{D} \approx 3.92 \times 10^4\,\mathrm{N\,.\,m^{-1}} \quad \& \quad k = \frac{(M+m)g}{d} \approx 1.68 \times 10^5\,\mathrm{N\,.\,m^{-1}}$$

By clamping a shock absorber upright in a vice, the author stood on a bathroom scale and pulled the shock absorber upwards and out at a steady speed of $2.25\,\mathrm{cm\,.\,s^{-1}}$. It was observed that the author had an apparent increase in mass of 44 *kg*. Consequently, the force $F_\Phi$ exerted to pull the shock out at $v = 0.0225\,\mathrm{m\,.\,s^{-1}}$ was thus $F_\Phi = 44 \times 9.8\,\mathrm{N}$. Because we assume that the damping force is viscous, that is $F_\Phi = \Phi v$, we determine that

$$\Phi = \frac{F_\Phi}{v} \approx 1.92 \times 10^4\ \mathrm{kg\,.\,s^{-1}}$$

If we let $\omega = \sqrt{\frac{K}{M}}$ and $2\lambda = \frac{\Phi}{M}$, then we observe (as is the case with good shock absorbers) that $\lambda^2 > \omega^2$ and the suspension spring is **overdamped** (see [4]).

The impressed force from the road $f(t)$ can be modelled along the lines of [6] and we do this in section 5.1. The only unknown quantity is $\phi$, the coefficient of (viscous) damping for the tyre.

## 4 Coefficient of damping for the tyre

Obviously we cannot measure $\phi$ directly because we have no "shock absorber" on which to pull. How does one set about determining the coefficient of damping $\phi$ by means of some simple experiment? In order to deduce what to do, remove a wheel from the motor vehicle. Now let the wheel drop through a height

$$h_1 = 1\,\mathrm{m}$$

above the road. Just as it strikes the road ($t = 0$) it has a downward velocity of, say $v_1$. A short time later (say $t = T$), the wheel rebounds off the road with an upward velocity of, say $v_2$. Because the velocities are not large, we can ignore air friction. However, because the inflated tyre is not perfectly elastic, some energy is lost on the rebound ($|v_2| < |v_1|$) and so the maximum rebound height will be, say $h_2$ with $h_2 < h_1$. We measured the rebound height

$$h_2 = 0.83\,\mathrm{m}$$

Consider that during the (small) time interval $T$ between the inflated tyre striking the road and leaving it on the rebound, the behaviour of the tyre can be modelled by the damped spring IVP:

$$m\ddot{x} + \phi\dot{x} + kx = 0, \quad 0 \le t \le T \qquad (4)$$
$$x(0) = 0, \quad \dot{x}(0) = v_1$$

As shown in [?], we must have **underdamped motion** ($\lambda^2 = \frac{\phi^2}{4m^2} < \frac{k}{m} = \omega^2$, see [4]). Using this information to solve this IVP together with the observation that

$$x(T) = 0 \quad \& \quad \dot{x}(T) = v_2$$

results in the fact that we can determine $\phi$ as:

$$\phi = \frac{-2\sqrt{mk}\ln R}{\sqrt{\pi^2 + \ln^2 R}} \approx 126\,\mathrm{kg\,.\,s^{-1}}$$

where the ratio $R$ with

$$R = \left|\frac{v_2}{v_1}\right| = \sqrt{\frac{v_2^2}{v_1^2}} = \sqrt{\frac{h_2}{h_1}} \approx 0.911$$

is well known in physics and is called the **coefficient of restitution** of the tyre.

## 5 Solving the coupled system

We have now determined all of the coefficients of the coupled spring problem of section 2. A model for the forcing function is next on the agenda.

### 5.1 Corrugations

Most corrugations (washboards) in the road, have a periodic profile which is practically never sinusoidal. We should be able to approximate this profile using a truncated Fourier sine series. Because the model is linear, we can take each term of the truncated series as the forcing function and, proceeding as described below, obtain the individual solutions and sum them, producing the solution to our model. However, in order to obtain some insight into what happens when a vehicle enters a series of washboards, **as a first approximation,** we assume that the corrugations have a sinusoidal profile. As in [6], we model the impressed forcing function $f(t)$, as follows. Say the corrugation profile is $G(t) = \mathbb{A}\sin\sigma t$; $0 \le t \le \frac{L}{u}$ where $L$ is the length of the corrugated section of the road and $u$ is the velocity of the vehicle. We assume that $L$ is large and consequently the time interval over which we are working is large. A series of (what appeared to be sinusoidal) corrugations was measured and it was found that the distance between two successive crests is about $\Lambda = 0.25\,\mathrm{m}$ and their maximum depth is about $0.03\,\mathrm{m}$. Consequently we assume that their amplitude is $\mathbb{A} = 0.015\,\mathrm{m}$. A vehicle tyre travelling at $u = 60\,\mathrm{km\,.\,h^{-1}}$ takes $\tau = 0.015\,\mathrm{s}$ between hitting successive crests of the corrugations. Hence

$$\sigma = \tfrac{2\pi}{\tau} \approx 419\,\mathrm{s}^{-1}$$

8According to Newton's Law, the force $f(t)$ acting on the wheel caused by the corrugations is

$$f(t) = m\ddot{G} = -m\sigma^2\mathbb{A}\sin\sigma t \tag{5}$$

We have now determined all the pertinent coefficients as well as the nature of a possible forcing function $f(t)$, and can therefore proceed with the solution.

## 5.2 Vehicle body vibrations

We first convert the coupled spring problem to a fourth-order non-homogeneous ODE involving only the displacement of the body $X$. From (2) we obtain:

$$x = \tfrac{M}{K}\ddot{X} + \tfrac{\Phi}{K}\dot{X} + X; \quad \dot{x} = \tfrac{M}{K}\dddot{X} + \tfrac{\Phi}{K}\ddot{X} + \dot{X} \quad \& \quad \ddot{x} = \tfrac{M}{K}\ddddot{X} + \tfrac{\Phi}{K}\dddot{X} + \ddot{X} \tag{6}$$

Substituting (5) and the pertinent expression from (6) into (1) and (3) we obtain the fourth-order non-homogeneous IVP with constant coefficients:

$$\alpha\ddddot{X} + \beta\dddot{X} + \gamma\ddot{X} + \delta\dot{X} + kX = -m\sigma^2\mathbb{A}\sin\sigma t \tag{7}$$

$$X(0) = 0; \quad \dot{X}(0) = 0; \quad \ddot{X}(0) = 0; \quad \dddot{X}(0) = 0 \tag{8}$$

where

$$\left.\begin{array}{ll} \alpha = \frac{mM}{K} \approx 0.207\,\mathrm{kg\,.\,s^2} & \beta = \frac{\Phi m + \phi M}{K} \approx 14.2\,\mathrm{kg\,.\,s} \\ \gamma = \frac{mK + \phi\Phi + M(k+K)}{K} \approx 1.67 \times 10^3\,\mathrm{kg} & \delta = \frac{(k+K)\Phi + K\phi}{K} \approx 1.01 \times 10^5\,\mathrm{kg\,.\,s^{-1}} \end{array}\right\} \tag{9}$$

Using **the method of undetermined coefficients**, the particular solution of (7) has the form

$$X_p = \Upsilon\cos\sigma t + \Theta\sin\sigma t$$

and it satisfies

$$\alpha\ddddot{X}_p + \beta\dddot{X}_p + \gamma\ddot{X}_p + \delta\dot{X}_p + kX_p \equiv -m\sigma^2\mathbb{A}\sin\sigma t \tag{10}$$

The complementary solution $X_c$ of (7) satisfies

$$\alpha\ddddot{X}_c + \beta\dddot{X}_c + \gamma\ddot{X}_c + \delta\dot{X}_c + kX_c = 0 \tag{11}$$

Although the differentiation on the left-hand side (LHS) of (10) is easy enough to do by hand, because we need to calculate $\Upsilon$ and $\Theta$ as well the roots of the auxiliary equation of (11), we immediately introduce the computing power of the CAS DERIVE®(see [2]). For instance, simplifying the RHS of (10) involves the expressions given in Figure 2 below.

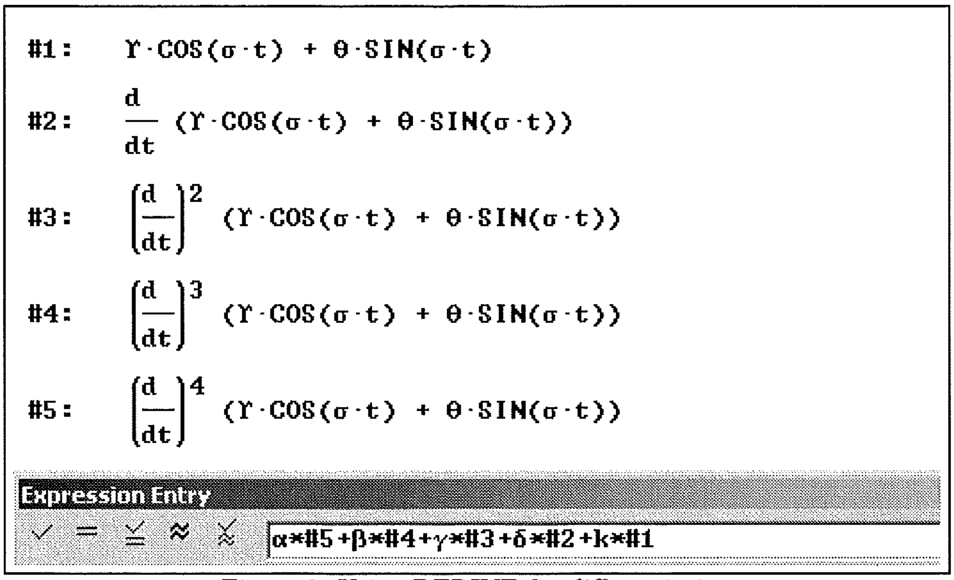

Figure 2: Using DERIVE for differentiation

By clicking on "=" in the "Expression Entry" box, we determine a simplified version of the RHS of (10). Equating coefficients on the (simplified) LHS and RHS of (10) we obtain the system of linear equations with "unknowns" $\Upsilon$ and $\Theta$:

$$\left(\alpha\sigma^4-\gamma\sigma^2+k\right)\Upsilon-\sigma(\beta\sigma^2-\delta)\Theta=0$$
$$\sigma(\beta\sigma^2-\delta)\Upsilon+\left(\alpha\sigma^2-\gamma\sigma^2+k\right)\Theta=-m\sigma^2\mathbb{A}$$

Using DERIVE® to solve and calculate we obtain:

$$\Theta=-\frac{\mathbb{A}m\sigma^2(\alpha\sigma^4-\gamma\sigma^2+k)}{\alpha^2\sigma^8+2\alpha\sigma^4(k-\gamma\sigma^2)+\beta^2\sigma^6-2\beta\delta\sigma^4+\delta^2\sigma^2+\gamma^2\sigma^4-2\gamma k\sigma^2+k^2}\approx-1.14\times10^{-5}\,\mathrm{m}$$
$$\Upsilon=\frac{\mathbb{A}m\sigma^3(\delta-\beta\sigma^2)}{\alpha^2\sigma^8+2\alpha\sigma^4(k-\gamma\sigma^2)+\beta^2\sigma^6-2\beta\delta\sigma^4+\delta^2\sigma^2+\gamma^2\sigma^4-2\gamma k\sigma^2+k^2}\approx-1.88\times10^{-6}\,\mathrm{m}$$

Forcing one's students to hand check the details of the differentiation and calculations above will make them appreciate the use of technology! From (11), the complementary solution $X_c$ has auxiliary equation

$$\alpha w^4+\beta w^3+\gamma w^2+\delta w+k=0$$

This is where we find it **necessary, not convenient**, to resort to a CAS. Using DERIVE® we determine, to three digits, the roots of this polynomial as follows:

$$w_1=-62.0\,\mathrm{s}^{-1};\quad w_2=-1.71\,\mathrm{s}^{-1},\quad w_{3,4}=\Psi\pm i\digamma$$

with

$$\Psi=-2.40\,\mathrm{s}^{-1}\quad\&\quad\digamma=87.6\,\mathrm{s}^{-1}.$$

Consequently, the solution is of the form

$$X=\underbrace{ae^{w_1t}+be^{w_2t}+e^{\Psi t}\left(\kappa\cos\digamma t+\lambda\sin\digamma t\right)}_{X_c=\text{transient solution}}+\underbrace{\Upsilon\cos\sigma t+\Theta\sin\sigma t}_{X_p=\text{steady-state solution}}\tag{12}$$

In order to determine $a$, $b$, $\kappa$ and $\lambda$, we need to determine $\dot{X}$, $\ddot{X}$ and $\dddot{X}$. If one attempts this by hand, keeping tabs on all of the resulting expressions becomes tedious. Figure 3 demonstrates that the expression for $\dddot{X}$ is quite technical, as one would expect.

$$\#28: \quad \left(\frac{d}{dt}\right)^3 X(t)$$

$$\#29: \quad \hat{e}^{\Psi \cdot t} \cdot ((F^3 \cdot \kappa - 3 \cdot F^2 \cdot \Psi \cdot \lambda - 3 \cdot F \cdot \Psi^2 \cdot \kappa + \Psi^3 \cdot \lambda) \cdot \mathrm{SIN}(F \cdot t) - (F^3 \cdot \lambda + 3 \cdot F^2 \cdot \Psi \cdot \kappa - 3 \cdot F \cdot \Psi^2 \cdot \lambda - \Psi^3 \cdot \kappa) \cdot \mathrm{COS}(F \cdot t)) + a \cdot w1^3 \cdot \hat{e}^{t \cdot w1} + b \cdot w2^3 \cdot \hat{e}^{t \cdot w2} - \Xi \cdot \sigma^3 \cdot \mathrm{COS}(\sigma \cdot t) + \Upsilon \cdot \sigma^3 \cdot \mathrm{SIN}(\sigma \cdot t)$$

Figure 3: The Derive expression for $\dddot{X}$.

DERIVE® effortlessly yields these three derivatives, and after applying the initial conditions (8), we obtain a system of linear equations which DERIVE® solves effortlessly. This yields:

$$\kappa = 5.89 \times 10^{-4}\,\mathrm{m};\ \lambda = 8.75 \times 10^{-4}\,\mathrm{m};\ a = 1.18 \times 10^{-3}\,\mathrm{m};\ b = -1.77 \times 10^{-3}\,\mathrm{m}$$

If all of the previous calculations were made in DERIVE® with working precision set to 16 digits, it is simple to declare their values and plot (12). Based on the a claim that for a CAS "accuracy = working precision - 10 digits" (see for instance [7]), we expect our solution to be accurate to at least 3 digits. Figure 4 shows a view of the transient state, while Figure 5 shows the steady state.

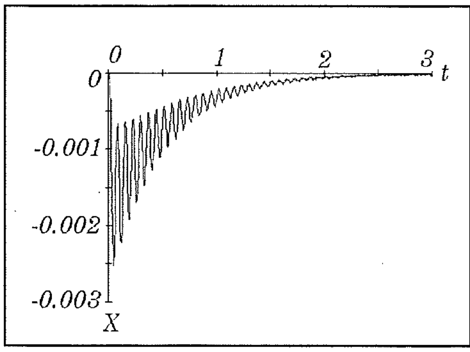


Figure 4: The transient state

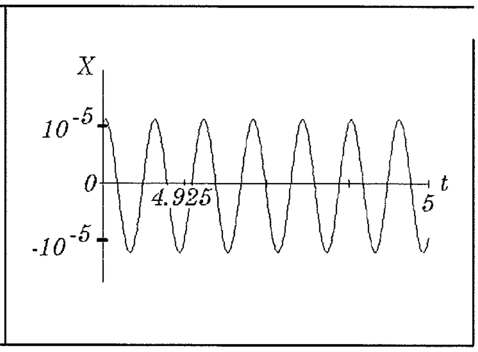


Figure 5: The steady state

Upon entering the corrugations, the vehicle body almost immediately has a 2.5 mm upward displacement. While undergoing rapid low-amplitude vibrations, the body returns to the position of rest (for all practical purposes) within 1.5 s. The low-amplitude vibrations persist while the corrugations (washboards) are present. The particular solution $X_p$ is the **steady-state solution** with amplitude of about 10 $\mu$m (microns). The frequency of vibration here is $f = \frac{\sigma}{2\pi} \approx 67$ Hz, well within the human hearing range.

## 5.3 Tyre vibrations

¿From (2) we obtain the function predicting tyre vibrations

$$x = \frac{KX + M\ddot{X} + \Phi\dot{X}}{K}$$

It takes some stamina to determine that

$$x = ne^{w_1 t} + oe^{w_2 t} + e^{\Psi t}\left(p\cos\left(Ft\right) + q\sin\left(Ft\right)\right) + r\cos\left(\sigma t\right) + s\sin\left(\sigma t\right)$$

where

$$
\begin{aligned}
n &= \frac{Ka+a\Phi w_1+Maw_1^2}{K}; \quad o = \frac{Kb+b\Phi w_2+Mbw_2^2}{K};\\
p &= \frac{K\zeta+\Phi(\Psi\zeta+\eta\digamma)+M\zeta\left(\Psi^2-\digamma^2\right)+2M\Psi\eta\digamma}{K};\\
q &= \frac{K\eta+\Phi(\Psi\eta-\zeta\digamma)+M\eta\left(\Psi^2-\digamma^2\right)-2M\Psi\zeta\digamma}{K};\\
r &= \frac{K\Upsilon+\Theta\Phi\sigma-M\Upsilon\sigma^2}{K}; \quad s = \frac{K\Theta-\Upsilon\Phi\sigma-M\Theta\sigma^2}{K}
\end{aligned}
$$

DERIVE® effortlessly calculates these coefficients for us and Figure 6 below shows that, upon encountering the corrugations, the model predicts that the axel displaces both up and down by about 7.5 cm shortly after entering the corrugations. A jeep tyre has about 16 cm of rubber which can be compressed before the rim hits the road, and so a downward displacement $x = 7.5\,\text{cm}$ is possible. This may explain why the rims of modern, low-profile tyres sometimes buckle on quite good dirt roads with corrugations but no potholes. However, close scrutiny reveals that the model needs to be refined, because the tyre would leave the road if the upward displacement of the axel was $x = -7.5\,\text{cm}$ and so $f(t)$ would "switch off". Indeed, the corrugations have an amplitude of $\mathbb{A} = 1.5\,\text{cm}$ and initial tyre compression is $d = 1.9\,\text{cm}$. In order to keep the surface of the tyre in contact with the road, the tyre surface would need to stretch radially towards the road by about 4.1 cm, an impossibility in the author's opinion. Consequently, the forcing function $f(t)$ cannot be accurately represented by (5). The refinement of $f(t)$ and/or the model is left for a future project which might not be suitable for undergraduate work. However, the model in question does illustrate the global behaviour of the tyre as well as explaining to some extent what a passenger feels in a vehicle entering corrugations. Figure 7 shows how the model predicts that the tyre steadies down to an oscillation which follows the pattern of the corrugations with a vibration frequency of $f = \frac{\sigma}{2\pi} \approx 67\,\text{Hz}$.

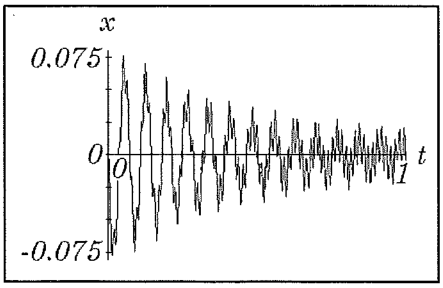


Figure 6: The transient state of $x$

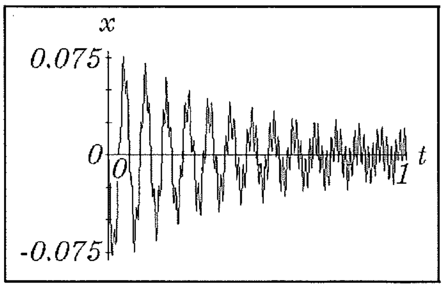


Figure 7: The steady state of $x$

## 6 Conclusion

The data used above were collected by the author during October 2002. The author's current (second semester 2003) ODE group of 31 students are undergoing "continual assessment" as opposed to "semester tests and a final examination" and so this project can be discussed with them during the last weeks of the term (usually reserved for examinations). During this time (approximately 8 contact hours), the author will strive for mastery of the model by all students. Currently (August/September 2003), five small groups (about six students in each group) have been given a soccer ball to play with as an introduction to the "bouncing soccer ball model" [?], which is entwined in the model under discussion. An evaluation of the students' experiences with the model under discussion will be reported at the Delta03 conference.

In the author's opinion, working through this student model will introduce and/or reinforce the following mathematical and/or educational concepts in a natural and topical setting:

> A realistic model of a well-known physical occurrence; a nontrivial (non-"plug and chug") problem; spring-damper pairs and the overdamped and underdamped situations;

small group work; an affordable, easy to perform experiment; a "sanity check" - does the model predict what we expect? If not, refine the model.

## References

[1] Temple H. Fay and Sarah D. Graham, "Coupled spring equations", International Journal of Mathematical Education in Science and Technology, **34**, (2003), 165–79.

[2] *DERIVE for Windows, Version 5*, Soft Warehouse Europe, http://www.derive-europe.com, 2003.

[3] T. H. Fay, K. A. Hardie and S. V. Joubert, "Dirt road corrugations", SA J. Science **97**, September/October (2001), 357-358.

[4] Dennis G. Zill, and Michael R. Cullen, *Differential Equations with Boundary -Value Problems*, Brookes/Cole, Pacific Grove, USA, 2001.

[5] Uwe Kiencke and Lars Nielsen, *Automotive Control Systems*, Springer-Verlag, Berlin, Heidelberg, New York, 2000.

[6] A. Bekker, T. H. Fay and S. V. Joubert, "The Shock Absorber in an Automobile Suspension System", Math. and Computer Ed. **33** (1999), 129-141.

[7] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer,* Second Edition, Addison-Wesley, 1991.

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Stephan V. Joubert and M. Anton van Wyk, "A simple non-trivial student model of a bouncing soccer ball", accepted to appear in the International Journal of Mathematical Education in Science and Technology.

# LEARNING MATHEMATICS TROUGH ORAL PRESENTATIONS

OWE KÅESTEN AND LOUISE BONTA

We will present a project aiming at the studying how to create an environment for peer leaning, where students teach students, by making oral presentations of problem solving and theories in connection with the teaching of mathematics on the master of engineering program at Campus Norrkþing, Linkþings universitet Sweden.

This project was designed to take the students through learning experiences involving the basic rules and tools of a good presentation in order to strengthen the students' understanding and perception of central mathematical concepts. By reflecting why the student has chosen to interpret and present a certain mathematical concept the way he has, he will gain deeper understanding and more qualified learning. The reward of this will be great when it comes to self-confidence and deeper and shared understanding.

We wanted to closely reflect how the environment should be created and designed for a more effective learning to take place in connection with these presentations. It seems that the focus had been more to go through with the presentation than to focus on the contents. As the presentations are part of the assessment we wanted to produce a guide with tools for the teachers as well as for the students - tools that will be a support when creating the environment where the presentations take place in order to strengthen the students' understanding of central mathematical concepts and improve the oral presentations from a learning point of view.

## Aims and background

The purpose of this project is to study how to create an environment for peer learning, where students teach students, by making oral presentations of problem solving and theories in connection with the teaching of mathematics, during the first year on the master engineer program.

Various forms of assessments have been carried out within the master engineer program during the last five years at Campus Norrkþing, arriving from the fact that the students perform different kinds of oral and written presentations. The purpose of these has been, apart from being a part of the assessment, to use the power that lies in students teaching students, so called þeer learning." The teacher has become more of a listener.

It is now time to closely reflect how the environment should be created for a more effective learning to take place in connection with these presentations. It seems that the focus has been more to go through with the presentation rather than focus on the contents of the presentation, which we intend to make the students do. We intend to analyse how these environments should be designed and to produce tools for the teacher as well as for the student. The purpose of these tools is that they should be a support when creating the environment where the presentations take place, in order to reach the target to strengthen the students' understanding of central mathematical concepts. Furthermore, the environment could probably be used when teaching other subjects.

The teacher will have an increased role; he or she will, apart from being a listener, set the limits and create the environment. Another very important role for him or her will be to observe and correct potential mistakes.

Such teaching can create a base for active and deep learning and aims at motivating the students to learn. Motivation and enthusiasm are conditional for active and deep learning. The memory is strongly dependent of your own activity - the more you are engaged in what you do, the more you remember.

Our experiences show that many see the power in using different means in getting the students to reflect.

Therefore we are anxious to create another method by using oral presentations. We also see a positive spin off effects; the students will acquire a spoken mathematical language. Often students only have a written mathematical language and they sometimes express themselves very poorly.

*The project has its base in the following theories and experiences:*

**The learning process and thinking.** What does the students' thinking look like? How does it change during their time at the universities? How do they learn? What does their different learning process look like? What do students remember and why?

The learning process differs between different students. Students understand in different ways simply because they are different; they come from different homes, they do not have the same necessary qualifications or knowledge of languages. There are many theories about different stiles of learning [1]. Thus, a good learning situation should consist of structure and surprises, details and overview, patterns, but above all a learning situation or a lecture must engage and create interest. As our brains are different the structure of the learning situation, the lecture, must adapt.

**Reflective thinking.** Reflection generally means to meditate, ponder and think over, to take a step back to see and to think over one self and one's actions in order to get perspective. Reflecting is supposed to give the student an overview, remind him what and how he has done and what the consequences are. It is a good idea for the student to write down his reflections in order to get structure to his thoughts [2].

**Learning loops.** Learning is intimately connected with a person's possibilities to have an influence on his surroundings in order to solve his problems or to reach his goal. Space is needed to experiment and try different alternatives and will be of great importance for a person to help to facilitate qualified learning. The person designs his own plan according to his knowledge, his notions and his interpretation of goals and tasks. He then acts or does not act according to his plan and observes and reflects on his action in the light of the consequences. The different steps in such a loop can be described as follows:

- Define task and goal
- Interpret task on the basis of knowledge
- Design an action plan (problem solving)
- Perform
- Observe the consequences
- Reflect and evaluate the consequences or the effect [3]

**Presentation skills.** It has been said that most people fear presenting to large groups even more than they fear death. American studies have shown that more than 40 %f Am erican university students consider themselves shy. We believe this is true of Swedish students also. Shyness and fear of public speaking do come down to the same terror of being judged and evaluated. It's classic performance anxiety, stage fright. For those afraid of public speaking the cure is usually the poison: Feel the fear and do it. Courage is not the absence of fear, but doing what one fears the most. We would like to reduce this fear by creating a confiding surrounding where the students can feel safe.

An engineer who doesn't know how to communicate with other people will not gain a hearing. Therefore, it is important that the students learn presentation skills. We think it is important for the students to learn how to argue, discuss and put forward their thoughts and their research both orally and in writing. In order to communicate the students must be able to know and explain what the different mathematical concepts mean. Each human being has a notion of his own what the world is like and tries to understand the situations according to his notion. His notion is like a map. If you do not know how the map has been drawn you will probably have a problem. He who does not know the map will get lost.

By making an oral presentation the student will be forced to follow a certain line of thought. Being a good speaker is a matter of attitude reflected in a speaker's preparation, commitment and enthusiasm. Being a good speaker requires thorough preparation and rehearsal. Time, patience and hard work are necessary and focusing on the task makes the student confident and takes attention away from fear.

Humour and entertainment figure prominently in the repertoire of a good speaker. A good speaker must be versatile and able to adapt to the different audiences he or she has to face. The speaker must learn how to adjust the material to the people to whom he or she is speaking [4].

This project is designed to take the students through learning experiences involving the basic rules of a good presentation and the presentation tools in order to strengthen the student's understanding and perception of central mathematical concepts. By reflecting why the student has chosen to interpret and present a certain mathematical concept the way he has, he will gain deeper understanding and more qualified learning. The reward of this will be great when it comes to self-confidence and deeper and shared understanding.

**Advantages.** Apart from pedagogical advantages students teaching students will be more effective. As the teacher is present at the presentations he will be able to be more on the alert. There will be an opportunity for discussion and the student can for example be asked to explain how he has interpreted the problem or the situation and find out what the student understands and not understands. Thus the teacher will be able to give more effective feed back and concentrate to help.

Another advantage is that the students learn a lot from each other when they prepare their presentations. They have to explain to the others how they think and they will have to try to understand others' misunderstandings. All teachers know that the best way to learn is to prepare and explain to other students. And students find pleasure in listening to other students' presentations (if they are not too boring that is).

This will create opportunities for active learning and help motivate the students to learn. This is about active understanding contrary to passive memorising. Active learning will help the students' ability to think, reason and apply and put their knowledge into practice. Motivation and enthusiasm are two important factors for deep learning and will probably help students to take more responsibility for their own learning. The memory needs activity and enthusiasm and the more you are involved in what you learn, the better you will remember. The memory is a network of facts, experiences and notions and in order to learn new things you have to find the ends of the threads and tie them together [5]

## Points of departure and prerequisites

This project is part of the visions for the master engineer programs at Campus Norrköping, visions that are described by Linköping Institute of Technology. Among other things it is stated that "Special care is devoted to teaching and working forms as well as to the structure, contents and organisation of the teaching. The assessment is carried out in various forms, not just traditional examination tests." *Furthermore, under the head line of "The Human Engineer "* (Den mänsklige civilingenjören) it says: " The students will meet with course elements which will form their character, elements such as role plays, rethorics/argumentation, critical analyses, management and responsibility and self criticism. These elements are in most cases integrated in the technical and scientific courses and are lead by a team of teachers consisting of both technical and arts teachers." This project is well in line with this vision.

## Design of the project and some results

**Stage one.** An enquiry and interview questions was formulated so that they, as much as it is possible, make the students not only reflect on their learning and but also if and how this might have anything to do with the presentations. We used older students to formulate the questions. These students have during previous years participated in oral assessments, consisting of presentations of mathematical material. We also had interviews with the teachers involved in the presentations.

*Students*

The students' questionnaire was about how the students felt, what they learnt and what expectations they had on the teachers and their behaviour in order to feel more confident and if they thought they got a deeper understanding by this kind of examination.

Here are some comments on how the students felt when presenting and what the listening students thought was needed in order to learn from the others' presentations:

- It is dull to listen to others' presentations
- It is fun if you know what you are supposed to talk about – even if you are nervous.
- You must put some fun to the presentation
- You mustn't talk too fast and the presentations mustn't be too long.
- We are not used to make oral presentations.
- Each group has to do its best to make the presentation more interesting.
- You must create a dialogue with the public.
- You have to talk about things that are of interest to the others.
- You need better presentation skills, eg how to use the white board and over heads

**Teachers**

One of the questions to the teachers was how, and if, they had prepared themselves.

- Not much – I pick things up during the presentations.
- I try to find out in advance where the students can go wrong and where there might be difficulties and how to tell them that in a positive way.
- I try to figure out alternative solutions if they have difficulties.
- I try to find a way to make the students feel confident.
- I try to figure out a way how to create an atmosphere where the other students can ask questions.
- What shall I look at? The logical consequences, the pedagogical level, the mathematical contents?

We found that it was not that easy for many teachers to change from being an ordinary math teacher to a more passive and listening teacher. Many of them didn't feel comfortable with the idea how to criticize in a constructive way, to keep silent and not interfere during the presentation and so on if and when they discovered a mistake. The teachers did not have the same attitude and objective regarding oral presentations despite the fact that they taught the same courses and that it was part of an examination.

We also noticed that information to the students was crucial. They wanted to know what the presentation was good for, was it a real examination or was it just some kind of learning situation. Furthermore it was essential to separate the examination from the learning situation, that separate the presentation itself and the questions from and the interference by the teacher.

Seen from the students' points of view the teachers' role and how he/she acts are of great importance. Therefore we thought that if the guide for the teachers was correctly written and used it would essentially improve the learning situation.

**Stage two.** As the presentations are part of the assessment we wanted to produce a guide with tools for the teachers as well as for the students - tools that will be a support when creating the environment where the presentations take place in order to strengthen the students' understanding of central mathematical concepts and improve the oral presentations from a learning point of view. We designed these guides based on stage one.

**Stage three.** We used the guides the second year and followed what happened through enquiries and interviews of the students and teachers. At this movement we can give some results from the students' questionnaires.

*How do you feel in relation to the oral presentations?*

50 % of the students feel happy
65 % of the students feel stimulated
59 % of the students feel nervous
62 % of the students feel it is a challenge

*What do you learn?*

95 % to speak to an audience
87 % to talk mathematics
45 % get a deeper understanding for mathematics
40 % get a better insight of their own way of learning
25 % get a better insight of other students' way of learning
46 % reflect more over mathematical problems

*What do you think of other students' presentations?*

70% find them interesting
54% find them informative
46 % find them stimulating
30 % find them boring

95 % of the students were happy with the way the teachers had presented the feed back.

## Conclusions

The evaluation is not yet finished but what we have seen so far is that among the positive reactions we find:

- Information to the students has become more clear and better
- Teachers' behaviour has affected the presentation in a more positive way
- Students' are more positive to the oral examinations than before
- Students prepare themselves more thoroughly in order to be able to answer questions
- Students feel more confident
- Students find other students' presentations more informative
- 95 % of the students find this form of examination good
- Teachers have found that the students have improved
- Teachers who use the guide find it useful

At the conference we will present our conclusions and experiences in the form of a check -list for students and teachers. The check -list is intended to offer concrete pieces of advice how a presentation environment can be designed in order to create a good teaching situation.

## References


1. Howard Gardner, 1994, Frames of Mind-The Theory of Multiple
2. Molander, B, 1993, *Kunskap i handling*
3. Granberg, O, Ohlsson, J, 2000, *Från lärandets loopar till lärande organisationer*
4. Louise Bonta, 1998, *Presentationer, Retorik och PowerPoint*
5. Marton, Ference et al, 1995, *The experience of learning*



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# MATHEMATICS AND NON-MATH MAJORS

A. BATHI KASTURIARACHI, AUSTIN MELTON, AND BEVERLY M. REED

Many universities require most or all of their students to take, at least, one mathematics course. For many non-math majors this generic mathematics course is either college algebra or a beginning statistics course. While these courses are beneficial and help the students learn useful skills, they do not give most students a good idea what mathematics is nor what mathematicians do. This paper reports on the authors' efforts to develop a mathematics course which is appropriate for beginning university students and which will enable the students to experience the excitement of doing mathematics while also learning what mathematicians do.

## Introduction

Approximately two years ago, we began developing an entry-level course which could be taken by non-math majors and which would introduce the students to mathematical thinking. Part of our motivation was realizing that after university students take an introductory course in a discipline, they usually have some understanding of the discipline and of what professionals in the field do. However, this does not seem to be the case after students take an entry-level mathematics course.

We first taught this course in our September-December term in 2002; it was taught under the title "Explorations in Modern Mathematics". Before the course was offered we reported on our preparations and plans for the course at the International Conference on the Teaching of Mathematics in Crete, Greece in July 2002; please see "Mathematics That Changes Lives" [2]. In this current paper, we summarize our original goals from the perspective of having taught the course once; we report on the first offering of the course; and we outline our plans for teaching the course again. Further, we report on the outcomes of a questionnaire that we gave to our students at the beginning and end of the course.

## Planning for the original course

When we were preparing the original version of the course, we believed we could develop a course in which the material would be understandable by beginning university students and which would also expose the students to "real mathematics". We did not believe that we could teach or explain advanced mathematics to beginning university students. However, we felt there is an "essence" of mathematics that is independent from the depth of mathematics. This essence involves studying a situation, i.e., a problem, which is not fully understood. From this studying and questioning, one begins to gain insight into the situation and then forms conjectures about the situation. These conjectures are expressed as or translated into precise questions which one tries to verify or disprove. Our goal was to develop such a learning environment in mathematics for beginning university students.

When considering content for the original course, we wanted material understandable to those students who would normally take college algebra. However, we did not want to use college algebra material because many of the students have seen this material multiple times, and we did not want the "baggage" associated with past mathematics classes to hinder our efforts. The topics we considered were cryptography, fuzzy set theory, geometric solutions to algebraic equations, graph theory, and spherical geometry.

We decided to cover several topics instead of focusing on only one or two. This decision was likely made due to concerns that too much depth in a subject would either make the subject too hard or too boring. Also, we felt by having more topics the class would "move" faster and, thus, more likely keep the students' interest. In the end we decided on cryptography, fuzzy set theory, graph theory, and spherical geometry with group projects. As our regular university terms are fifteen weeks long, we decided to spend three weeks on each topic and to have the students give presentations of group projects during the final three weeks of the term. These projects were to be worked on by groups of three to five students, and the topics were to be chosen from the four course topic areas.

As part of our motivation for developing and teaching this course was a belief that beginning university students have little or no understanding of what mathematics is or of what mathematicians do, we developed a questionnaire which would help us evaluate the students' understanding of what mathematics is and of what mathematicians do. We based our questionnaire on the works of Aiken [1], Liu [4], and Schoenfeld [5] and gave it to the students at the beginning and end of the course. The questionnaire is reproduced in Appendix A of this paper.

## Background information on the students and comments on the questionnaire

Twenty-five students took the course, and of those, twenty-four finished the course. There were a few other students who signed up for course but dropped it relatively early in the term. The students were fairly evenly distributed among first-year through fourth-year students, and one of the students was even a graduate student who simply wanted to take more mathematics. As mentioned earlier, the course was taught under the title "Explorations in Modern Mathematics". A course by this name has been taught for approximately six years. The reasons for developing this Explorations course were to give students an alternative to college algebra for satisfying a basic mathematics requirement and to expose the students to non-college algebra topics. Thus, the reasons for developing the Explorations course were related to the reasons for developing our course.

However, the explicit motivation to allow students personally to experience mathematics was not part of the motivation for developing the original Explorations course. Our offering of the Explorations course was not advertised as being any different from other offerings of the Explorations course. Thus, the students who signed up for our Explorations course did not know our motivation for teaching the course was different from those teaching other sections of the course, but the students did know the course would not be a college algebra course. This knowledge that the course would not be a college algebra course, no doubt, motivated some students to take the course, and it may also have been a significant contributing factor to the near even distribution of students – first-year to fourth-year – taking the course. It is not uncommon for students who do not want to take mathematics to put it off as long as possible, and at least some of these students would avoid college algebra if possible. Thus, the students taking this course were not the "average" beginning university student, and some of them would likely know less about mathematics and mathematicians than the "average" beginning university student. However, some of the students – especially those taking this course to get a different perspective to mathematics – may have been more aware of what mathematics is and what mathematicians do than the "average" beginning university student.

## The presentation of the original material

We[1] began with fuzzy set theory, which is regular or classical set theory with membership functions. In fuzzy set theory, one begins with a universal set and defines subsets in terms of membership functions whose domain is the universal set and whose codomain or range is usually taken to be the closed unit interval, [0,1]. The image of an element under a given subset membership function determines to what degree the element is a member of the subset. If A is a classical subset of a universal set U, then A's membership function is its characteristic function. A membership function for "excellent students" in a class might map students who earn A's to 1, those who earn B's to 0.75, those who earn C's to 0.5, those who earn D's to 0.25, and those who earn F's to 0.

We introduced fuzzy set theory as a modification or enhancement to classical set theory. Membership functions, as such, were not mentioned until we were well into the fuzzy set theory section. In place of explicit membership functions, we talked about membership values being assigned to each element of a subset. For the most part the students did well with the manipulations needed to perform fuzzy unions (defined in terms of the maximum of the two membership functions/subsets being unioned), intersections (defined in terms of the minimum), complements (defined as 1 minus the given membership value), and relative complements (defined as the maximum of the difference and 0); and thus, they seemed to understand these concepts. Further, we did some simple applications involving the use of fuzzy sets to

[1] Though we use the pronoun "we" throughout this paper, the second author was the sole instructor of the course.

select teams of people who had certain language and technical skills. Also, we studied a detailed example of using fuzzy if-then statements to help people decide how much they should invest for retirement based on their salary, number of dependents, amount of current debt, collateral, age, and whether or not the company where they are employed has a retirement program.

The next section we covered was cryptography. We discussed additive ciphers, multiplicative ciphers, affine ciphers (which consist of an additive cipher followed by a multiplicative cipher), and key word ciphers (which involve selecting a specific word and letter to determine how the alphabet is bijectively mapped to itself for the encoding). The additive and multiplicative ciphers allowed us naturally to introduce addition and multiplication mod 26, in particular, and mod k, for a positive integer k, in general. We talked about primes and relative primeness, and we did interesting arithmetic mod p, for p a prime.

While covering cryptography, the class started to become rushed in that we were having trouble covering all the material while also allowing the students time to understand it relatively well. Further, when they took the cryptography exam and did not do well, we clearly realized that we were moving too fast. We decided to let the students retake the cryptography exam. Of the twenty-five students in the class, sixteen retook the exam, and each of these sixteen did better on the second attempt, many significantly. Also, at this time, we decided to cover only spherical geometry in place of spherical geometry and graph theory in the remainder of the course.

Our final section of material was spherical geometry, and our main resource was "Non-Euclidean Adventures on the Lenart Sphere" by Istvan Lenart [3]. We had one Lenart sphere with "straight edge" and compass. Usually, the Lenart sphere was used by the instructor while the students used clear plastic spheres which were about twelve centimetres in diameter and on which they could write with colored marking pens. We mentioned Euclid's axioms, and we covered great arcs, triangles, and (not necessarily great arc) circles.

The last three weeks of the course were to be devoted to presentations of group projects. After the pace of the course slowed down, however, we had only about four class periods – a little more than a week – for presentations.

## Results of the first offering of the course

We discuss the results from two perspectives. First we discuss the course in terms of which topics the students understood well and which topics they enjoyed studying, and then we also discuss the course in terms of how well it met our objectives of allowing and enabling the students to experience mathematics and to learn what mathematicians do.

The students understood the fuzzy set theory concepts in the sense that they could work problems involving fuzzy set operations. However, they did not seem to understand that these concepts would allow them to manipulate sets in new and possibly exciting ways. They did not understand that they could now manipulate formal sets in ways that are considered analogous to humans reasoning with vague concepts like "almost" and "very".

Though the section on cryptography was more difficult for most students than the fuzzy set theory section, once the students understood the concepts they were able to do well on the exam. In reconsidering the first two sections, it now seems clear that more concepts, including more traditional mathematical reasoning and ideas, were presented in the cryptography section than in the fuzzy set theory section. We covered arithmetic mod m, for an arbitrary positive integer, and we worked with a field in mod p, for a prime p. In fact, we solved general equations of the form $ax + b \equiv c \mod p$. Also, the students were usually comfortable in working through encoding and decoding scenarios and in keeping straight when each process should be used. Further, the students enjoyed the cryptography section; this was their favourite section. Evidence for this claim includes their comments in and out of class and the fact that most of the group projects were about cryptography.

The students' mathematical skills seemed to improve during the study of cryptography. For example, it seemed that their understanding of the importance of the additive and multiplicative identities and their understanding of solving linear equations increased by solving linear equations mod p. Also, their general problem solving skills likely improved though the breaking of simple codes. These are positive and worthwhile results, but the students were not doing mathematics nor were they developing new conceptual insights.

It was difficult for the students to visualize some of the spherical concepts and constructions. For example, when we first studied three non-collinear points, no two of which were a "polar pair", it was not easy for the students to see that these points could be the vertices of eight different triangles. In time, they became skilled enough at drawing spheres on paper and the chalkboard that they could "see" all eight, but in the beginning it was hard. These difficulties would probably have been less severe if we had had enough Lenart spheres with "straight" edges and compasses for every two or three students. The students probably enjoyed this section more than the fuzzy set theory section but less than the cryptography section. However, better visuals might have allowed this section to be as popular as the cryptography section.

One student who was better than average but certainly not one of the best students for the first two sections was able to "see" spherical constructions that most of the students could not see. For him, this section allowed him to create and gain new insights. He saw new concepts and ideas on the sphere. Creativity and new understanding to this degree did not seem to happen for the other students though the realization that there are no distinct parallel lines, that triangles can have multiple right angles, and that the sum of the interior angles of triangles always equals at least 180 degrees really seemed to impress some of the students.

## Problems with the first offering of the course

After the fuzzy logic unit, we realized that we should have spent time at the beginning of the semester introducing some math tools such as functions. The instructor felt uncomfortable with standard explanations of the fuzzy set operations. For example, when unioning two fuzzy subsets, the normal rule is the membership function of the union is the function whose value for each element in the universal set is the maximum of the corresponding images of the two functions/subsets being unioned. However, if an element x is in subset A to a degree of 0.6 and in subset B to a degree of 0.4, then it seems possible that x could be in the union of A and B anywhere from a degree of 0.6 to 1.0 because the part of x in A and the part of x in B could overlap or could be complementary.

A problem for the students was the lack of an appropriate text. The instructor prepared notes for the students.

## Student feedback and analysis of data

Informally, students didn't appear to get excited about fuzzy set theory; found spherical geometry interesting but hard to visualize; and enjoyed cryptography. Evidence of their enjoyment was the number of students who chose cryptography as the topic of their final group project.

Analysis of student questionnaires revealed a significant difference in nine of the 21 pre- and post-responses on questions 6-26. We conducted a one-tailed *t*-test for the difference between means and found a significant difference ($p < .05$) in student responses to questions 12, 15, and 25. The differences for questions 6, 7, 10, 15, 17, 18, 23, 25, and 26 were significant at the $p < .10$ level. All differences in means indicated a positive change.

Interestingly, four of these questions dealt with the nature of mathematics (questions 6, 7, 10, 15). We were pleased that at the end of the course, students were less likely to think that "there is only one correct way to solve any mathematics problem, usually using the rule the teacher demonstrated in class"; or that "students who have understood the mathematics they have studied will be able to solve any mathematical problem in five minutes or less"; or that "there is nothing creative about mathematics, it's just memorizing formulas and things". One of our principle objectives for teaching the class was to broaden students' views of mathematics and the course appears to have accomplished this goal to some degree. Heartwarming to the authors was our success in attempting to dispel the notion that mathematicians can solve problems in five minutes or less. Perhaps our students will appreciate the fact that struggling with mathematics is a normal, to-be-expected part of the problem-solving process.

Also refreshing was the apparent change in attitude toward mathematics that some students exhibited, as evidenced by the significant positive changes in the following questions: 17, 18, 23, 25, 26. If students can struggle with an interesting problem and feel less "uneasy...uncomfortable and nervous", we feel we have made significant gains.

Analysis of the qualitative questions was equally interesting. At the beginning of the course, one student consistently thought of mathematics as being a "course taught" rather than a way of thinking: (italics added)

> Mathematics is a *course* that is a continuous learning process, for which you build on previous skills. You use knowledge of earlier *courses to* help you in your current *course.* It requires a sense of discipline .It is a *course* that is needed in everyday activities and forces everyone to use a form of mathematics to live their lives daily.

Contrast the above with her views at the end of the semester: "Mathematics are components which help people to function on a day to day basis. It requires an understanding of basic principles and requires an open mind and understanding." Though the student's view of mathematics as a practical tool is evident both excerpts, her sharp distinction between "school math" and "real math" is not evident in the latter.

Another student had a similar epiphany. At the beginning of the semester she thought that "Mathematics is part of the core instruction of a well rounded education", and near the end, she referred to it as "The formula of ideas with theorems and numbers."

We weren't as successful with all students, however. One student, in response to question # ("W hen you are stuck on an unfamiliar mathematics problem, what are your reaction and response?") revealed "frustration" as her typical response as well as "quitting". Ten of 19 students responded in a similar, negative fashion to question 4 initially, and 6 of 17 did so near the end of the semester. So though we appear to have made some progress with some students, we do have a way to go with others.

## Conclusions and plans for the future

Our plan for a new course is simple. We'd like to get the students' attention by discussing geometry including spherical geometry. The basic ideas are to help the students see i) that mathematics is more than what they have been exposed to, ii) that it is understandable, and iii) that it is interesting. This should take three to four weeks. Then we would move into game theory, which is something (in its simple forms) they can understand and through which they can, we think, create mathematics. We would spend the rest of the term on game theory with some history of math. The history part would be interwoven as follows. As students learn new things about game theory, the instructor would introduce other "new" ideas similar to the game theory, arisen in different branches of mathematics, which now have significant impact on our lives.

In order to more carefully assess the effectiveness of this course, we hope to study our students' thinking as they progress through it. One possibility is to interview students as they work with a mathematical problem, studying their strategies, and perhaps comparing them to those of a more mature mathematician. Though we certainly do not expect our students to achieve the level of a practicing mathematician, we would expect some growth as the semester progresses.

## Appendix A: Student questionnaire

PART I. For the first five questions, please give a written response. If you need more space, please write on the back of one or more of the pages.

1. In your opinion, what is mathematics? If you feel it is different from other disciplines, please explain how and/or why.
2. In your opinion, how does mathematical knowledge develop? Does the development of mathematical knowledge follow any rules? Please explain your answers.
3. In your understanding and imagination, how do mathematicians think when solving a mathematics problem?
4. When you are stuck on an unfamiliar mathematics problem, what are your reaction and response?
5. What do you think are the main differences between what a mathematician does when he/she does mathematics and what you do when you study and do mathematics?

PART II. For the remaining questions, please respond by circling *1, 2, 3, 4*, or *5* where *1* means *strongly disagree* and *5* means *strongly agree.*

6. **1 2 3 4 5** Mathematics problems have one and only one right answer.
7. **1 2 3 4 5** There is only one correct way to solve any mathematics problem, usually the rule the teacher demonstrated in class.
8. **1 2 3 4 5** When learning mathematics, I really don't expect to understand it; I prefer to memorize it.
9. **1 2 3 4 5** Mathematics is best done by oneself.
10. **1 2 3 4 5** Students who have understood the mathematics they have studied will be able to solve any mathematical problem in five minutes or less.
11. **1 2 3 4 5** The mathematics I have learned in school has little or nothing to do with the real world.
12. **1 2 3 4 5** Mathematics is less important to people than art or literature.
13. **1 2 3 4 5** An understanding of mathematics is needed by artists and writers as well as scientists.
14. **1 2 3 4 5** Mathematics is needed in designing practically everything.
15. **1 2 3 4 5** There is nothing creative about mathematics; it's just memorizing formulas and things.
16. **1 2 3 4 5** I enjoy going beyond the assigned work and trying to solve new problems in mathematics.
17. **1 2 3 4 5** Mathematics is enjoyable and stimulating to me.
18. **1 2 3 4 5** Mathematics makes me feel uneasy and confused.
19. **1 2 3 4 5** I am interested and willing to use mathematics outside school.
20. **1 2 3 4 5** I have never liked mathematics and it is my most dreaded subject.
21. **1 2 3 4 5** I have always enjoyed studying mathematics in school.
22. **1 2 3 4 5** I would like to develop my mathematical skills and study the subject more.
23. **1 2 3 4 5** Mathematics makes me feel uncomfortable and nervous.
24. **1 2 3 4 5** I am interested and willing to acquire further knowledge of mathematics.
25. **1 2 3 4 5** Mathematics is dull and boring because it leaves no room for personal opinion.
26. **1 2 3 4 5** Mathematics is very interesting and I have usually enjoyed courses in the subject.

## References


1. L.Aiken,.,*Two scales of attitude toward mathematics,* Journal for Research in Mathematics Education (**5**) (1974), 67-71
2. A.B. Kasturiarachi, A.Melton, & B.Reed, *Mathematics that changes lives,* Proceedings of the Second International Conference on the Teaching of Mathematics, Crete, (2002).
3. I.Lenart, *Non-Euclidean Adventures on the Lenart Sphere*, Key Curriculum Press, 1996
4. P.Liu, *Developing college students' views on mathematical thinking in a historical approach, problem-based calculus course,* Proceedings of the Second International Conference on the Teaching of Mathematics, Crete, (2002).
5. A.Schoenfeld, *Learning to think mathematically: problem solving, metacognition, and sense-making in mathematics,* In D.Grouws (Ed.), Handbook for Research on Mathematics Teaching and Learning, (1992), 334-370.



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# FUNCTIONS AND OPERATORS IN MAPLE AND MATLAB

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**Abstract**

Some of the most distinctive features of computer algebra systems and numerical packages are their implementations of the concept of a function. When thoughtfully matched and integrated into the curriculum these realizations may be extremely helpful for the students' development of a multi-faceted concept image. On the other hand, thoughtless use of either kind of package may cause more confusion, do more harm than good.

This article contrasts two major features: On one side are the overloaded definitions that make MATLAB so powerful, convenient, and beloved by the advanced user. Typical examples include the sine and reciprocal of a vector, the exponential of a matrix, and the division by a matrix, in particular, when it is not square. At a time when students are supposed to learn distinguish different objects such as scalars and vectors, and different rules for algebraic operations among them, such overloaded functions are potentially hazardous – but when purposefully integrated, these features can much facilitate learning to work with different algebras.

On the other side, both MAPLE and MATHEMATICA distinguish between pure functions and expressions. For the unprepared user this can be a nightmare, but when thoughtfully integrated, these features are wonderful aides to help students develop powerful ways of working of functions. We focus on the key operation of function composition (including taking inverses), and on derivatives $f'$ without a need to name the variable of differentiation. This is very different from the usage in physics.

The main objective of this article is to develop awareness for the consequences of choosing a specific technology, and both its hazards for curricular integrity as well as exciting opportunities to improve the curriculum.

## 1 Introduction: Functions, technology, and clients' preferences

The concept of a function is arguably the central object of studies in mathematics at both the late secondary and beginning post-secondary level. A vast body of literature has been assembled on this topic, and it is simply impossible to survey it here in detail. Instead, we just mention a few perspectives that are directly related to our concerns. Functions appear at the introductory level in the form of co-varying quantities, i.e. special classes of relations that satisfy an abstraction of the vertical line test, compare [2]. Modern computer technology allows one to replace the traditional static descriptions with dynamic interactive notations, some of which even allow the study of functions at the elementary level [6]. Among these, new forms of graphical representations, going as far as entirely visual languages such as the graphical MATLAB environment SIMULINK, provide many new challenges for the educational community. Clients such as engineering colleges press forward with expectations that such new environments are implemented at a rapid pace. The mathematics education research naturally lags behind in evaluating the merits of such innovations. We refer to [9] for exemplary investigations of the role of visualization.

The concept of a function has many different faces. Students develop understanding of such different aspects only over the course of many years. One extreme is the abstract definition of a function $f$ from a set $X$ to a set $Y$ as a subset $f \subseteq X \times Y$ that satisfies $(x, y_1), (x, y_2) \in f \Longrightarrow y_1 = y_2$. From this point of view every function is a function of a single variable. While found in most textbooks at various levels, this formal definition is next to irrelevant for the large majority of mathematics courses at the high school and introductory collegiate levels. Instead, the vast majority of class-time and of examination questions is devoted to descriptions in terms of algebraic formulas.

An important alternative or intermediate way of looking at a function is as a mapping $f\colon X \mapsto Y$. Several modern calculus texts, e.g. [4] employ this language and use the term map (or mapping) as a primitive that needs no definition. In contrast the earlier mentioned formal definition starts with the notion of set with remains undefined at this level.

However, for the learner the notation $f\colon x \mapsto f(x)$ is fraught with possible misconceptions. This article revisits some of these dangers in relation to the notation employed in computer algebra systems. This notation of mapping is practically synonymous with the practical notions of function, procedure, subroutine, method, etc., as employed in computer science and programming. In contrast to the abstract mathematical notion of a function, here the usual understanding is that the $f(x)$ in $f\colon x \mapsto f(x)$ stands for some effectively computable expression, i.e. basically a formula. This formula may take the form of several pages of nested subroutines etc., but it is usually understood as computable in a technical sense.

The approach of constructing functions in some basic programming language, whether ISETL or MAPLE or any other, has been explored and studied in depth by Dubinsky and his school [3]. In this article we come to some closely related observations. But our point of departure is entirely different, starting pragmatically: Teaching introductory collegiate level courses in calculus, linear algebra and differential equations at a large public university in the USA, the preferences and wishes of the client disciplines are of importance to the mathematics department. The clients are foremost a diverse collection of engineering programs. While very different in their individual needs and expectations, they all share the accountability via regular strict reviews by the accreditation agency ABET [1]. Consequently, the rate at which innovations are integrated into their curricula is much faster than that in typical mathematics departments. For example, since 1990 this author has taught practically all courses in fully mediated classrooms, in which all students have access to computers, usually shared by pairs of students. This is accompanied by routine teamwork and interdisciplinary curricula, and, to keep it all honest, full Internet access in most exams. For over a decade the preferred software for mathematical tasks at the calculus level were the spreadsheet EXCEL and the computer algebra system (CAS) MAPLE. Calculators were not considered professional tools by our engineers. In differential equations we kept, for many years, a truce with about equal numbers of sections using CAS and using MATLAB. The first linear algebra course relies predominantly on MATLAB. However, increasingly the engineers are asking that we start right away in the first calculus course with MATLAB instead of CAS.

After having spent the last decade fine-tuning the way CAS may be used to enhance the learning of calculus (which, of course, in turn includes substantial changes to the calculus curriculum), the first reaction is one of disbelief and shock. On one side one loses so many CAS features that have become integral components of our calculus courses. On the other side one has to face a completely different collection of potential problems such as heavily overloaded functions. Thus initially we saw it as out primary task to educate the engineers why their preference simply made no sense: "you can't have both integration of technology and demand that we use MATLAB in calculus". But after studying the details, we have begun to wonder whether the real demand is for a radically revised mathematics curriculum. They say MATLAB, but mean new mathematics courses. It is just that the desired courses are most simply characterized as being such that MATLAB can be integrated!

Discussions with colleagues at many institutions suggest that our situation is typical. Moreover, very few engineers and few mathematicians seem to be familiar with the pedagogical implications of adopting MATLAB versus adopting CAS as primary software for calculus level courses. Finally, the available research literature in mathematics education appears to lag far behind the incessant cycles of major innovations of either software.

This article aims at laying out some of the fundamental issues that relate the study of functions (as in calculus level curricula) to the opportunities offered and dangers posed by using either CAS or MATLAB as the primary supporting software at the first and second year college levels. We hope that this article will serve both as a reference for faculty who need to make adoption decisions and communicate with client faculty, and as a starting point for more quantitative education research that investigates the possible benefits and hazards of professional software.[1]

The next three sections survey some distinguishing features of MATLAB and CAS that relate to their use in the first college level courses (calculus, differential equations, and linear algebra). They address plotting, overloaded definitions, and encapsulating functions as objects. The final section concludes with some observations about the use of either software in different style calculus courses.

## 2 Functions as graphs in MATLAB and MAPLE

From an abstract point of view a function is a graph – it is defined as a subset of the Cartesian product of its domain and range that satisfies the vertical line test. Taking a closer look at a function means investigating its graph. In the centuries before easy access to computing technology became standard, much effort was devoted to finding this graph, i.e., tabulating pairs $(x, y) \in f$ that were published as logarithm tables etc. The drudgery of this task, given an $x \in \mathrm{dom}_f$, find a $y$ such that $(x, y) \in f$, motivated analytical approaches that described the function from the formula without having to tabulate large sets of such pairs. On the other hand, due to its repetitive structure, this task is ideally suited for automation. Indeed, the task of manually creating tables has almost vanished from our courses. But instructors at advanced levels deplore that when students encounter new complicated functions, operators, simulations, etc., very few now instinctively first think of (an abstracted) table of pairs. A comparatively benign, but indicative example is presented by the strange machine-generated graphs of $x \mapsto \sin(nx)$ as $n$ increases. Few students understand what is going on – apparently due to their lack of experience of first tabulating values and then transferring them to a plot. In MAPLE, the first interesting patterns on the standard domain $[-10, 10]$ appear at $n = 26$ and $n = 29$. See [8] for a detailed description of these phenomena that are due to aliasing effects.

From the pedagogical side one may consider as the most important learning goal that students develop a good intuition of when to trust a computer generated graph. Desired is an uncanny ability to apply appropriate tests that will either confirm that the graph is trustworthy, or that will refute it and explain why it can't be right. From this side, the sophisticated plotting routines of CAS such as MAPLE, but also the new `fplot` of MATLAB are less desirable: they employ adaptive procedures that increase the number of points as needed – but such decisions are not at all transparent, and failures are often hard to predict or to recognize. On the other hand, the classical way of plotting in MATLAB forces the user to explicitly construct the table of pairs of values to be plotted – albeit all of them at one time with a few keystrokes, not one point at a time. Typical syntax is `xx=-10:0.1:10; yy=sin(26*xx); plot(xx,yy,'ro')`. In this case, the user gets exactly what (s)he asks for. We found that students, who routinely have to explicitly think about what is being plotted, are much more at ease with new problem situations, especially at advanced levels and complicated applications.

There are many other features and specialized plots in either package that provide major learning opportunities or present major possible pitfalls. But this is not the place to address these individually.

## 3 Learning about mathematical structures: CAS versus MATLAB

At the elementary and secondary level students work with very few, very similar algebraic structures: They start with the commutative semi-groups of the natural numbers with addition and

[1]A preliminary version of some sections of this article has been presented at the ICTME 2 in Beirut, Lebanon, in June 2003.

multiplication. Slowly they develop working knowledge of the field of rational numbers, and eventually also a naive understanding of the field of real numbers. The algebras of functions considered at this level naturally inherit the commutative ring structure from their range of real (or rational) numbers. At the post-secondary level students are confronted with a rapidly growing collection of objects that exhibit different kinds of algebraic structures. All of a sudden $a \cdot b$ need not be equal to $b \cdot a$. Similarly, $a \cdot x = a \cdot y$ need not imply $x = y$ even if $a \neq 0$. Consequently, division by such objects is not admissible. Even worse, $a \times (b \times c)$ no longer needs to equal $(a \times b) \times c$. On the same single line of a calculation one may have to use that $c^{-1} = \frac{1}{c}$ while recognizing that $f^{-1}$ is not the same as $\frac{1}{f}$. For many students at this level it is a major challenge to become comfortable and fluent working in these different algebraic structures. Don't divide by a vector. Don't cancel a matrix (unless one has verified that it is invertible). Don't forget the lack of associativity of the cross-product. Closely related is the increased importance of paying attention to the domains of functions. A silly example is the function $f(x) = \log \log \sin x$ and the statement $\frac{d}{dx} f(x) = \cot x / \log \sin x$. In the context of functions of a real variable the domain of $f$ is empty, whereas the domain of its purported derivative has nonempty interior. More important are the different domains of `atan` and `atan2`, only the second one being a partial inverse of the change to polar coordinates. Another typical example of the importance of domains is that only for finite, square matrices $A \cdot B = I$ implies $B \cdot A = I$.

Now add to this tumultuous situation, in which students for the first time really have to sort out a multitude of different mathematical (algebraic) structures, the request of the engineers for MATLAB as opposed to CAS. We claim that the major CAS MAPLE and MATHEMATICA help as organizing tools, providing much needed stability. Neither one is as strongly typed as classical first programming languages like PASCAL and C, but both force the student to carefully distinguish between constants and variables, between expressions and (pure) functions, between scalar and vector quantities, between commutative associative products and noncommutative and/or nonassociative products. The feedback is immediate and unrelenting. An almost silly example defines a function `y:=x^2-6*x-7;` Differentiate `dy:=diff(y,x);` and solve to find the critical points `x:=solve(dy=0);` Any subsequent attempt to use the second derivative test `ddy:=diff(dy,x);` must fail as MAPLE does not take derivatives with respects to constants. But even earlier, the typical first conflict is the distinction between unordered sets (as returned by `solve(...);`) and ordered lists: The complex roots of a polynomial are naturally an unordered set. There is no meaningful way to extract a first and a second root. This instructor's experience is that after a short period in which students learn to accept the software's refusal to work with ill-defined commands, students very quickly raise all their work to such higher standards. In such classes, e.g. division by vectors becomes unseen very quickly, simply because the routine use of a CAS forms the habit of constant awareness of the algebraic structures.

Contrast this with MATLAB whose quest to minimize the number of keystrokes needed for any task is legendary. Almost all functions in MATLAB are severely overloaded. This means that there are many different ways in which the functions may be called, i.e. with different numbers of parameters and different types of parameters. Abstractly this means that the domains of most MATLAB functions are often nontrivial unions of sets, with nontrivial rules of which definition applies when. As a simple example, the sine of a vector is defined to be the vector of the sines of its components. In contrast, in a traditional class students learn to evaluate the sine function only on scalars! More caution is advised when calculating `exp(A)` for a matrix `A`. MATLAB interprets this as the matrix whose entries are the exponentials of the entries of `A`, which is different from the exponential of `A` as it is used in the first courses in differential equations and linear algebra.
Our favorite example is the interpretation of the symbol " \" for the binary function of left division. In the case of an invertible matrix `A` and a column vector `b` of matching length, `A\b` is beautiful syntax for the solution `x` of the linear system `Ax=b`. After all, at the end of the first course in linear algebra we want all students to think of solving linear systems to be a routine process that generalizes division of numbers. But what if `A` is not an invertible matrix? MATLAB would not want to waste the precious single keystroke " \". After all, there could be no confusion for the mature user. Arguably, the best interpretation of `A\b` in that case is the least squares solution $x$

of $\|b - Ax\|^2 \stackrel{!}{=} \min$. Indeed, in MATLAB the following is perfectly legal, and meaningful: After defining column vectors `v = [ 1 2 3 4 ]'` and `w = [ 1 4 9 16 ]'`, the command `v\w` yields 3.333. This is, of course, the slope of the line through the origin that best fits (in the least squares sense) the data points $(i, i^2)$, $i = 1, 2, 3, 4$. MATLAB does the thinking for the user – always interpreting to the best of its ability what an assumedly intelligent user must have meant. This is completely opposite to the behavior of all the standard CAS, which play dumb and return error messages that try to convey in which sense the requested operations are illegal.

Teaching first year college level courses with MATLAB can be done – and we have done it – but it is a major challenge. It requires a much higher level of mathematical maturity of the students, a constant high level of alertness by the instructor, and a willingness to constantly bend the rules that the textbooks proscribe. It is a fantastic experience to get a student to bite her/his tongue and smilingly say: "Sure, I can divide by a vector – as long as I understand what I want to do with it." But it is not easy to get to this level.

In direct contrast, both students and instructor take much more passive rules if supported by a CAS – the CAS becomes responsible for meaningful syntax, and even encourages trial-and-error approaches.

# 4 Functions as objects

At the beginning of the first calculus course students analyze one specific function at a time. It is a major step forward to work with generic functions. At the end of the first semester all students are expected to reliable differentiate any specific formula, with $x$'s, e.g., by repeatedly applying chain and product rules. But simplifying $(f \circ g)''$ is a completely different task, which most students at this stage cannot handle. In our experience, CAS can make a major difference in getting significant numbers of students to this higher level where they can work with generic functions `f` and `g` as above. On the other hand, it is unclear how a predominantly numerical package such MATLAB can be utilized effectively in such settings.

Both MAPLE and MATHEMATICA distinguish between expressions such as `y:=x^2` and pure functions such as `f:=q->q^2`. The fundamental difference is that one may "plug in" (substitute) a specific value for a variable in an expression whereas one evaluates a function at a point In MAPLE: `subs(x=3,y)` versus `f(3)`. A significant number of students who experienced only an algebra-oriented calculus course at their secondary school often have difficulties with pure functions: We have met numerous students who misunderstand the juxtaposition in $\sin x^2$, leading them to use the product rule when differentiating this expression to erroneously obtain $\cos x^2 + \sin 2x$. The same students typically also read $f(3)$ as "ef - three" as opposed to "ef-at-three" or "ef-of-three". It is here where the arrow notation employed by CAS for pure functions has its biggest impact: It immediately catches such misconceptions about functions, and allows all students to proceed to the next level.

Understanding that the primary operation on functions is function composition, we have found CAS a very effective tool for studying calculus topics addressing functions in general. The first major point is that a pure function need not make any reference to any input variable. E.g., `sin` is a function and its derivative is `cos`. This is very different from the way scientists differentiate one quantity always with respect to another quantity (the $b$ in $\frac{da}{db}$). Without requiring names for input variables, CAS invite one to do analysis in this new algebra of function. For example, in MAPLE `D(D(f@g));` immediately returns the desired output corresponding to $(f \circ g)'' = (f'' \circ g) \cdot g'^2 + (f' \circ g) \cdot g''$ – the test question so many graduates fail. Other test items on which we observed improved performance after systematic use of pure functions in CAS are of the form: "If $f$ is decreasing and convex, what can you say about its inverse?" (decreasing and concave), and "If $f$ and $g$ are both decreasing, what can you say about their composition $f \circ g$?" (increasing).

The predominantly numeric character of MATLAB makes it much harder to work with compositions of functions – just consider the difficulties of generating a table for the composition of two functions each given by a table. This immediately forces one to reconsider teaching interpolation, which arguably is much underrated in current curricula. A special case in the context of function composition is finding (a table for) the inverse of such a function. The easy answer is to simply switch the columns. However, this is a much harder problem if the sample points in the new domain are prescribed and are different from the sample output values supplied for the original function. Clearly, MATLAB takes one here into entirely different direction when addressing compositions of functions.

We note on the side that the symbolic toolbox in MATLAB does provide access to MAPLE – but at the high cost of cumbersome notation that distinguishes numeric and symbolic objects. In particular, the conversion of large symbolic expressions (that result from symbolic work) into executable function calls is a nontrivial obstacle for any beginner. On the other hand, MATLAB's function handles, and encapsulation of user defined functions into traditionally external "m-files" provide some intriguing material for further study regarding its impact on students beginning to manipulate functions as objects or as black-boxes.

# 5 Conclusion and outlook: CAS versus MATLAB in calculus

We claim: Traditional first year calculus courses are courses in algebra, at least as determined by the content of the best-selling textbooks published in the USA and distributed worldwide. For clarification we recall the standard distinction that mathematicians make between derivations and derivatives. Derivations are a purely algebraic object defined as linear operators (on an algebra) that satisfy the Leibniz rule $D(fg) = (Df)g + f(Dg)$. On the other hand, derivatives are analytic objects defined as limits via approximability by linear objects. Limits themselves belong to the topological realm, but in the context of derivatives they are usually defined in the setting of metric spaces.

Every calculus textbook that we are aware of proclaims an analysis point of view. Yet immediately after the first definitions of limits, practically all exercises on limits are entirely in the domain of algebra. This includes all those limits that can be evaluated after factoring and cancelling common factors. Even further, every limit that can be evaluated using Taylor expansions of real analytic functions (and anything using L'Hopital's Rule) is basically algebra. Differentiation and integration in the somewhat generalized algebras of elementary functions clearly amount to almost purely algebraic operations [5].

Most every mathematics instructor will include a fair dose of analysis in her/his lectures – but the students know what really matters is what is on the final exam. With this the ability of any CAS (operated by a comparatively unskilled data-entry person) to earn a grade of A on practically every traditional calculus exam, constitutes proof that the traditional calculus course is basically an algebra course. After all, algebra is basically all that CAS can do – the A stands for algebra!

Now add computer software to this course: MATLAB can add some pictures, and can perform some numerical checks of algebraic work – but by itself it is almost useless on most traditional exams. As such it remains an add-on that will not even interfere with exams. There is not even any need to discuss rules for what the software may be used for in the exam – and what has to be done by hand. On the other hand, a CAS reduces typical traditional exams to trivial data entry – as such it cannot be allowed on a traditional final exam without rendering them meaningless. Students understand this well. They generally will not invest much effort into mastering a tool that may not be used in exams, when it counts.

But our client engineers not only demand that our students use professional computer technology, but that we even integrate these into the courses (as opposed to "add-on". In view of the above discussion on how either software conflicts with standard goals and examinations, this can only be possible with a radically reformed curriculum such as [4], which no longer places almost exclusive emphasis on algebra. The first step, taken widely in the 1990s allowed for quite sensible integration of CAS into a reformed curriculum: Much of the new emphasis is on modeling, approximation etc. Routine symbolic calculations are done by hand, while messier computations that result from realistic models motivated by real applications are relegated to the CAS.

This leaves us to wonder about the next phase: Our engineers are very clear about their preference that on one hand CAS be replaced by MATLAB, and on the other hand MATLAB be integrated with the course content. We are looking at another major shift of the courses, and a further move away from algebraic manipulations that apply only to very narrow classes of functions, e.g., those that can be integrated in closed form. Any, even only cursory, look at modern engineering courses, and, even more, engineering practice exhibits substantially larger universes of functions than those studied in traditional calculus classes. Many of these functions are implemented as often quite large scale simulations – i.e., basically very large formulas – which often include many black boxes. At first sight, black box components that implement numerical solutions of differential equations may seem out of place in first year calculus. But upon some reflection, one notices that such black boxes are not very different from those utilized for generations in calculus such as roots, logarithms and trigonometric functions. It really does not matter if the building block is implemented as a logarithm table, or as an m-file in MATLAB. It is just a component function inside a bigger problem whose analysis demands calculus.

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## References

[1] `http://www.abet.org/` ABET, Accreditation Board for Engineering.

[2] M. Carlson, *A Cross-Sectional Investigation of the Development of the Function Concept*; Research in Collegiate Mathematics Education III, Conference Board of the Mathematical Sciences, Issues in Mathematics Education Volume 7; American Mathematical Society, (1998) 114–163.

[3] E. Dubinsky and G. Harel, *The nature of the process of function*, in E. Dubinsky and G. Harel, eds., *The concept of function.* MAA Notes **25** (1992).

[4] D. Hughes-Hallet et. al., *Calculus* (2001). Wiley, Boston.

[5] K. Geddes, S. Czapor, and G. Labahn, *The Risch Integration Algorithm*, in: *Algorithms for computer algebra*, (1992) 511–573. Kluwer, Amsterdam, Netherlands.

[6] J. Kaput, *Technology and mathematics education*, in D. Grouws, ed., *Handbook of Research on Mathematics Teaching and Learning*, (1992), 515–556. Macmillan, NY.

[7] M. Kawski, *CAS or MATLAB in first year collegiate math?* Proc. ICTME 2 (2003) Beirut, Lebanon (to appear).

[8] D. Hardin and G. Strang, *A Thousand Points of Light*, Coll. Math. J. **21** (1990) 406–409.

[9] D. Tall, *Functions and Calculus*, in: A. J. Bishop et al (Eds.), International Handbook of Mathematics Education, (1997) 289-325. Kluwer, Dordrecht.

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# USING HISTORY IN TEACHING

PIETER MARITZ

This talk deals with attempts by the author to employ, when the occasion arises, a combination of history of mathematics and certain aspects of cultural importance to elucidate undergraduate teaching of mathematics.

The point of departure is to learn from a history of mathematics that mathematics is created by human beings. Some of these human beings led quiet lifes, while others found themselves in invidious positions because of their political, cultural or religious agendas and affiliations. Students should not only learn about the great mathematical discoveries, but they should also be made aware of the lifes of some of the people behind these achievements.

## Cape Colony

The teaching of mathematics in South Africa at university level spans a period of about 130 years, while serious research in mathematics commenced about 80 years ago. As part of their policy of Anglicisation of the Cape Colony during the years 1806 – 1910, the English authorities introduced a number of Scottish ministers, teachers and academics to the Cape Colony. The first professor in the 'Arts Department' at the Stellenbosch Gymnasium/Stellenbosch College was the Scottish mathematician George Gordon, from November 13, 1874 until his sudden death on August 30, 1882, leaving behind his pregnant wife and three young children.

> In (Gordon's) favourite study, Mathematics, he had achieved a high standard, but his personality had also the charm of great variety of knowledge and singular sweetness and kindness of manner. He was ardently beloved by his students, who, after his sudden death, erected a handsome monument over his grave with an inscription on it as few teachers earn. (James Mackinnon: *South African Traits,* see [2, p. 21] )

Gordon's grave had been neglected over the years, but was repaired to a reasonable state in 2001, as will be illustrated by some photographs.

In 1892, Cecil John Rhodes, Prime Minister of the Cape Colony, appointed the Scottish mathematician and educationist Dr. Thomas Muir as Superintendent General of Education in the Cape. Coming to South Africa did not offer Muir any decided financial advantages, but he had in any case been advised to seek a warmer climate for the sake of his wife's health. Muir will be remembered as one of the greatest organisers and reformers in the history of Cape education. He was enthusiastic for the building of beautiful, almost palatial schools; "... to whatever little village you may go, you will find there no better building than the school", in Muir's own words. He was one of the most important public figures in Cape Town, until his retirement in 1915, the year in which he was knighted. Most of Muir's about 320 papers and books were on determinants and allied subjects. His *magnum opus* was a five-volume work *The Theory of Determinants in the Historical Order of Development* (London, 1890 – 1930).

Before his move to Cape Town in 1892, he had already published four books: the first one in 1874 on quadratic surds and continued fractions, the next one in 1878 on arithmetic, and then two on determinants, one in 1882 (to be revised and enlarged later by Willian H. Metzler), and one in 1890 (which was destined to grow into the first volume of his m*agnum opus*). When he came to South Africa, Muir was working on the second volume of his *magnum opus.* Due to the pressure of his official duties, however, he had to put it aside, but during the Anglo-Boer War, as a kind of relief from the strain of those days, he resumed his work upon determinants. Muir introduced the term 'wronskian' into the terminology of determinants. He was working on a sixth volume on determinants when he died on March 21, 1934. Terms like: *the Bauer-Muir theorem, the Muir-Ramanujan approximation of the perimeter of an ellipse, a well-known theorem of Euler and Muir,* ... are common in the modern literature.

That Thomas Muir was a man of distinction among his countrymen, was reflected by Lord Kelvin's highly extravagant laudation:

> Thomas Muir was not only the greatest living mathematician, but the greatest mathematician that ever lived. (*Cape Times*, March 22, 1934, Leading Article)

Photographs of Muir, some of his schools, streets named after him, and his grave will be presented.

## Real Analysis, Measure Theory and Topology

In teaching real analysis, measure theory and topology at third year level, the names of Bolzano and Cauchy, and of Luzin, Suslin, A.A. Lyapunov, and others from the Moscow School of Mathematics will surely feature.

Students are usually fascinated by the story of the Bohemian priest Bernard Bolzano's (1781 – 1848) life and work in Prague – that he was a 'voice crying in the wilderness' and that many of his results had to be rediscovered in the second half of the ninteenth century. He spiritually belonged to the Enlightenment, with the result that the Austro-Hungarian authorities became displeased with his liberal views. In December 1819, Bolzano was suspended from his professorship, forbidden to publish, and put under police supervision. His publications (from 1804) and manuscripts have revealed a wide range of remarkable insights. Bolzano gave the first topological definitions of line, surface, and solid, and he stated the Jordan curve theorem. He also constructed, in the early 1830's, the first example of a function everywhere continuous and nowhere differentiable. This example preceded by some forty years that of Weierstrass. Whether or not Bolzano and Cauchy had contact during the latter's stay in Prague, whether or not Cauchy had plagiarized Bolzano's definition of continuity, or whether or not Bolzano's work was just a curious dead-end in the historical maze, are interesting projects for third year students to work on [6].

> From an historical point of view, Bolzano's mathematical work raises an interesting issue: For all its originality and fruitfulness, it seemingly had not the slightest significant influence on the work or thought of any subsequent mathematician. It was either misunderstood, unread, or unknown until much too late to have any effect. ([9, p. 52])

A photograph of the Bolzano House in Prague (25 Celetná Street) will be shown.

A short introduction to the early years of the Moscow School of Mathematics with its famous mathematicians and mutual ructions in the uncertain years after the Revolution usually meets the students' approval. In courses on real functions, topology, set theory, measure theory and functional analysis it is in any case inevitable that the following names be mentioned: Egorov and his student Luzin, and some of Luzin's students, like P.S. Aleksandrov, Suslin, Uryson, Kolmogorov, Lyusternik, P.S. Novikov, L.V. Keldysh, and others. The importance of the work done by the Moscow School of Mathematics is of course well-known. However, it is also important to know how and why Egorov was pushed out of his positions by 'The Initiative Group for the Reorganization of the Mathematical Society', consisting of, among others, Lyusternik, Shnirel'man, Gel'fond and Pontryagin. In the years 1916 – 1917, Suslin discovered a class of sets different from Borel-sets (B-sets), and called them A-sets (sometimes also called Suslin sets). These sets were produced by means of the operation that Suslin proposed to call the A-operation. There are two versions of what motivated Suslin to call his sets A-sets: (1) in honour of P.S. Aleksandrov, and (2) to parallel the common use of B-sets for Borel sets. In a paper of Luzin's, published in 1926, Luzin called A-sets analytic sets, an act that filled Aleksandrov with great bitterness toward Luzin; that was perhaps the major cause of the deteriotion of the relationship between student and teacher. Luzin's trial in 1936 by the USSR Academy of Sciences was an integral part of Stalin's Great Terror of 1936-1937.

Famous mathematicians formed the interrogating commission at the trial. Of these, Lyusternik, Shnirel'man, and Gel'fond already belonged to the 'Initiative Group'. They were joined by Sobol'ev, P.S. Aleksandrov, Kolmogorov, Schmidt, Vinogradov, Khinchin, and others. The major charges against Luzin boiled down to the following: (1) he systematically gave undeserved favourable reviews of the works of other mathematicians; (2) he published his best works abroad while only minor papers were published in the USSR; (3) he misappropriated results of his own disciples, like those of Suslin, Aleksandrov, Kolmogorov, Lavrent'ev and P. S. Novikov. The campaign against Luzin was abruptly stopped sometime between July 11 and July 13, 1936, by somebody high up, perhaps by Stalin himself. A photograph of the *Luzin Tree,* taken in 1992 in the Department of Mathematics of Moscow State University, will be shown. Ludmila.V. Keldysh, wife of P.S. Novikov, was according to some sources, the only one of Luzin's students to remain faithful to him during the difficult years 1936 – 1937 when considerable pressure was applied on her to denounce him publicly in the framework of the campaign against him [5, 8].

One of P. S. Novikov's students was Alexei Andreyevich Lyapunov (1911 – 1973). The students are usually fascinated by the history of the Lyapunov clan, which goes back at least to the Times of Trouble in the Russian history. A. A. Lyapunov's *Convexity Theorem* (1940) on the range of a vector measure states that the range of a bounded measure with values in a finite-dimensional topological space X is compact, and in the non-atomic case, convex. The proof of this theorem in the one-dimensional case is a good exercise for the students, but what is more fascinating to the students is the phenomenal explosion of this particular theorem into a variety of fields, like in mathematical economics, optimisation theory, dynamical systems, time-optimal control problems, statistics, probability theorems, functional analysis, differential equations, and so on. One should be aware of the fact that a phrase like 'a theorem of Lyapunov' does not always refer to the *Convexity Theorem,* or to any other theorem of A. A. Lyapunov. It might just as well refer to a theorem of the famous A. M. Lyapunov, a nephew of both of A. A. Lyapunov's grandfathers. Of course, A. M. Lyapunov is famous for his work on stability in systems, and he is the only Lyapunov mathematician mentioned by some important scientific dictionaries and encyclopaedia. There is a general mix-up between these two Lyapunovs in the literature. Expressions like: 'Liapunov constants', 'Lyapunov functions', 'Lyapunov spectrum', ... all refer to the work of A. M. Lyapunov, while '... which proves Lyapunov's theorem' can refer to a result of any one of the two. Fortunately, it is usually quite clear from the context which one of the Lyapunovs is under discussion.

> It seems to be well forgotten that many of the first ideas in geometry, basis theory and isomorphic theory of Banach spaces have vector measure-theoretic origins. Equally well forgotten is the fact that much of the early interest in weak and weak* compactness was motivated by vector measure-theoretic considerations. ([3] )

A photograph of Lyapunov as well as some maps concerning the Lyapunov ancestors will be presented.

## Discrete mathematics, combinatorics

The magic square on the Passion Façade of the Sagrada Família in Barcelona is a good starting point to discuss the history, theory and classification of the 880 essentially different normal magic squares of order 4 [7]. This magic square on the Passion Façade (photographs will be shown) has some connection with the *Dürer square* in the famous engraving *Melancholia* (1513).

When the formula of Stirling for the approximation of n!, or the Stirling numbers of the first and second kinds, or some of his methods are under discussion, one is more than tempted to tell the students something about James Stirling's interesting life, his work and his disappointments. James Stirling (May 11, 1692 – December 5, 1770) was a member of an old Jacobite family, the Stirlings of Garden. He went to Balliol College, Oxford University as a Snell Exhibitioner in 1710, where he displayed an aptitude for mathematics, and became friendly with Sir Isaac Newton and many of the leading mathematicians of the day. He left Oxford in 1717 without taking a degree because of his refusal to take the oath of allegiance to the House of Hanover. Stirling went to the University of Padua in Italy, but on arrival there found that one of the prerequisites of a post was that the holder must be a Roman Catholic. So, Stirling found himself stranded in Venice without the necessary financial resources to enable him to return to London. Little is know about his life in Italy for the next five years, but he certainly continued his mathematical research. In late 1724, Stirling returned to London, where for the next ten years he taught mathematics and the Newtonian theories at an academy in Lower Tower Street.

During this period he was elected a Fellow of the Royal Society, and was recognised as one of the most advanced mathematicians both at home and on the Continent. He had made several original contributions to the literature of higher mathematics, enjoyed the friendship of Newton, and was corresponding on equal terms with most of the leading European scientists. In 1735, Stirling was asked by the Scotch Mining Company, whose headquarters were in London, to act as manager of their unprofitable mines at Leadhills. He successfully managed the mines for thirty-five years, during which time he transformed them into one of the most profitable industrial enterprises in Scotland.

With the aid of Allan Ramsay (the poet), who as a boy had worked in the mines, a library was instituted which is still in existence today – a photograph of this library will be shown. One of the practical outcomes of Stirling's association with the mines was his invention of a machine to blow the furnaces for melting lead and for the supply of fresh air to ventilate the mine workings. In the year 1752, at the request of the magistrates of Glasgow, Stirling carried out a survey and submitted proposals for the deepening of the upper reaches of the river Clyde, and which paved the way for the later schemes of such well-known engineers such as Goldburn and Watt. For his work, which he carried out gratuitously, he was presented by the Corporation of Glasgow with a silver tea kettle, of which a photo will be shown. The mining company built him a house (photograph) supplied with a well stocked library and wine cellar, and provided him with a carraige and a pair of horses. Stirling died on December 5, 1770, and was buried in Greyfriars Churchyard, Edinburgh. A photograph of the grave will be shown.

> This gentleman may be regarded as an excellent specimen of the Scotsmen of the last age, who began their course without patrons and without money, yet being well taught, and obliged to avail themselves of time and chance, their spirit of industry and address enabled them to surmount every difficulty, raising them to eminence, and commanding the esteem of all who knew them. (Ramsay of Ochtertyre, in *Scotland and Scotsmen of the Eighteenth Century*, see [4, p. 34] )

## Cultural break

The paintings by the Italians  Pellegrino Tibaldi and Nicolas Granello on the seven liberal arts (*Artes Liberales*) (several photographs to be presented), in the library of the monastery of San Lorenzo at El Escorial near Madrid are usually well received by the students. A short talk on this topic is usually given in a geometry or analytic geometry class, where they also learn that Pope Sylvester II and the mathematician Gerbert of Aurillac (~ 940 – 1003) was one and the same person. He was decisive in the establishment a new institutions of higher education at cathedrals and monasteries in the tenth century.

> Is Theodorus an expert in geometry?
> Of course he is, Socrates, very much so.
> And also in astronomy and arithmetic and music and in all the liberal arts?
> I am sure he is. ( From Plato's dialogue *Theaetetus*, see [1, p. 40] )

## References


1. B. Artmann, *The Liberal Arts*, The Mathematical Intelligencer **20(3)** (1998), 41.
2. G. F. C. de Bruyn, *Eerste professor 125 jaar gelede aangestel* (*First Professor appointed 125 years ago*), Matieland, Desember 1999, 20 – 21.
3. J. Diestel & J. J. Uhl (jr.), Vector measures, Mathematical Surveys 15, A.M.S., Providence, R.I., 1977.
4. W. D. Hendry, James Stirling 'The Venetian', Scotland's Magazine, October, 1965, 33 – 35.
5. G. G. Lorentz, Mathematics and Politics in the Soviet Union from 1928 to 1953, Journal of Approximation Theory 116 (2002), 169 – 223.
6. P. Maritz, The Bolzano House in Prague, The Mathematical Intelligencer 23(2) (2001), 52 – 55.
7. P. Maritz, The Magic Square on Sagrada Família, The Mathematical Intelligencer 23(4), 2001, 49 – 53.
8. P. Maritz, Around the Graves of Petrovskii and Pontryagin, The Mathematical Intelligencer 25(2) (2003), 55 – 73.
9. S. Russ, Bolzano' Analytic Programme, The Mathematical Intelligencer 14(3) (1992), 45 – 53.



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# TEACHERS ASSESSMENT: A COMPARATIVE STUDY WITH NEURAL NETWORKS

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## Introduction

A considerable amount of research has been done in the area of Teacher's Assessment, especially at the university level. Some of the studies focus on the diverse assessment strategies, and on different ways to obtain accurate results from carefully chosen questionnaires. The goal of most of these studies is to find the essential qualities of a teacher to be successful, from a student viewpoint, in an attempt to formalize our knowledge in this area. As in many social studies, we sometimes crash with the difficulties of trying to formulate rules that should apply to human behavior, which is beyond any doubts a complex domain to analyze.

In this paper, first we focus on a comparative analysis of different domains of data and also different technologies from Artificial Neural Networks (ANN). We choose examples of questionnaires from different universities, with different number of questions, also considering the impact on the prediction ability of the ANNs between domains of crude data and domains of pre-clusterized data. On the other hand, we compare not only different domains, but also different tools, mainly using Perceptrons and Hopfield ANNs. As a Result of this study we establish comparisons in order to determinate the best predictors, validate different questionnaires designs, and thus, optimize the assessment process.

## Background

According to this, an interesting approach has been done by some researchers in this area, focusing on exemplary teachers, analyzing their performance and expertness, specially their exemplary practices [1]. Most of these studies showed that it seems to be better to avoid experiments with no certain results, and that is preferable to adopt methods and practices learned from successful teachers [2].

In the Republic of Uruguay, some researchers have been working on this topic for years using different approaches and different evaluation instruments. In our case, some years ago a descriptive study was made as a first step on this research line [3], later another study analyzing data with mathematical tools like Multivariate Analysis [4], and recently introducing a different approach with Artificial Neural Networks, specially Multi-layer Perceptrons ([5] and [6]).

## Convenience of using ANN

Why using ANNs in connection with Teachers' assessment instead of other techniques? It is well known the ability of ANNs to recognize patterns in a wide range of areas. When the relation among data is difficult to represent as a mathematical function, because is not easy to find it, or the expression is too complex, an interesting choice is training ANNs with sets of data, and once it is trained, approximate the results using this tool, without having a symbolic expression of the function that relates the inputs with the output.

In our case, questionnaires are given to the students with several questions about the teachers, in order to let the students assess them in their qualities and practices, filling the questionnaires with numerical values from a fixed range, for each quality of the teacher. They also include a numerical value to express the global assessment for that teacher. Once finished, the vectors of numerical values of partial assesses are the inputs, and the global assess is the output. Although we don't have a mathematical expression that relates the inputs with the output, those statistical studies mentioned before showed that this domain of data remains stable for a considerable period of time, so training ANNs to approximate the function between them seems to be a good strategy.

In addition, they've also showed some patterns in the students behavior referred to the teachers' assessment. Despite the questionnaires are personal, students use to fill them in a way that could be defined as a collective behavior. This fact made us think in the convenience of introducing a different approach in this research line, by applying well known methods of recognizing patterns such as ANNs.

## Comparison of different domains of data

The first part of the study consists in a comparative analysis of different domains of universities and questionnaires. We choose examples of questionnaires from two different faculties from the Republic University of Uruguay (Economics and Chemistry), in one case with a relatively high number of questions, and in the other case, with a low number. The following table shows the characteristics of the two questionnaires:

TABLE 1
*Questionnaires for teachers' assessment*

| | Economics | Chemistry |
|---|---|---|
| ***Number of questions*** | 2 | 12 |
| Number of teachers assessed | 47 | 27 |
| Number of students participated | 1015 | 1165 |

As we can see, in Economy Faculty, only two questions were made to the students and 47 teachers were assessed this way with values from 0 to 5. A Perceptron ANN was trained with this data, and could approximate de global assess with a media relative error of 3.91%. We can see in the graphic real values (blue column) and absolute error (red column) for each teacher assessed.

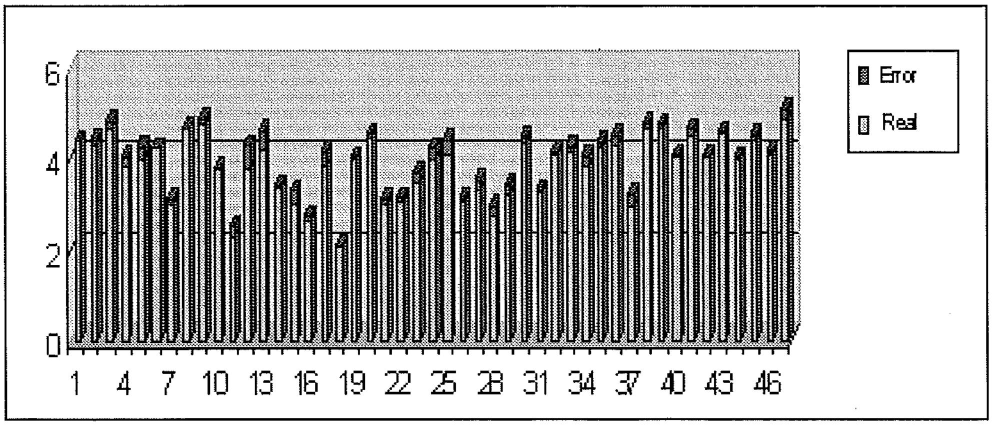


*FIGURE 1.* Real values and error in Economics Faculty

The same process was repeated for the other Faculty, in this case of Chemistry, with questionnaires of 12 questions for each of the 27 teachers assessed. The results are shown in the same way, with a media relative error of 1.33% as we can see in the following chart:

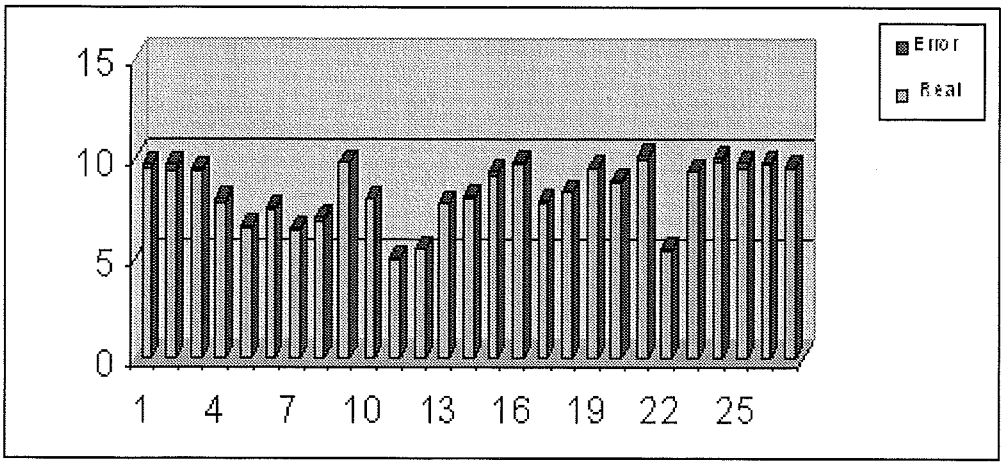


*FIGURE 2.* Real values and error in Chemistry Faculty.

As we could see, the more teachers assessed, the more tends to stabilize the relative error. Standard deviation of the error is low and decreasing, specially for Chemistry Faculty. Relative error in the two-questions version (3.91%) stabilizes in higher values than the twelve-questions version (1.33%) which suggest, that the Chemistry questionnaire has a better design, as exists some kind of function that relates the partial assessments with the global one. It can also be noticed that in the case of teachers with few students consulted (less than 10), the error in the approximation of the global assess increases, so the number of students assessed has to be taken into account, as we did in the case of Economy Faculty, with better results in the prediction ability of the ANN.

Let's see now a statistical analysis of both sets of data:

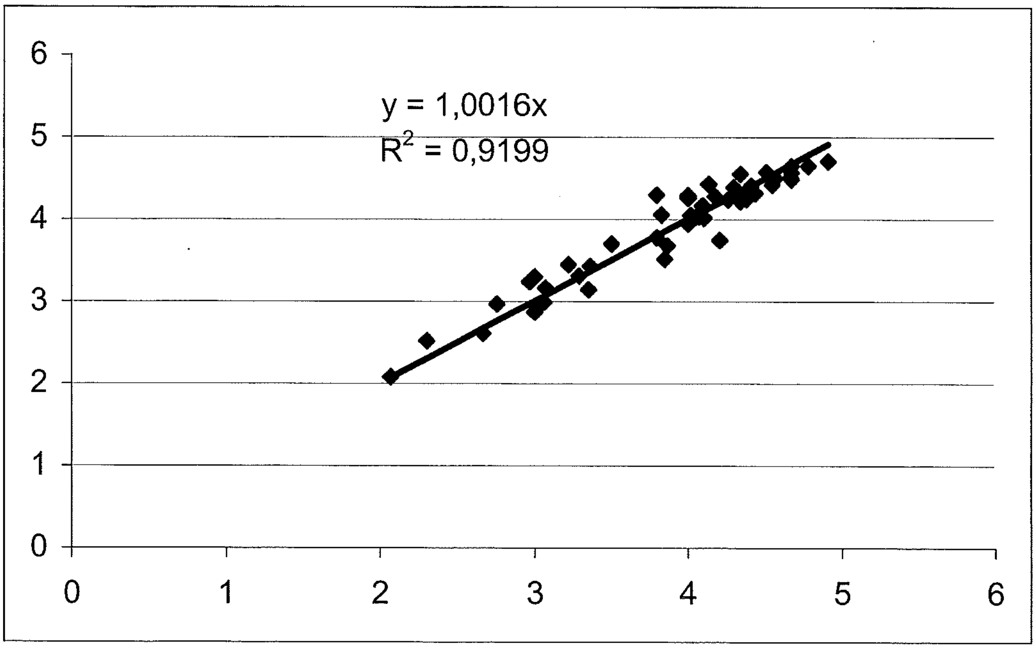


*Figure 3.* Linear co-relation (Economics Faculty)

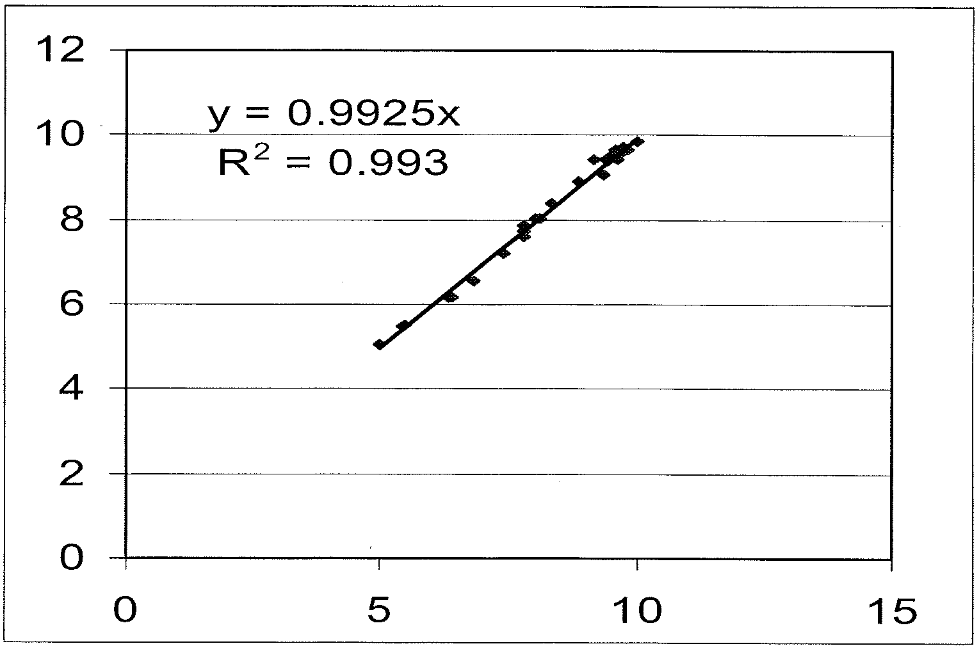


*Figure 4.* Linear co-relation (Chemistry Faculty).

The charts show values predicted by the ANN against real values in both institutions. The results can be seen in each chart. In both cases, co-relation coefficients obtained are considerably near 1, specially in Chemistry Faculty. According to Lothar Sachs, this let us establish that exists a linear co-relation between both sets of data [7].

On the other hand, we also applied the t test for linked sets of data analysis. To be able to apply this kind of test, the differences must be independent, from random sets of data with a normal distribution. The hypothesis of independence is guaranteed in our case, because different teachers were assessed, from different courses, different groups, and so on. The normality of the differences was also checked using the D'Agostino test (one of the most powerful ones) at the significance level of 5% [8].

Considering the $t$ test of linked sets of data, this test checks the null hypothesis of the differences. We calculate a statistical value $t$, and then compare it with the critical value obtained from the statistical tables. In Economics Faculty, the statistical value was 0.586, considerably lower than the critical value upon 2 (with a significance level of 5%) , therefore, there are no significant evidences for reject the null hypothesis, that is, that the two sets of data are equal from a statistical point of view. In Chemistry Faculty, the same analysis conduces to a $t$ value of –2.647, while the critical value is 2.05, therefore, in this case, the experiences cannot be considered equal at the significance level of 5%, but it could be possible to arrive at that conclusion with a more restrictive significance level, of 1%. This probably is a result of the lack of variability among data, with few teachers and high number of questions for each. This, also made us take into account the importance of analyze data in institutions with large number of teachers and reduce to a reasonable basic set, the number of questions included in the questionnaires. Recently, these techniques are being implemented to analyze historical data of teachers' assessment in ORT University of Montevideo, Uruguay using ANNs.

## Comparison of ANNs

Another important thing to consider in the ability of the different ANNs of approximating results for this particular case. There are different kind of ANNs, with different topologies and also different algorithms of convergence. We analyzed Madalines, Perceptrons and Hopfield networks as some of the most important of the ANNs, in order to determine the most suitable tool for this area, implementing them in $C^{++}$ language instead of using existing software, in order to be able to optimize several parameters and adapt the architecture for our particular domain of data.
Madaline ANNs are well documented but mostly using boolean entries, so it would be difficult to adapt

our continuous domain to this schema. Multi-layer Perceptrons are recommended for static domains, a good reason to choose it and train it with our set of data. Another reason was that this type of ANNs has strong mathematical support for the convergence algorithm, called Delta Rule. Hopfield ANNs are also popular for recognizing patterns, although an important constraint is that the amount of variables must remain low. This forces us to do a pre-clusterization of data before training the net, with the risk of loosing information that is only available on the crude data set.

## Use of Multi-layer Perceptrons

We designed the architecture of the Perceptron with three layers which is the most common choice in this kind of ANNs. Each neuron receives information from the previous layer and sends it for the next. When a neuron receives information, makes a pondered add of all the inputs and their weights assigned in the neuron. The result is transformed before passing it to the next layer. Sigmoidal function is used as the transformation function, because as we mentioned before, Perceptrons use a learning rule called *Delta Rule*, which is based in determine the propagated function error, and for calculating the gradient of the function, it is necessary to use a differentiable function as the Sigmoidal, one of the most often used successfully in Perceptrons [9].

We also used a training coefficient, to minimize the oscillations of the solution predicted. Because oscillations are the main problem to solve in Perceptrons, we also took into account the previous state of all the neurons, and by monitoring the rate of change of the weights of each neuron, we can accelerate the convergence when we are still far from the solution, and decelerate the process when its near, once again avoiding oscillations.

Randomizations of the initial weights, and also the load of data along the training epochs was also made, to increase de inference ability of the Perceptron, which is also a know method in this ANNs. We trained the three-layer Perceptron (13-8-1 neurons) for some thousands of cycles (called epochs) until the error stabilized around the values mentioned before, and so the train process stopped. If not, oscillations take place, and we could not arrive to the best approximation, when the error function is minimized.

## Use of Hopfield ANNs

This kind of ANNs has only a single layer, where all the neurons are fully connected each other. In the case of Hopfield, we used the Ramp function instead of the Sigmoidal, as we don't need to differentiate the function in this kind of ANNs.

Different from Perceptrons, here we use symmetric weights between each pair of neurons, without auto-exiting, as the main characteristics of the Hopfield model. The adjust of the weights is also different from Perceptrons, and in this case is made asynchronically, once at a time, as it is also considered the best choice to train this ANNs [10].

As we did before in the case of Perceptrons, we also validated this ANN with testing-domains and later trained it with both sets of data from the two institutions considered in this work.

## Results of the comparison

As we could see from the results, Perceptrons seem to be more accurate in the approximation of global assesses. Hopfield ANNs have a great potential of memorization of patterns and are also suitable for the same domains, if data is pre-clusterized, but Perceptrons are better to inference new sets of data with a low level of error.

## Conclusions

As we could see, the use of ANNs showed that the number of questions included in the questionnaires has great influence in the results. A low number of questions could not be enough to obtain a global assessment with a functional relation with the partial evaluation given as an answer to the questionnaire. We could also see that it's necessary to test this relation with a large set of data to avoid high levels of error in the approximation. Several topologies of ANNs can be used in this area, specially Perceptrons, and if the data is clusterized in a few set of patterns, also Hopfield ANNs can be applied.

There are several uses that can be proposed for these techniques, like predict results based in partial evaluations, provide a powerful tool for decision support systems that could help to increase assessment precision, and also help in the evaluation of possible candidates in the educational area. As in other social areas, the use of non-linear computation like ANNs, is useful to detect anomalies, also to generate hypothetical scenarios, (in this case, defining "hypothetical teachers" and testing the impact of input-variations on the net output). The increased knowledge that can be obtained this way, can later be used designing adequate questionnaires in order to minimize the evaluations error by measuring only the important teachers' qualities, also adopting planned policies of promotion, and discussing the results of the assessment with them, getting a constructive feed-back in a context of continuous improvement.

## References


1. Lovitt, C., Clarke, D. *The winds of change are sweeping through Mathematics Educations.* Curriculum Development in Australian Schools, **3**, (1987) 37.
2. Kennedy, K.J., *National initiations in social education in Australia*, in G. Mc. Donald & B.J. Fraser, Eds. Issues in social education, Perth: Western Australian Institute of Technology. 1986.
3. Martinez Luaces, V.. *Matemática como Asignatura de Servicio: algunas conclusiones basadas en una evaluación docente.* Números. Revista Española de Didáctica de Matemática, **36**, (1998), 65-74.
4. Gómez, A. y Martinez Luaces, V., *Evaluación Docente utilizando Análisis Multivariado*, Acta Latinoamericana de Matemática Educativa, **15-2**, (2002) 1016-1021.
5. Martinez Luaces, F. & Martinez Luaces, V.. *Sistemas de apoyo a la decisión: un caso tomado de docentes de CC.EE.* , TELEDUC 2003 La Habana, Cuba. 2003
6. Martinez Luaces, V. & Martinez Luaces, F., to apear. RELME XVII , Santiago de Chile, Chile. 2003
7. Sachs, L. *Estadística Aplicada* Ed. Springer 1967. 263-264, 341-342
8. Martinez Luaces, V.. *Estadística Aplicada a la Ingeniería Ambiental* Ed. IMFIA 1999. 21-2248
9. Picton, Phil *Neural Networks* p. 37-46 Ed. Antony Rowe Ltd. U.K., 2000
10. Hecht-Nielsen *Neurocomputing p.96-98* Ed. Addison-Wesley, U.S.A., 1991



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# COGNITIVE DIAGNOSIS OF ALGEBRA ERRORS

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Automated fine-grained cognitive diagnosis in the domain of algebra skills was achieved with a taxonomy of algebra errors that exploits the structural relationship between an erroneous answer and the solution path that led to it. The approach has been evaluated by applying case-based reasoning to implement a system for standard algebra skill problems. This paper provides an overview of the error analysis that underpins the system, describes the design of the system and includes an evaluation of its performance.

## Introduction

Universities around the world continue to report declining levels of mathematical understanding particularly in the domain of algebra by students entering tertiary studies [4]. Hence, many universities have introduced diagnostic tests and increasingly these are being conducted on-line. This has exposed the need for dedicated diagnostic systems that have greater explanatory power than existing systems such as those described by [1]. The task of ascribing causes to errors made in the performance of cognitive tasks is called cognitive diagnosis [2]. Early research showed good results for tasks where only a single solution technique is available. However, for problems that can be solved by a number of techniques, finer-grained diagnosis can only be achieved if the system has a means of identifying which solution technique the student adopted, because different solution paths require different skills and hence expose the student to different error sources [5].

The current research developed techniques that automated a fine-grained cognitive diagnosis. This was achieved by developing a taxonomy of algebra errors that exploits the relationship between an erroneous answer and the solution path that led to it. The approach has been evaluated by applying case-based reasoning to implement a system for standard algebra problems. Case-based reasoning enabled us to use partial matching during search and retrieval. This is important because arithmetic slips could lead to an infinite variety of answers and hence prevent the system from ever finding an exact match for a student's answer.

This paper provides an overview of the error analysis that underpins the system, describes the design of the system and includes an evaluation of its performance.

## Diagnostic Testing at the University of Ballarat

Responses on a diagnostic test performed by students entering the Bachelor of Engineering program at the University of Ballarat in 1996 was analysed to discover preferences for particular solution techniques, inconsistencies in problem solving and factors that impact on these characteristics. The test was the same as that used at the University of Melbourne[7]. The 1996 diagnostic test was modified and presented to students in the following three years. The modified test, illustrated in Table 1, included the ten algebra questions from the original test, plus an extra twenty-three related questions at different levels of difficulty.

TABLE 1

*Diagnostic Test at the University of Ballarat, 1997-1999*

| Question | Question |
|---|---|
| 1. Evaluate: $\frac{2}{3}[-18+4\times 9-(6\times 8)]$ | 2. Express $2\sqrt{2}+4\sqrt{18}+3\sqrt{48}+5\sqrt{3}$ in the form $a\sqrt{2}+b\sqrt{3}$ |
| 3. Expand using positive indices only: $\left[\frac{3b^3c^{-2}}{5a^3b^{-2}}\right]^3$ | 4. Simplify: $2\log_{10}26+\log_{10}75-\log_{10}3-2\log_{10}13$ |
| 5. Factorise: $(x+3y)^2-y^2$ | 6. Expand: $(x+y)^3$ |
| 7. Solve for a in terms of b: $e^a=3b$ | 8. Solve for x: $x(x-5)=-6$ |
| 9. Solve for x: $\frac{3}{x}-\frac{4}{a}=\frac{5}{b}$ | 10. Solve for x and y: $x+3y=-6$, $x-3y=9$ |
| 11. Express as a single fraction: $\frac{3}{a}+\frac{3}{b}$ | 12. Write in simplest form: $\sqrt{a^2-b^2}$ |
| 13. Expand: $(a-b)^2$ | 14. Simplify: $\log_{10}8+\log_{10}5-\log_{10}2$ |
| 15. Simplify: $\frac{100!}{98!}$ | 16. Simplify: $\frac{x^2-4}{x^2+4x+4}$ |
| 17. Simplify using positive indices only: $5^{-1}+5^{-1}$ | 18. Simplify using positive indices only: $\left(\frac{1}{2}\right)^{-3}$ |
| 19. Simplify using positive indices only: $3^x\times 3^x$ | 20. Simplify: $\frac{x^2+3x+1}{x^2+3x+2}$ |
| 21. Simplify: $\frac{3}{4}\div\frac{2}{3}$ | 22. Write in simplest form: $2^x+2^x$ |
| 23. Write in simplest form: $a\sqrt{a}$ | 24. Write in simplest form: $\frac{x^3+3x^2+2x}{x^2+2x}$ |
| 25. Solve for x: $\frac{2}{x}=\frac{1}{3}$ | 26. Solve for x: $x^2-4x=0$ |
| 27. Solve for x: $\frac{x^2-1}{x-1}=0$ | 28. Solve for x: $2x+4<5x+10$ |
| 29. Factorise: $x^2+6x+8$ | 30. Solve for x: $e^{3t}.x=\frac{1}{3}e^{3t}+e^t+2$ |
| 31. Solve for x: $(x-1)(x^2-3x)=0$ | 32. Solve for x: $\log_a x=b$ |
| 33. Solve for x: $x^2=x$ | |

## Error analysis

For every question on the diagnostic test, each student's working and the resulting one-line answer were analysed. Answers that were produced by the same solution technique were grouped to enable the identification of structural similarities between answers that were produced by the same technique and structural differences between answers that were produced by different techniques. For example, the question Factorise: $(x + 3y)^2 - y^2$ can be solved by a variety of techniques. Common incorrect responses, the associated solution technique and the structure of the final answer are shown in Table 2.

TABLE 2
*Solution Technique and Resultant Answer for Factorising* $(x + 3y)^2 - y^2$

| **Solution Technique** | **Erroneous Answers** | **Answer Structure** |
|---|---|---|
| Template | $(x+3y+y)+(x+3y-y)$<br>$(x+4y)+(x+2y)$<br>$(x-3)(y+3)$ | 2 terms, both linear_bracket<br>2 terms, both linear_bracket<br>1 term, linear_cross_bracket |
| Generalised Distributivity | $(x+2y)^2$ | 1 term, quadratic_bracket |
| Expand and Factorise (Expansion step is performed by template or repeated application of the distributive law and prematurely halted) | $x^2$ and one of $[+6xy, +3xy, +9xy, +18xy]$ and one of $[+2y^2, +5y^2, +8y^2]$ | 3 terms, two are quadratic_pronumeral, one is linear_cross_pronumeral |
| Expand and Factorise (Expansion step is performed by generalised distributivity and prematurely halted) | $x^2$ and one of $[+2y^2, +5y^2, +8y^2]$ | 2 terms, both are quadratic_pronumeral |
| Expand and Factorise (Expansion step is performed by generalised distributivity, but collecting like terms involves the malrule: $+ny^2 - y^2 \rightarrow n$ and prematurely halted) | $x^2$ and one of [+3, +6, +9] | 2 terms, one is numerical and the other is quadratic_pronumeral |

The results of this analysis indicated that the two most important features for identifying a student's solution technique from the structure of the final answer are the number of terms in the answer and the structure of these terms (or term types). This information enabled us to construct a dedicated cognitive diagnostic system for algebra skills problems, the design and operation of which are detailed in the next section.

## Design of the diagnostic system

The automated system enables algebra problems to be set and classified, to receive student answers and to infer a student's solution technique from the structure of the final answer to a problem. From this, the system can also identify errors that the student made whilst executing their solution plan. The basis of the system's operation is a computational model of algebraic problem solving, which is based upon Polya's four-stage model of general problem solving as follows [6]:

- *Interpreting the problem*: The selection and relative weighting of features relevant for a given "Question Type" are determined. This is a classification task and the output from this stage is the value of the attribute "Question Type". For example, the factorisation problem $(x + 3y)^2 - y^2$ is recognised as a difference of two squares problem. Features such as the number of terms and term types are identified and assigned importance values.
- *Planning a solution*: Plans used by past students (who have interpreted the problem in the same way as the current student) are accessed. These are ranked by generating similarity scores between the current student's answer and those given in the past. This is a planning task and the output from this stage is the value of the attribute "Solution Technique". For example, applying the difference of two squares template could be the solution technique adopted.
- *Executing the plan*: The accessed plans are adapted in terms of the current problem and then executed in order to determine which one most adequately describes the error(s) that the current student has made. For example, surface features such as the values of coefficients, pro-numerals and powers are extracted from the question and are applied to the solution technique to generate the answer the student obtained.
- *Reviewing the process*: Plans that successfully represent the current student's misconceptions are stored for later use along with an explanation of any errors. The output from this stage is the diagnostic report.

The final system has two main modes of operation: question entry/classification and error diagnosis. These tasks are each performed by a dedicated case-based reasoner, but both tasks also require the system to represent mathematical expressions in a unique manner. The parser performs this latter task. The three main system components are now discussed.

## Parser

To reduce redundancy within the diagnostic system and to enable the use of string-matching functions, each input string was converted into its equivalent form where the operands of each commutative operator are ordered by ASCII value. The reordered expression is then stored as part of the final data structure. The input to the parser is a single expression (e.g $(x + 3y)^2 - y^2$) and the output is a data structure comprising the original string ('(x+3*y)^(2)-y^(2)'), the ordered string ('+(+3*y+x)^(+2)-y^(+2)'), the number of terms (2), and a list corresponding to each decomposed term. Each of these term lists contains the ordered string for the term, its sign, the number of sub-terms, the value of the coefficient, the number of pro-numerals, a list containing each pro-numeral and its power, the number of bracketed terms, and a list containing each bracketed term and its power (e.g.,
[ '+(+3*y+x)^(+2)', +, 1, 1, 0, [], 1, [+3*y+x, +2] ]).

## The Question Classifier

Within the system, algebra questions have the basic structure [keyword, expression, pro-numeral, correct_answer], where the feature "pro-numeral" is only required for equation-solving questions and is ignored otherwise. Two questions are identical only if there is an exact match on all of these features. The system maintains an index file of every question that it encounters, so that classification only needs to take place once.

If a question has no exact match in the index file, the system must allocate it a case number and perform the classification. Classification of a question is a multi-stage process. Initially, the parser decomposes a question into the following set of features: keyword, expression, number of terms and the term list for each term. Because questions that have different keywords require different sets of features to be used for representation, the classifier is currently structured in three sub-libraries of "atomic" Question Types, one for each keyword. A new question is compared with all the cases in the relevant case base and a similarity value for each case is generated using nearest neighbour matching. The highest-ranking Question Types are then added to the appropriate examples index file.

For example, the first step in classifying a factorisation problem is the identification of a common factor (whether this is numeric or algebraic). The presence or absence of a greatest common divisor of coefficients, a common algebraic factor and a common bracketed term are detected and represented as Boolean features (gcd_coeff, common_pron and common_bt). The second step is to identify the type of each term (e.g. numeric, quadratic, cubic etc.). A list of term types is generated and the query is represented by the vector [number of terms, term types, sign list, gcd_coeff, common_bt, common_pron]. Thus the expression $64x^2 - 16(x + 2b)^2$ is represented as: [Factorise, 2, (quadratic_pro-numeral, quadratic_bt),(+,-),0,0,0].
At this point, test entry and classification are complete. The second stage of case-based reasoning occurs during the diagnostic phase that occurs when a student answers one or more questions incorrectly. This is discussed in the next section.

## The Diagnoser

The diagnostic component uses two sets of inputs (the ordered hypothesis list of Question Types generated by the classifier and a set of features extracted from the student's answer) and produces output including the solution technique, a complete worked solution for the problem and diagnosis of associated errors.

For each question that has been incorrectly answered, the student's answer is sent to the parser where the relevant set of surface features is extracted. This set of features is used to match the current answer with others resident in the case library. The features are determined by the attribute Question Type and include the number of terms in the answer, the types of terms and the coefficients and signs on each of the terms. For each question type, the Solution Technique case library contains an exemplar case that contains a sample question and all the answers that have previously been obtained.

Once the query for a single problem has been generated, the system retrieves and opens the exemplar case corresponding to the most similar "Question Type" from the hypothesis list. If the question under investigation is its own exemplar, then exact string matching can be used to determine which answer in the exemplar case corresponds to that given by the current student. If the query problem is not an exemplar case, the system uses the value of "Question Type" to retrieve and open the highest ranked exemplar case from the hypothesis list generated during the classification phase. Each answer in the exemplar case is associated with a particular solution technique. The system compares the structure of each answer in the exemplar file with that of the student's answer by using the feature sets extracted from each of these answers. It then ranks the exemplar answers using nearest neighbour calculations, and finally it chooses the nearest match. We now provide details of these calculations in terms of an example.

Consider the problem *Factorise* $(2a + 3b)^2 - 25b^2$. The associated exemplar problem retrieved is *Factorise* $((x + 3y)^2 - y^2)$. Suppose that for this question a student produced the following working: $(2a + 3b)^2 - 25b^2 = (2a + 3b)^2 - (5b)^2 = (2a + 3b - 5b)^2 = (2a - 2b)^2$. This answer has the features number_terms = 1 and term_types = [quadratic_b]. However, if this feature set alone were used to match the student's answer with those resident in the case base, the system would fail to distinguish between variations of the solution technique (what is referred to as generalised distributivity in [3]. For example, the answer above could be further "factorised" to give either of the two common answers $2(a - b)^2$ or $4(a - b)^2$, both of which have a single term of type quadratic_b. Therefore, to achieve better discrimination between answers, the system also decomposes the student's answer to obtain a list of term signs, a list of the number of sub-terms in each term, a list of term coefficients, a list of the pro-numerals (and their powers) for each term, and a list of the bracketed terms (and their powers) contained in each term. The first two features (the number of terms and the term types) are the most important in matching answers and hence are given the greatest weight. The other consideration in generating a similarity score is to determine how features are compared and differences are calculated.

Assessing the similarity between query and case values for integer features is performed by applying a symmetric quadratic function to the absolute value of the difference between the case and query values. The same method is used for the lists that comprise the fourth, sixth and eighth features. A symmetric table of similarity values was derived to represent the similarity between different symbolic types. For example, a term that has type linear_pro-numeral is considered to be more similar to a term that has type quadratic_pro-numeral than to a term that has type cubic_pro-numeral.

Answers that have been obtained using the same solution technique have very similar if not identical structure. For the above example, the stored answers most similar to that given by the student are the three that were obtained using the generalised distributivity approach, each of which returned the maximum similarity value of 1. Therefore, the system must have a means of distinguishing between minor variations. For this reason, each answer stored in an exemplar case has an associated adaptation (or generative) mechanism. All of the candidate solution techniques (i.e. all of those with the highest similarity value) are placed on a hypothesis list. The system cycles through the candidate solution techniques on the hypothesis list and instantiates the associated generative mechanisms in terms of the query case (the problem). These are then executed and the emergent answer string is parsed to obtain the ordered answer string. The system then performs string matching of the student's answer with that produced by the generative mechanism. The generative mechanisms can also trap each line of working; these are presented on screen to the student as part of the session. When there is an exact match between the student's answer and that produced by a generative mechanism, it is probable that the system has identified the solution method adopted by the student. However, it is often the case that different solution techniques lead to identical answers, hence it is best that the system enables the student to reject a proposed diagnosis. For this reason, the system presents the line-by-line working to the student for verification. If there is not an exact match, the system must invoke a dialogue with the student to determine the exact method that the student used.

## Evaluation

The case libraries contain 24 expansion and 37 factorisation problem types respectively (including NULL). To determine the optimal feature weights, the final system was trained using a set of 198 algebra problems (97 expansion problems and 101 factorisation problems). This set was chosen to include at least three examples for each case within the "Question Type" case library, including the NULL cases. The features required for matching problem types had already been determined, but the final weights that are assigned to particular features required fine-tuning.

When this was completed, the system's performance in classifying algebra problems was tested using a new set of 279 problems, containing 134 expansion problems and 145 factorisation problems. A minimum of 5 examples were included for each "Question Type" contained in the case libraries, and at least 5 examples were included for each compound factorisation problem type (i.e. those that can be correctly classified on structure but also contain a numerical common factor). The test set also contained some problems that did not exactly fit any cases in the libraries.

Two experienced secondary-level teachers were involved throughout the testing of the system. Their task was to identify irrelevant cases that were returned and relevant cases that were not returned. Classification of the 134 Expansion problems returned a total of 424 hypothesised types, whilst the classification of the 145 Factorisation problems returned a total of 374 hypothesised types. The data used to test the Diagnoser included problems of each "Question Type", answers corresponding to each answer contained in the exemplars and some answers that did not correspond to any known cases. The "answers" were generated in collaboration with the teachers. A total of 429 question-answer pairs were tested.

Accuracy of performance of the case-based reasoners was measured in three ways: correctness, precision and recall. Correctness refers to whether or not the system assigns the highest similarity value to the most on-point case. Precision is defined as the proportion of retrieved cases that is deemed to be relevant and the final measure, recall, is defined as the proportion of relevant cases in the case library that the system retrieved. Results for both the Classifier and the Diagnoser are illustrated in Table 3.

TABLE 3

*Performance measures for the Classifier and Diagnoser*

| Measure | Expand | Factorise | Value |
|---|---|---|---|
| Correctness | 94% | 93% | 62% |
| Precision | 71% | 80% | 64% |
| Recall | 77% | 89% | 94% |
| | *Classifier* Results | | *Diagnoser* Results |

The results illustrate that the system can classify algebra problems with a high level of accuracy. This was to be expected because of the hierarchical nature of the algebra domain and was important because the classification task provides the link between an algebra question and a student's answer. The diagnoser assigned the highest similarity score to the most on-point case for 62% of the queries. Precision was assessed at 64% (of the 1408 retrieved cases, 924 were relevant) whilst recall was measured at 94% (the library contained 979 relevant cases and 924 of these cases were retrieved by the system). However, closer examination of the cases that were not retrieved as similar indicated that the cause was incomplete implementation of generative mechanisms.

## Conclusion

This paper has presented a new approach to cognitive diagnosis within the domain of algebra skills. A cognitive diagnostic system for algebra that has greater explanatory power than existing systems was developed to test the approach. Previous approaches to cognitive diagnosis have achieved high levels of success when modelling the error-making processes for procedural problems (i.e., those that can only be solved by a single solution technique) such as multi-digit subtraction. This success resulted from the fact that the solution of such problems can be decomposed into a single set of skills, the execution of which is then analysed. However, for problems that can be solved in a variety of ways, it is important that the student's chosen technique is identified because different techniques require different skills and hence give rise to different sources of errors. If the solution technique is not identified, then cognitive diagnosis is limited to simply determining that an individual student has not mastered the given type of problem but cannot provide an explanation of how the student interprets it and attempts to solve it. This, in turn, limits the capability of an interactive learning environment in tailoring remediation and instruction to the needs of the individual.

Future research aims to apply the diagnostic method to other mathematical domains (e.g. calculus or trigonometry), to other scientific domains (e.g. physics), and also to other non-scientific domains that share some characteristics with that of algebra, viz. a well-defined and hierarchical domain, well-documented errors and solution techniques, and a structural relationship between an answer and the solution technique that led to it.

## References


1. J. Appleby, P. Samuels, & T. Treasure-Jones, *DIAGNOSYS - A knowledge-based diagnostic test of basic mathematical skills,* Computers In Education, **28 (2)** (1997), 113-131.
2. K. De Koning, J. Breuker, & B. Bredeweg *Cognitive diagnosis revisited* In Greer, J. E. (Ed.) Artificial Intelligence In Education, 1995 Proceedings of AI-ED95 Washington, DC: August, (1995). 115-122.
3. M. Matz, *Towards a process model for high school algebra errors* In Sleeman, D. And Brown, J. S. (Eds.) Intelligent Tutoring Systems. Academic Press, London. (1982) 25-50.
4. H. Mays, *A cognitive diagnostic system for explaining algebra errors* Unpublished Ph. D. Thesis, School Of Information Technology, University Of Ballarat. 2002.
5. S. Ohlsson. & P. Langley *Psychological evaluation of path hypotheses in cognitive diagnosis,* In H. Mandl & A. Lesgold, (Eds.) Learning Issues For Intelligent Tutoring Systems Springer-Verlag, New York. (1988) 42-62.
6. G. Polya, *How to solve it,* Princeton University Press. Princeton, NJ. 1948.
7. P. Swedosh, *Mathematical misconceptions commonly exhibited by entering tertiary mathematics students*, In Clarkson, P. (Ed.) Technology In Mathematics Education Melbourne: MERGA. (1996) 534-541.



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# AN INVESTIGATION INTO THE PERFORMANCE OF STUDENTS IN APPLIED MATHEMATICS AT THE UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

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The School of Computational and Applied Mathematics has this year (2003) embarked on a complete overhaul of the teaching of applied mathematics to students in the undergraduate years of study. The school offers courses in the following areas: Numerical analysis, optimisation, control theory, mathematical modelling, differential equations and mechanics. The motivation for this change is due to the consistently poor relative performance of black students in the third year of study. As a result of this poor performance the participation of black students in the honours and higher degrees programmes has declined. To improve the performance of all students, the School has in 2003 adopted a problem solving approach to teaching in all areas of study. In this paper we discuss some of the challenges we have faced as well as the responses of students to the change in teaching methodology.

## Introduction

At the end of each academic year, the School of Computational and Applied Mathematics at the University of the Witwatersrand, Johannesburg, reviews applications from students at Wits and other universities in South Africa for entry into the Honours Program (Fourth year of study). At the end of 2002 we noted that 82% of black students who applied for entry into the Honours program did not meet the minimum entry criteria. As a result of this disturbing situation, the School decided to analyze the performance of undergraduate students by race between 1997 to 2002. Some of the results from this analysis are indicated in Tables 1,2,3 and 4 and Figures 1, 2 and 3 below.

From Table 4 we observe that the percentage of black students shows a steady increase from 1998 toward normal demographic levels. Figures 1,2 and 3 indicate the distribution of marks of third year students between 2000 and 2002. We note that the performance of black students was on average, across all years of study, 11% worse than students from other race groups. This is clearly indicated in Tables 1,2 and 3.

Many factors have been attributed to the poor performance of black students in Universities, the predominant reason being poor preparation at high school, a legacy of Apartheid [1-3]. The damage done may be appreciated by the fact that in 2002 about 3000 of 20,000 high school matriculants with Higher Grade mathematics are black. Matriculation is thus a severe filter. A minimum mark of 60% in mathematics is required to gain entry into the Faculty of Science. We take the view that our intake is of high intrinsic ability.

TABLE 1

*Average Mark of Race Group In the First Year of Study*

| Year | White | Black | Indian |
|---|---|---|---|
| 1997 | 64 | 52 | 53 |
| 1998 | 64 | 48 | 52 |
| 1999 | 66 | 54 | 59 |
| 2000 | 65 | 53 | 59 |
| 2001 | 65 | 57 | 64 |
| 2002 | 65 | 49 | 57 |

TABLE 2

*Average Mark of Race Group In the Second Year of Study*

| Year | White | Black | Indian |
|---|---|---|---|
| 1997 | 59 | 57 | 55 |
| 1998 | 58 | 55 | 61 |
| 1999 | 66 | 55 | 62 |
| 2000 | 69 | 54 | 66 |
| 2001 | 58 | 58 | 60 |
| 2002 | 59 | 49 | 54 |

TABLE 3

*Average Mark of Race Group In the Third Year of Study*

| Year | White | Black | Indian |
|---|---|---|---|
| 1997 | 70 | 57 | 72 |
| 1998 | 64 | 45 | 62 |
| 1999 | 73 | 63 | 77 |
| 2000 | 66 | 57 | 72 |
| 2001 | 73 | 54 | 73 |
| 2002 | 66 | 52 | 67 |

TABLE 4

*Percentage of Students from Different Races In the Third Year of Study and Total Student Numbers*

| Year | White | Black | Indian | Total Student Numbers |
|---|---|---|---|---|
| 1997 | 41.4 | 48.3 | 10.3 | 29 |
| 1998 | 47 | 38 | 15 | 34 |
| 1999 | 45 | 31 | 24 | 42 |
| 2000 | 56 | 35 | 9 | 46 |
| 2001 | 41 | 46 | 13 | 54 |
| 2002 | 30 | 54 | 16 | 50 |

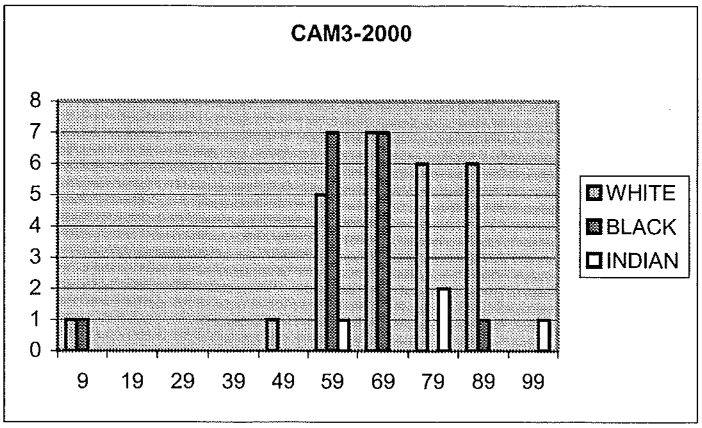


*FIGURE 1. Distribution of marks by race in the Third Year of study in 2000.*

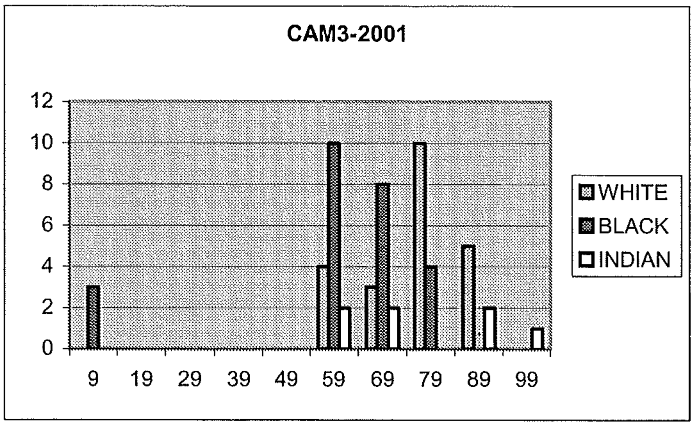


*FIGURE 2. Distribution of marks by race in the Third Year of study in 2001.*

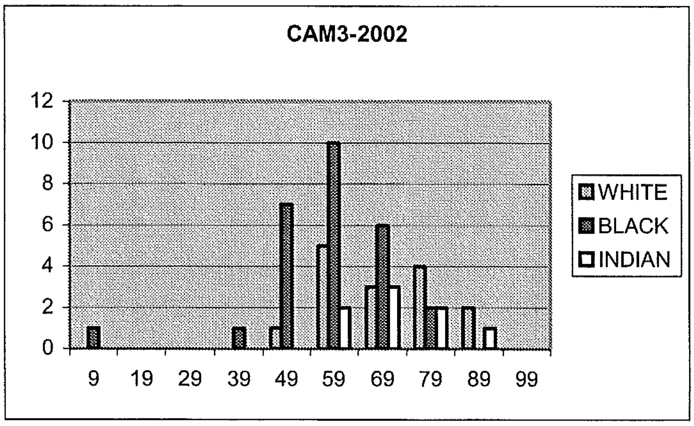


*FIGURE 3. Distribution of marks by race in the Third Year of study in 2002.*

## Teaching Situation Pre 2003

Before 2003 the School offered formal academic support to all students in their first year of study. The University centrally funded tutors who offered this support. In spite of this academic support, the performance of students across all race groups in the School did not significantly change (see Tables 1,2, and 3).

Most of the teaching taking place in the School was standard chalk and talk together with computer laboratory sessions. Solutions to tutorial problems were routinely handed out to students. Examinations usually included "seen" questions with little original content.

At the end of 2002 the School went on a retreat to discuss the poor relative performance of black students. Educationalists were invited. The basic problem was identified as the under-preparedness of students with respect to mathematics. Students were accustomed to rote learning. Their manipulation and logical reasoning skills were under-developed. Also, English is not the first language of the majority of our students. All teaching in the School takes place in English. The solution was to adopt a problem solving approach to learning on the grounds that black students had no exposure to this. It was felt that the problem solving approach would not place a big demand on all communication skills required in the first instance. Stress could be placed on reading and oral communication.

It was decided that this approach would be of obvious benefit to all our students. We note from Figures 1,2 and 3 that the results for white and indian students are skewed toward high marks. This is a clear indication that well prepared students were not being intellectually challenged. It is clear that by introducing more difficult problems to correct this that the mean for white students could be reduced without affecting the mean for black students. This would help to close the gap in performance and at the same time raise standards. The pass rates should remain acceptable while the global mark distribution tends to normal.

It was decided that lecturers were not to spend more time on teaching since this problem solving approach would ensure that students were active participants in their own education.

## Changes made in 2003

The School had a series of retirements and resignations before 2003. At the same time the university reduced the number of academic posts available to the School. The School adopted a new strategy of appointing staff to lecturer track positions and stopped academic support in 2002 as it appeared to have little impact on student performance as indicated above.

We interpreted problem solving as hands-on learning, rather than rote learning. Emphasis is placed on tutorials rather than lectures, in pursuit of active learning [4]. Across all years of study tutorial solutions were no longer handed out. Tutorials were structured to include at least one "hard" problem per section. This was done to challenge the brighter students and prepare students for more challenging examinations. At the same time adequate simpler problems were used to cement the basic ideas. Tutors agreed not to give solutions but guide students toward solving problems for themselves. All lecturers agreed to set at least one difficult examination question.

In First Year students were given a complete set of notes. They had to put themselves into groups no bigger than five or six. This led quite naturally to groups that were racially mixed. At the beginning of each lecture the lecturer would discuss with the students what that particular topic was about. Students then worked through the material on their own. At the end of the lecture period the lecturer summarised what was done and what the main points were. Most teaching periods are double periods (90 minutes long). One tutorial was held per week with two periods per week being allocated to computer labs.

What we noticed was that the lectures became like tutorials with the lecturer actively discussing many concepts with smaller groups of students. Lectures became very lively and enjoyable for lecturer and students, with students actively participating in the learning of the material. Weekly tests were held during the tutorial period so that students could have on-going feedback on their performance.

## Some preliminary feedback

The feedback we received from the first year students in lecturer evaluations was positive. Many of those who were fresh out of school enjoyed this way of teaching.

A similar approach was taken with the second year students. These students were familiar with the School's old teaching style. These students have resisted this problem solving approach to teaching.

The third year students were in addition directed to do extra reading. In some cases they did projects and assignments on material not directly covered in class. The feedback from the third years was very positive. They enjoyed this mode of teaching.

Initial lecturer feedback indicates that no additional time has been spent on teaching. This approach does however require more energy from lecturers and this has proved exhausting for many, especially in a big first year class. In many cases the rate at which we work through material has increased.

By November, we will be able to do a mark analysis by race to determine if this mode of teaching has made any difference to our distribution of marks.

## Challenges

One of the biggest challenges we have faced was from other Schools within the mathematical sciences. Some schools have been very supportive while others have not. In November, when we present our results to the other Schools within the Faculty of science we hope to convince many people that this was a good strategy to follow.

The open hostility of the second-year students to this problem solving approach to learning has been disappointing. We will lose many of these students as they do not have to take Computational and Applied Mathematics through to the third year of study. We expect a much smaller third year class in 2004.

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## References

1. M. Kahn & M. S. Rollnick, *Science Education in the New South Africa, Reflections and visions,* International Journal of Science Education **15(3)** (1993) 261-272.
2. F. Mumba, M. S. Rollnick & M. White, H*ow wide is the gap between high school and first year university chemistry at the University of the Witwatersrand,* South African Journal of Higher Education **16(3),** (2002) 148-157.
3. M. Kahn, *Some Policy Process and Implementation Issues facing science education and training in post Apartheid South Africa*, International Journal of Science Education **17(4)** (1995) 441-452.
4. Dale, E *Audio-Visual Methods in Teaching (3rd Edn)* Holt, Rinehart and Winston, 1969.

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# THE INFLUENCE OF SECOND LANGUAGE MATHEMATICS TEACHING ON CALCULUS PERFORMANCE

ANS NAUDÉ

This paper reports on the findings of a quantitative study concerning the supposed disadvantage of Afrikaans first language students who attend calculus lectures presented in English at the University of Pretoria. The motivation for the study and possible explanations for the results are discussed.

## Introduction

Mastering mathematics could be considered to be a two-step process: Firstly, one has to understand the mathematical concepts. Secondly, one has to be able to communicate these concepts using the language of mathematics.

A teacher or lecturer clarifies concepts using commonly spoken language and a subject-specific, scientific language. These explanations comprise an integral part of the first step. In the second step, the students have to familiarize themselves with the scientific manner of communicating the acquired concepts. The second step is especially important if one considers the fact that passing tests is normally strongly related to the students' ability to write down mathematical notions using mathematical notation. This interpretation supports the underlying notion of this study, namely that successful completion of a mathematics course relies heavily on two issues of language:

- Effective communication between the lecturer and student
- The student's ability to "translate" abstract concepts into written mathematics.

If the communication between the lecturer and student takes place in the second language of the student, effective communication could become more difficult.

## Background

Proper understanding of mathematical ideas entails a deep understanding of fundamental concepts (DelMas et al, 1994, Garfield & Chance, 1994, Richards, 1982, Thurston, 1995). In order to achieve this in-depth understanding (or "basic mental infrastructure" as Thurston calls it), he suggests that effective communication of mathematical ideas is the key. Language is an integral part of this communication. McLean (2000) goes so far as to say that

> ... these fundamental concepts are simply a problem of language. (p. 38)

Bohlmann (2001) also stresses the importance of language:

> It [language] is the medium by which teachers introduce and convey concepts and procedures, through which texts are read and problems are solved. (p. 6)

However, having to communicate these ideas in a second language adds numerous difficulties. In a citation of recent studies on second language learning in science, Rollnick (2000) rightly states

> ... it is acknowledged that expecting students to learn a new and difficult subject through the medium of a second language is unreasonable, giving them a double task of mastering both science content and language. (p. 100)

This double task entails the acquisition of two conceptually difficult and different skills at once – one being related to language and the other to mathematics (Bohlmann, 2001). Chen and Donin (1997) suggest that successful tertiary science learning depends more on domain-specific knowledge than on language proficiency. (Their findings were based on a study of Chinese speaking postgraduate students of Biology.

In this study the students studied texts in both their home language and in English.) According to Heugh (1999) the minimum vocabulary necessary to cope with English as instruction medium is 5000 words. After four years of second language teaching, a student would have acquired only about 800 words.

In the case of Afrikaans first language students in the South African schooling system however, students get formal exposure to English from the first grade and they complete at least the English Second Language curriculum up to grade twelve. Moreover, should they be proficient in both Afrikaans and English, their bilingualism could be an advantage in their studies, helping them to see different representations of a single idea (Rollnick, 2000, Bohlmann, 2001).

Proficiency in conversational English is not the only prerequisite for these students to master mathematics. They also need to be familiar with scientific English. According to Lemke (1990):

> ... the mastery of a specialized subject like science is in large part mastery of its specialized ways of using language. (p.21)

Mathematical English entails the use of abstract generalisations and logical relationships (Lemke, 1990) that both first and second language English students have to master. According to Rollnick (2000):

> ... the difference between everyday language and science or mathematics terminology also leads to first language speakers learning a new language when learning science. (p. 100)

She also refers to Sutton (1996), who claims that learning science is "... learning to talk in new ways..."

The first part of the underlying notion of this study as stated in the introduction was based on the discussion above. It should however be added that poor language proficiency could be the result, rather than the cause, of poor understanding (Bohlmann, 2000).

The second part of the underlying notion refers to the students' ability to write down mathematics. The written mathematics includes the genres of proof, definitions and theorems (Marais, 2002, Lemke, 1990, Wheeler & Wheeler, 1979). In mathematics students need to learn to formalise these concepts, using the *alphabet* of mathematical labels and symbols. According to Brown (1994)

> For someone learning mathematics there is a similarity with learning a language in that there is a need to grapple with an inherited mode of symbolization and classification, arbitrarily associated with some pre-existing world. (p. 142)

When one considers that reading and understanding mathematical text influences one's ability to study textbooks and completemathematics tests, the importance of proper mathematical literacy becomes evident (O'Toole, 1996).

## The South African context

Political changes in South Africa forced the language issue at universities under the spotlight. South African universities and governing bodies need to decide on policies concerning the language(s) in which lectures are presented. The outcome of this and similar studies will contribute to the making of informed decisions on this subject. Many of the traditionally Afrikaans universities (such as the University of Pretoria) now find the need to – at least partially – convert to English as a teaching medium. In an article on the performance of grade twelve students of 2002, (Mboweni-Marais, 2003, p. 15) it is suggested that grade twelve students, receiving tuition in a language other than their mother tongue, are at a disadvantage.

In reaction to this and similar reports, many prospective students and parents from the Afrikaans community are dissatisfied with this shift in teaching medium from Afrikaans to English, believing that having to change from Afrikaans-medium schools to English-medium tertiary education will negatively impact on the students' academic achievement. This has been one of the factors motivating some universities to present – as far as practically possible – separate lectures in English and Afrikaans. This solution is not efficient when one considers the extra financial and logistical burden it casts upon the university. Code switching (the teaching method by which the speaker switches between the first and the second language (Rollnick, 2000) could possibly be another solution, but this process is time consuming.

Referring to the literature cited, answers to the following questions would assist in leading policy makers to a reasonable solution:

- Is Afrikaans first language students' understanding of English adequate for the mastering of tertiary mathematics if they attend English lectures?
- Do Afrikaans first language first year students have sufficient understanding of mathematical concepts (resulting from primary and secondary education) to grasp more advanced concepts introduced at tertiary level?

In this study, the hypothesis that Afrikaans first language students have sufficient understanding of English to successfully complete a tertiary mathematics course is investigated.

## Objectives

The primary objective is to investigate the difference in performance of Afrikaans first language students who attend Afrikaans lectures and Afrikaans first language students who attend English lectures. Factoring out some of the influence of cultural background, previous exposure to mathematics education and the mathematical ability of the student (using a co-variate), a presumably fair comparison can be drawn.

As secondary objectives, the following comparisons are made:

- the performances of all students who receive first language lectures with that of all students who attend second language lectures.
- the performances of all non-Afrikaans first language-students that attend English second language lectures with that of all the Afrikaans first language students attending English lectures.
- the performances of all second language learners with that of all first language learners.

## Research Design

The results of a total of 836 first year engineering students of 2002 and 2003 were analysed. With each research participant a grade twelve mathematics result (Y) and a final calculus mark (X) were associated. The grade twelve mark was considered the co-variate in the ANCOVA-analyses. All students used an English textbook.

ANCOVA relies on the assumption that the datasets are homogeneous. When permitted, an ANCOVA-analysis was performed. In cases where the prerequisite was not met, ANOVA tests for difference in the observed means were performed. For ANCOVA a five percent level of significance was used. In the ANOVA analyses a one percent level of significance was implemented.

There are a (small) number of Afrikaans first language students who attended English high schools, thus familiarising them with the second language teaching medium. The influence of this could not be accurately assessed, since such data was not available. The abbreviations used in the reporting of the results are given in Table 1.

TABLE 1
*Abbreviations used in the Report*

| Description | Abbreviation |
|---|---|
| *Afrikaans first language students attending English (second language) lectures* | $A_1$ |
| *Afrikaans first language students attending Afrikaans (first language) lectures* | $A_2$ |
| *English first language students attending English (first language) lectures* | $E_1$ |
| *Other first language students attending English (second language) lectures* | $C_2$ |
| *All students that attend first language lectures* | F ($A_1 \cup E_1$) |
| *All students that attend second language lectures* | S ($A_2 \cup C_2$) |

## Statistical Analysis

Details on the composition of the sample are reported in Table 2 and Table 3.

TABLE 2
*Groups that Constitute the Sample*

| | $A_1$ | | $A_2$ | | $E_1$ | | $C_2$ | | Total | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year | 2002 | 2003 | 2002 | 2003 | 2002 | 2003 | 2002 | 2003 | 2002 | 2003 |
| n | 129 | 337 | 3 | 25 | 40 | 125 | 41 | 136 | 213 | 623 |

The regression coefficients for the classes are given in Table 4 and results of the preliminary hypothesis tests are given in Table 5. Table 6 contains the ANCOVA-results.

TABLE 3
*Groups that Constitute $C_2$*

| | 2002 | 2003 |
|---|---|---|
| Other (including Japanese, French and Chinese) | 9 | 15 |
| African | 32 | 121 |

TABLE 4
*Regression Coefficients*

| | $A_1$ | $A_2$ | $E_1$ | $C_2$ | F | S | Total |
|---|---|---|---|---|---|---|---|
| □ | 0.561 | 0.39 | 0.496 | 0.58 | 0.527 | 0.385 | 0.511 |

TABLE 5
*p-values for Hypothesis Tests for Equal Regression Coefficients*

| | $A_1, A_2, E_1, C_2$ | $A_1, A_2$ | $A_2, C_2$ | $A_1, E_1$ | F, S | F, $C_2$ |
|---|---|---|---|---|---|---|
| p | 0 | 0.671 | 0.068 | 0.029 | 0.995 | 0 |

□ = 0.05

TABLE 6

*p-values for ANCOVA*

| | $A_1, A_2$ | $A_2, C_2$ | F, S |
|---|---|---|---|
| p | 0.018 | 0.494 | 0.361 |

□ = 0.05

ANOVA was performed on the mean grade twelve mathematics and calculus results respectively (refer to Table 7), identifying differing means using the Sheffé post hoc comparison tests. These results are given in Table 8.

TABLE 7

*p-values for ANOVA*

| | $A_1$ | $A_2$ | $E_1$ | $C_2$ |
|---|---|---|---|---|
| Mean (Y) | 72.794 | 77.107 | 71.479 | 69.847 |
| p = 0.007 | | | | |
| Mean(X) | 62.547 | 61.071 | 57.703 | 59.073 |
| p = 0.001 | | | | |

F(□;3,832) □ = 0.01

TABLE 8

*p-values for Sheffé*

| | $A_1, A_2$ | $A_1, E_1$ | $A_1, C_2$ | $A_2, E_1$ | $A_2, C_2$ | $E_1, C_2$ |
|---|---|---|---|---|---|---|
| Y: p | 0.364 | 0.713 | 0.066 | 0.178 | 0.041 | 0.688 |
| X: p | 0.963 | 0.003 | 0.055 | 0.72 | 0.898 | 0.851 |

F(□;3,832) □ = 0.01

## Discussion of Results

ANCOVA-analyses revealed that at a five percent confidence level there seems to be no difference in the performances of the two groups of second language learners (Afrikaans and Other (mainly African)). This suggests that none of these groups is more likely to be disadvantaged due to second language instruction at tertiary mathematics level. Also, there is no notable difference in the performance of the group of all students attending first language lectures and those attending second language lectures. These results support the hypothesis that mathematics at the advanced level is a language in itself, which, once one is familiar with it, seems to be adequate to master the subject.

Although the hypothesis of equal performance of Afrikaans students attending Afrikaans lectures and Afrikaans students attending English lectures is rejected at five percent significance level, the relatively high p-value of 0.018 suggests that it is worth while to do more research into this subject, before making conclusions.

The ANOVA-analysis and Sheffé comparison test, however, showed that there seems to be no statistical difference in the performances of Afrikaans students attending Afrikaans lectures and Afrikaans students attending English lectures (either at secondary or tertiary mathematics level). It is interesting to note that Afrikaans students attending first language lectures performed significantly better than their English counterparts – in spite of the fact that their secondary mathematics performances do not differ significantly. A source for this variation might be the difference in lecturers' styles.

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## References

1. C. Bohlmann, *Reading Skills and Mathematics,* Communications: Third Southern Hemisphere Symposium on Undergraduate Mathematics Teaching, (2001) 5-14.
2. T. Brown, *Towards a Hermeneutical Understanding of Mathematics and Mathematical Learning.* In P. Ernest (Ed.), Studies in Mathematics Education (1994), The Falmer Press, 141-150.
3. Q Chen & J. Donin, Discourse processing of first and second language biology tests: effects of language proficiency and domain specific knowledge, Modern Language Journal, **81(ii)** (1997), 209-227.
4. R. DelMas, J. Garfield & B. Chance, A model of classroom research in action: Developing simulation activities to improve students' statistical reasoning, Journal of Statistics Education, **7(3)** (1994).
5. G.B. Garraway, Language, culture and attitude in mathematics and science learning: a review of the literature, Journal of Research and Development in Education, **27(2)** (1994), 14-20.
6. K. Heugh, *Languages, development and reconstructing education in South Africa.* International Journal of Educational Development, **19** (1999), 301-313.
7. J.L. Lemke, Talking Science: Language, Learning and Values, Ablex Publishing Corporation, 1990.
8. C. Marais (2002), *Comparing first years' reasoning styles in formal logic and other mathematical contexts*, Dissertation submitted for Master of Science, Mathematics Institute, University of Warwick, 2000.
9. K. Mboweni-Marais, *Moedertaal vuurpyl vir uitmunt in 'pynvakke'*, Beeld, 26 June 2003, 15.
10. A. McLean, *The predictive approach to teaching statistics,* Journal of Statistics Education,**8(2)** (2000).
11. J.U. Opoku, Learning of English as a second language and the development of bilingual representational systems, International Journal of Psychology, **18** (1983), 113-143.
12. M. O'Toole, Science, Schools, Children and Books: Exploring the Classroom Interface between Science and Language, Studies in Science Education, **28** (1996), 113-143.
13. P. Richards, *Difficulties in Learning Mathematics,* in M. Cornelius (Ed.), *Teaching Mathematics,* Biddles Ltd, Guildford, Surrey, 1982, 59-133.
14. M. Rollnick, Current Issues and Perspectives on Second Language Learning of Science, Studies in Science Education, **35** (2000), 93-121.
15. C. Sutton, *The scientific model as a form of speech* in G. Welford, J. Osborne & P. Scott (Eds.), *Research in Science Education in Europe,* London: Falmer Press, 1996, 143-152.
16. M. Swain & J. Cummins, Bilingualism and cognitive functioning and education in language teaching, Linguistic Abstracts, **12**(1) (1979), 4-18.
17. D.O. Tall, *Advanced Mathematical Thinking,* London: Kluwer Academic Publishers, 1991.
18. W.P. Thurston, *On Proof and Progress in Mathematics,* For the Learning of Mathematics, **15** (1995), 29-37.
19. R.E. Wheeler & E.R. Wheeler, *Mathematics: An Everyday Language,* John Wiley & Sons, Inc, 1979.

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# TERTIARY MATHEMATICS INPUT: CAN IT PLAY A ROLE IN PROMPTING TEACHERS' THINKING ABOUT TEACHING?

JUDY PATERSON

A group of senior mathematics teachers from four low decile schools took part in a number of professional development opportunities offered by the University of Auckland. The four schools are currently part of the Mathematics Enhancement Project that is operating under the auspices of the Mathematics Education Unit at the university. This paper considers the teachers' responses to two of the professional development afternoons. The stimuli for discussion in the two encounters were different. In one the teachers attended lectures on advanced mathematical topics from two University lecturers, in the other they watched a video of a member of the research team, who is an experienced high school teacher, teaching a year twelve mathematics class. This paper analyses the subsequent discussions and considers the possibilities of using mathematics content as a stimulus for fostering productive discussion of both content and pedagogy amongst classroom teachers.

This paper reports on the early investigation into a possible model of professional development for mathematics teachers. In this small pilot study the teachers' responses to two different forms of professional development intervention are explored. On two occasions a group of eight teachers attended a lecture on an aspect of mathematics with which they were not familiar. After the lecture the teachers were asked what they had found most interesting about the presentation. The lecture and discussion were recorded and transcribed and the responses analysed. On the second occasion they also viewed a video recording of one of the research team teaching a Year 12 class from one of the schools involved in the project.

In the investigation for which this is a pilot study the possibility of using mathematics discussions to encourage debate about learning and teaching will be explored. In this phase the question being examined is the extent to which placing the teachers in a situation where they are learners of mathematics gives rise to their voicing thoughts about learning, understanding and teaching. Comparison with the teachers' responses to the video of classroom teaching proved useful but is not the focus of the investigation for which this is a pilot study.

## Content based professional development

The use of a content base for professional development has the support of a number of researchers. "Content base" can mean a number of different things: learning mathematics, learning mathematics in a new manner or learning about the content pedagogy of mathematics. In all of these meanings however the teachers are engaging with the structures and connections that contribute to understanding mathematics.

Garet, Porter, Desimone, Birman, & Suk Yoon **[10]** in a large scale empirical comparison of effects of different characteristics of professional development on teachers, list content focus as one of the three core features of effective professional development activities. They state, "In particular, an emerging body of work suggests that professional development that focuses on subject-matter content and how children learn it may be an especially important element in changing teaching practice". This is supported by other researchers (**[3]**, **[6]**, **[15]**) who call attention to the importance of content focus in professional development.

Shulman **[16]** discusses the development of what he has called 'pedagogic content knowledge'. His work focuses on the particular understanding of content that teachers need in order to be effective. Focus on specific content knowledge and how students learn that knowledge has been identified as an important aspect of effective professional development by a number of researchers (**[9]**, **[13]**) Kennedy **[13]** states, "compared to more general professional development, professional development that focuses on specific content and how students learn that content has larger positive effects on student achievement outcomes, especially achievement in conceptual understanding."

## Teacher reluctance to discuss pedagogy

The aim of basing discussion of learning, understanding and teaching on the teachers' own learning is an attempt to support the development of a cohort of professionally critical teachers that is teachers who are willing to critique their own practice and to consider whether it is enhancing student learning. Teachers are frequently reluctant to discuss pedagogy **[3]** and are resistant to change (**[2]**, **[4]**, **[5]**, **[7]**) . It has been reported in the literature **[15]** that there is a lack of critical pedagogical reflection in many teacher discourse communities. This resistance may be because teachers enter the profession with a wealth of personal experience and strong views on teaching and learning, built up over a lifetime of learning experiences. Mathematics teachers, almost always successful learners of mathematics at school, frequently find critiquing the system in which they succeeded difficult. This can lead to a reluctance to consider the possible merit of different perspectives on teaching and learning. Furthermore once employed, the teacher is often isolated, particularly in schools where the opportunity to work with colleagues and foster a culture of on-going professional development is severely limited. In such situations opportunities to re-examine one's beliefs about learning and one's approach to teaching are rare.

## Teacher growth and development

The manner in which teachers view growth and change, in particular and in general, is influenced by a number of factors. The teacher, the environment in which they teach and the nature of professional development opportunities all contribute to determining the teacher's response to professional development. Professional development programmes are "systematic efforts to bring about change in the classroom practices of teachers, in their attitudes and beliefs, and in the learning outcomes of students" **[11]**. It is generally accepted that the relationship between the changes in these domains is complex and probably cyclical. He argues that there is evidence that the change in practice is often the initial change but acknowledges that "participants attitudes must at least change from 'cynical' to 'skeptical' for any change in practice to occur".

This research seeks to establish that the teachers found that the learning of new mathematics stimulated them into thinking about the mathematics, their own thinking, student learning and their teaching. The research for which this is a pilot will explore the possibility of developing and extending similar discussions to encourage teachers to examine their pedagogies in the light of their expressed views on learning and understanding.

While the importance of time in fostering change is frequently reported the results are, according to Pehkonen and Torner **[14]**, "partly contradictory" as it has been shown that significant change can be shown in short periods of time. It is suggested that stimulating opportunities to study new mathematics could act as catalytic energisers, giving the teachers the chance to re-view themselves as learners. Feiman-Nemser **[8]** argues strongly that "if we want schools to produce more powerful learning on the part of students, we have to offer more powerful learning opportunities to teachers."

## Two Professional Development Experiences

The Mathematics Enhancement Project is described in detail in Alangui, Autagavaia, Barton and Poleki **[1]**. A group of teachers from the schools involved attended two lectures given by University of Auckland lecturers, one on codes and the other on packing, and watched a video of one of the research team, an experienced school teacher, teaching a Year Twelve lesson on integration. The four schools in the project are decile one schools, from the Manukau region, Auckland, New Zealand.

On the first afternoon only fifteen minutes were available for discussion after the "Introduction to Encryption" lecture. This kept the feedback focussed as each participant only had two minutes to answer the question "I'd like to ask, if you could think quickly, what was the most interesting thing that you learnt, if you could just toss it out for me? Anybody go for it. Most interesting bit of that."

On the second occasion the afternoon started with a lecture on "Spherical packing in $n$-dimensions" and then two researchers and the teachers watched the video and discussed it. The feedback on this lecture was in response to the question "Can I get some feed back for L2?" and this feedback was obtained after the video discussion and became partly entangled with issues arising out of the video. The feedback cues for the video were less clear and more part of a general observation conversation.
It is clear that there is a difference between asking teachers to recall personal learning and thinking and discussing a video which can be re-run and is thus still in the present. However the difference between the mood and tone of the responses was so substantial that it appears that the two stimuli generated very different types of thinking on the part of the teachers.

## Data Collection

The data were collected by audio recording the group discussion. This led to the loss of some data because of over-talking. The recordings were transcribed and subsequently analysed by a single reseacher. Discussion with another researcher present during the interventions confirmed the observations made in this paper. It is intended that subsequent research will involve the collection of data in a manner which will allow the use of "multiple lenses" when examining it. **[12]**

## Results

The transcribed data was analysed and categorised. (See Table One.)

The detailed analysis supports the general impression of the researchers that the responses to the two interventions were markedly different. The teachers thoughts on the two lectures were almost all in Categories 1, 2 and 3 from the framework while discussion during and after the video focussed on Categories 4 and 5. The mood of the post lecture discussions was positive, the teachers sounded stimulated. The video discussion was more negative and will be discussed in brief later.

The discussion following the two lectures appears to have made the teachers see themselves as learners and to encourage their thinking about students' learning. In the following selections from the teachers' responses it is very clear that they are thinking about their own thinking, about how to use the ideas in their teaching, about what they know and what they could learn and about how it feels to be a learner.

> T5: I was just going to add on about sitting here and listening to somebody teaching or saying things, it's like being a student. You're there and there's somebody trying to explain. You're always looking for the one thing that catches you and then you start thinking and I think for me, kind of both the times, right at the end when he talked about the signals coming from Mars and that's something you always think how can a rocket go up and how can they communicate etcetera and when he said that that's when I started really listening and really started paying attention and then he ended so that's what students do, I think they sit there half asleep or just looking out the window but when they listen something and you capture them but its so hard in a classroom of thirty kids to have to say something that will attract them
>
> (...)
>
> T5: I think I just wait and I wait and I wait and then finally I grab something and that's it (laughing) I mean if he had started from Mars and then moved on

The following excerpt from the responses to the first lecture on encryption has numerous examples from Categories 1, 2 and 3 of the framework.

Responses to
"teacher own" are <u>underlined</u>,
"mathematics" are in **bold**
"teaching" category are in *italics*
"student" are <u>*underlined in italics*</u>. .

T6: I think the **modulus**. They're bringing back in school..

R1: Bringing the **modulus** back in and seeing where it was used OK. Anybody?

T7: Maybe for me is the last part where Paul talked about the **polynomial** making a story and giving it to the kids.

R1: OK

T7: But you've got to know it really well before and that its problem solving task putting them in groups and

R1: so using it as a problem solving

T5: just putting them in groups and see if they talk and

R1: mmhmm

T5: And can come up with the **graph**

R1: OK

T3: But you've got to know it really well

R1: (T4) what grabbed you?

T4: I don't know R1, my own ignorance grabs me (ha ha)

R1: Absolutely....

T4: I think I'll do it on **number theory**, I don't know **number theory** at all quite honestly, I'm always interested to hear some

R1: Yeah. (T4)

T1: Probably in everyday, probably the first two, first box – Who needs to keep a secret? It's very to the point

R1: So attaching to it

T2: Yeah so I can connect pretty quickly and I feel I have to stop you know, L1 says stop and I start thinking yeah well I stopped probably two seconds ago. Then I start thinking again and

T8: Even that example there. That when you're teaching **prime, the greatest common divisor** and all that with third and fourth form. I mean that's not too hard for third and fourth form

R1: No

T8: If you chose you know really small numbers

R1: Hmm and you could actually use it

T8 Because that's quite a neat little thing you know give them a secret code to

In marked contrast the discussion prompted by the video of classroom teaching focussed on themes from Categories 4 and 5. One focus of attention was on reasons why the investigative type of teaching being done on the video could only work "for that type of class".

T1: I don't know a lot of his of T5's class but I know two or three that have been up there are the people that you can rely upon to do well. J would know what he was doing, P's a bright man he would

T2: That boy of T5's pretty good

Or why it could not work with other classes.

T3: Not this year. And they're much the same. The fourth form's the giddy limit you can do nothing of this with mine. One of them you could do nothing like this.

It is possible that the fact that the teachers knew the particular students encouraged them to respond by discussing particular classroom situations. However even when encouraged to focus on the students' engagement with mathematics the teachers generally continued to dwell on particular teaching detail or systemic issues. It is not the contention of the researcher that the systemic issues were minor, simply that it seemed that in viewing the video the teachers had trouble seeing past them to consider the pedagogy. Even when one teacher brings the conversation around to pedagogy the flow is taken back to constraints of time and an over-full curriculum.

## Conclusion

The positive responses to the lecture format interventions encourages my continued exploration of the utilisation of mathematics content as a possible initiator of productive teacher conversations. In particular I am interested in its effectiveness when used with teachers who for various reasons are particularly resistant to considering the merits of pedagogies other than those to which they are accustomed.

It is also suggested that there are possibilities for collaboration between mathematicians and mathematics educators in the creation of post-graduate courses for teachers which focus on both the opportunity to learn new mathematics and to reflect on their learning and the implications both for the delivery of the content and the teachers' own pedagogy.

Table One

| **The teachers talked about the following** | |
|---|---|
| **In General** | **Specifically** |
| 1. Their own | Thinking<br>Understanding, knowledge, lack of knowledge<br>Learning<br>Ideas<br>Reasons for listening<br>Feelings |
| 2. Mathematics | Uses<br>Concepts and ideas<br>Metaphors, transformations and representations |
| 3. Teaching | Where to use ideas in the curriculum<br>Approaches<br>Presentation of ideas<br>Pedagogy |
| 4. Student | Learning<br>Motivation<br>Behaviour<br>Class management<br>Interest |
| 5. Constraints | Class size<br>Systemic issues<br>Time<br>Possible pedagogies |

## References


1. Alangui, W., Autagavaia, J., Barton,B., & Poleki, A. (2001). The Mathematics Enhancement Project: Combining research and development. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and Beyond.* (Proceedings of 24th Conference of Mathematics Education Research Group of Australasia, Vol 1 pp35 – 42). Sydney : MERGA
2. Ball, D.L. (1994). *Developing mathematics reform: What we don't know about teacher learning—but would make a good working hypothesis?* Paper presented at Conference on Teacher Enhancement K-6, Arlington VA
3. Ball,D.L. (1996). Teacher Learning and the Mathematics Reforms: What We Think We Know and What We Need to Learn. *Phi Delta Kappa 77* (7) 500-508
4. Beeby,C. (1970). Curriculum Planning. In G. Howson (Ed) *Developing a New Curriculum.* London Heinneman.
5. Brickner, D.L. (1995). *The effects of first and second order barriers to change on the degree and nature of computer usage of secondary mathematics teachers: A case study.* Unpublished doctoral dissertation, Purdue University, West Lafayette, Indiana . referenced in a report at http://www.math.purdue.edu/highSchool/technology/index.html (viewed October 2002)
6. Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms a learning environment for teachers and researchers. In Davis, R.B., Maher. C.A. and Noddings, N. (Eds), *Constructivist views on the teaching and learning of Mathematics,* (pp 125-146., JRME Monograph No 4, Reston, VA: National Council of Teachers of Mathematics.
7. Edwards, B. (2000). *Teachers doing what they're told? The fate of centrally mandated change.* Paper presented at Australian Association for Research in Education Conference. Sydney, La Trobe University.
8. Feiman-Nemser, S. (2001) From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record* 103 (6), pp 1013-1055
9. Fennema, E., & Carpenter, T.P. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27* (4), pp403-435.
10. Garet, M.S., Porter, A.C., Desimone, L., Birman, B.F., & Suk Yoon, K. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915-945.
11. Guskey, T. (2002). Professional Development and Teacher Change. *Teachers and Teaching: theory and practice, 8*(3/4), 381 –391.
12. Hoban, G.F. (2002) *Teacher learning for educational change:A systems thinking approach.* Open University Press. Buckingham Philadelphia
13. Kennedy, C.A. (1998). *Turning the Tables: Engaging Teachers in the Learning Process.* http://www.smeed.net/kennedyc/rsch/paper2.htm (viewed October 2002)
14. Pehkonen, E., & Torner, G. (1999). Teachers' professional development: What are the key change factors for mathematics teachers? *European Journal of Teacher Education, 22*(2/3), pp259-267.
15. Putnam, R.T. & Borko, H. (2000).What do New Views of Knowledge and Thinking Have to Say About Research on Teacher Learning? *Educational Researcher, 29*(1), pp4-15.
16. Shulman, L. S. (1992). Merging Content Knowledge and Pedagogy: An Interview with Lee Shulman. *Journal of Staff Development 13*(1), pp14-17.



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# MATHEMATICAL EXPECTATION: AN ANALYSIS OF THE OPERATION OF A PROFESSIONAL GAMBLING SYNDICATE

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The concept of mathematical expectation has a variety of practical applications and is central to the application of probability to decision making. The topic is now included specifically in many secondary school mathematics subjects, including the Queensland Year 11/12 syllabi. Earlier research by the author demonstrated that relatively sophisticated applications of this concept can be performed by students with relatively little mathematical background. Consequently, the author has developed a mathematical content elective unit in probability for BEd primary preservice teachers at Queensland University of Technology based around the mathematics of games of chance and mathematical expectation. The concept of mathematical expectations in a variety of gambling situations and games of chance is central to the unit. One application in the subject shows how professional gambling syndicates can operate with a positive mathematical expectation. This paper illustrates the development of the concept within the unit and includes a mathematical analysis of a report in The Brisbane Courier Mail, 3/10/00, "Betting sting on ice after TAB hit" by the class. The analysis shows how the syndicate operated with a positive mathematical expectation.

The concept of mathematical expectation has a variety of practical applications and is central to the application of probability in many decision making situations. The use of probability in decision-making is a topic that is now included specifically in many secondary school mathematics curriculums. Earlier research by the author (Peard, 1995) demonstrated that relatively sophisticated applications employing the concept of mathematical expectation can be performed by students with relatively little mathematical background in appropriate circumstances. Consequently, the author has developed a mathematical content elective unit in probability for B.Ed. primary preservice teachers at Queensland University of Technology based on this research.

Recent curriculum developments in school mathematics have seen a much greater emphasis on the role of probability in the classroom worldwide (Borovcnk & Peard, 1997). In Australia, "Chance and Data" has featured as a strand in the National Statement on Mathematics for Australian Schools, since 1991 (Australian Education Council, 1991), and the Queensland curriculum includes both topics in all years from 4 to 12. These changes have come about largely in recognition that students be taught how to deal realistically with uncertainties so that they may respond to probabilistic situations without preconceived notions, emotive judgments or even a lack of awareness that chance effects are operating. However, instruction in probability has been described as "a very difficult task, fraught with ambiguity and illusion" (Garfield & Ahlgren, 1988, p. 57), and many difficulties in its instruction have been reported in the Australian literature (See, for example, Peard, 1996a, 1996b; Peard, 2001; Truran, 1997).

Furthermore, the mathematical content knowledge of pre-service primary teachers has been a recent field of study of the author (Peard, 1998). Concern with the general mathematical competencies of pre-service primary teacher education students has been expressed for some time now (See for example, Carroll, 1994; Truran, 1997; Relich & Way, 1992). In an attempt to improve the general mathematical competencies of pre-service primary teachers, the pre-service B. Ed (primary) at QUT provides for the selection of a number of elective content units. The objectives of the probability unit include:

1. Understand the importance of game playing in the development of mathematical concepts.
2. Recognise mathematical strategies used in the playing of games of skill and formulate optimal strategies in simple games of skill.
3. Describe the role and importance of games of chance and other probabilistic concepts in modern mathematics.
4. Recognise the role of probability in simple games of chance and apply basic probability theory to analyse and make decisions in these games.

5. Determine mathematical fairness and compute mathematical expectations in games of chance and in gaming situations including: Casino, track, TAB and on-line betting. Compute bookmaker margins and betting profit margins.
6. Perform simple hypothesis tests in gambling and non gambling situations.

## The Content of the Unit

The unit begins with an informal approach, building on the students' intuitive understandings, interest in and familiarity with the probability without assuming any prerequisite knowledge other than the ability to convert fractions to decimals and percentages, fractional equivalence and basic operations. One of the major objectives of the unit is to ensure that the students do not hold any of the misconceptions about probability that are reported as common. These misconceptions, including the "gamblers' fallacy", are not confined to naive subjects and are prevalent among tertiary students (Shaughnessy, 1992) and pre-service teacher education students (Peard, 1996b). Key to the remediation of these misconceptions is the development of the concept of independence and mathematical expectation or expected return. The study of basic probability begins with the examination of the mathematics of simple games of chance and skill with the emphasis on the concept of mathematical expectation and its use in the decision making process in such games. Towards the end of the semester, more complex probabilities are introduced and the content extends to the applications of Binomial and Poisson probabilities using Microsoft Excel. This includes the use of these techniques in simple hypothesis testing.

## Different Approaches to Probability

The literature commonly identifies three different approaches to probability; classical (symmetrical or axiomatic), frequentist (experimental) and intuitive (See, Shaughnessey, 1992, p. 469). Difficulties are associated with each approach. Most introductory courses in probability begin with situations in which the outcomes are equally likely. In doing this there is an assumption of equal likelihood based on symmetry (coins, dice etc.) without the formal recognition of the axioms underlying such assumptions. The frequentist approach suffers from the clear difficulty that often short term frequencies give vastly differing results from long term, and to make inductive conclusions in probability is fraught with danger. Nevertheless it is important to include frequentist probabilities in any course as the applications of this occur in situations where there is no symmetry and the axiomatic probabilities might not be available. It is well documented that probability is the one field in which our intuition is at its least reliable. The familiar "birthday problem" is an example of the unreliability of intuition. Shaughnessy (1992) reported that "most people are surprised to learn that the probability (of at least two people in a group of 30 sharing the same birthday) is about 0.7 since this figure is counter-intuitive" (p. 479). "Monty's Dilemma" is another well documented situation of unreliable intuition (Borovcnik & Peard, 1997). Peard (2001) confirmed that this lack of intuition was widespread among a sample of tertiary students. Nevertheless, there are situations in which we need to use some intuition, based on prior experiences, in the estimation of probabilities. In insurance, for example, where the probabilities are clearly not symmetrical and there may be inadequate data from which to draw frequentist estimates, a degree of intuition is often used. In the analysis of the case in the present study, it is necessary to use both frequentist and intuitive estimations of probability.

## Mathematical Expectation and Expected Return

The concepts of mathematical expectation and expected return (ER) as the product of probability and the return or consequence, have a variety of practical applications and are key concepts to the application of probability to decision making. The decision of an airline to overbook flights, for example, involves computing the various probabilities of the numbers overbooked and forming the product of these and the associated cost of each eventuating. These are then compared with the probability and costs of empty seats. Other applications include insurance, warranties, restaurant overbooking, cloud seeding, and a variety of situations in gaming and betting. The computation of mathematical expectation (Ex) and expected return (ER) in different situations form a fundamental component of the unit.

## Expected Return

We begin the unit with some simple situations such as the analysis of the common casino game of roulette. If you bet say $10 on the "red" on a roulette wheel, we show that the probability of winning is 18/37. If you win you will receive $20 (your $10 bet plus your $10 win). We say that your expected return is the product of the probability of winning and the return from such a win.

$$ER = p A$$

where p is the probability of winning , and
A is the total amount of payment you will receive if you win.
In this case ER= 18/37 x $20 = $9.73

That is to say, in the long run you can expect to get back $9.73 for every $10 bet. Although on any one particular play you will get either $20 or nothing, the figure of $9.73 is what you expect to get on the average. Sometimes we refer to this as the mean or average return. This figure is better expressed as a % of the outlay. In this case:

ER = 9.73/10 x 100 % = 97.3 %

We then examine the playing of roulette and show that all bets on roulette have an expected return (ER) of 97.3%. This means that for every $100 bet, the house pays out $97.30 and keeps $2.70. In a lottery with several prizes, the expected return can be computed by considering the product of the probability of each and the amount of each. Mathematically,

$$ER = \Sigma p_i A_i$$

Alternatively, the expected return can be computed more easily as:

ER = the total amount paid out in winnings / the total amount taken in

## Mathematical Expectation

This is the probability of winning times the amount won less the probability of losing times the amount lost.

$$Ex = p W - q L$$

where p is the probability of winning and q = 1- p the probability of losing,
W is the amount won, and L is the amount lost.

Using the roulette example again:
p = 18/37, W, Amount won = $10
q = 19/37, L, Amount lost = $10
So Ex = 18/37 x 10 - 19/37 x 10
= -$ 0.27
This means that on a $10 bet you "expect" to lose 27c.
Or on any roulette bet you expect to loose 2.7%
Ex = -0.027 or - 2.7%

We show that if Ex = - 2.7 %, ER = (100 - 2.7) % and that in general:

$$ER - Ex = 1$$

Thus the two terms are measures of the same concept and either can be used. A negative expectation implies an expected return of less than 100%. The expected return of all Casino games (with the exception of blackjack) is determined. Since these are always less than 100%, (or negative expectation) there can be no long term system of winning. Thus, mathematically there can be no "system" of winning. The unit includes the use of this idea of mathematical expectation to analyse common gambling fallacies.

We conclude by showing that if ER<100% or Ex is negative, there can be no system of winning. Common misconceptions held regarding betting systems are explained and we then examine how professional gamblers and syndicates operate.

## Mathematical Fairness

Before proceeding we examine the concept of mathematical fairness and its relationship to gambling and games of chance in bookmaker betting, the setting of bookmaker odds and the determination of "fair" odds.

## A Fair Game

In everyday or colloquial use, this term can have different meanings. A football game is "fair" if there is no foul play. A roulette wheel is "fair" if each number has an equal chance of showing. A teacher is "fair" if all children are treated equally. “Fairness” does not imply equally likely outcomes, although when we use the term with reference to coins and dice, it is generally means this. In these situations it is better to speak of "unbiased" coins rather than "fair" coins. A mathematical definition of fairness can be formulated from expectation:
A game is fair if the total paid out by the losers is the same as the total collected by the winners. Each player (including the house, if any) has the same mathematical expectation.
That is to say, the Expected Return for all concerned is 100%. In this sense none of the Casino games examined are "fair", since the players' Expected Return is always less than 100%. On the other hand, the play of the games is always "fair" in the non-mathematical sense in that it is free from interference and the roulette wheels, coins, dice etc. are "unbiased".

## Totalisations Systems and Mathematical Expectation

In situations such as horse racing where the probabilities of the various outcomes can be only estimates generally based on both intuitive and frequentist ideas, the mathematical expectation of the bettor is best determined by considering:

ER = the total amount paid out/ the total taken in

We examine the operation of a totalisation system and see how it cannot lose since the % profit is subtracted before the winnings are distributed. The following example, using Microsoft Excel, shows a totalisation system in which $20 is bet on each of win and place, with a 15% take for the system.

Table 1
*An example of a totalisation system*

| Horse # | Win bets | Win pays | | place bets | place pays |
|---|---|---|---|---|---|
| | | | | | |
| 1 | $5.00 | $3.40 | | $5.00 | $1.13 |
| 2 | $1.00 | $17.00 | | $1.00 | $5.67 |
| 3 | $2.00 | $17.00 | | $2.00 | $5.67 |
| 4 | $1.00 | $17.00 | | $1.00 | $5.67 |
| 5 | $2.00 | $8.50 | | $2.00 | $2.83 |
| 6 | $1.00 | $17.00 | | $1.00 | $5.67 |
| 7 | $5.00 | $3.40 | | $5.00 | $1.13 |
| 8 | $3.00 | $5.67 | | $3.00 | $1.89 |
| total | $20.00 | | | $20.00 | |
| Win | | | | | |
| pool | $17.00 | | | $5.67 | (each) |

In the above example a total of $20 is bet on the 8 horses, 15% profit deducted and the remaining $17 is placed in the win pool to be paid to the winner according to the amounts in the adjacent dividend column. In the same way, in the place bet, the $17 is divided by 3 then paid to the three places according to the amounts in the column.

Note that it is possible that with such a system, it is possible for the return to be less that the bet. For example:

Table 2
An example of a totalisation system with a less than bet payout

| Horse # | Win bets | Win pays | | place bets | place pays |
|---|---|---|---|---|---|
| | | | | | |
| 1 | $10.00 | $2.13 | | $10.00 | **$0.71** |
| 2 | $1.00 | $21.25 | | $1.00 | $7.08 |
| 3 | $2.00 | $21.25 | | $2.00 | $3.54 |
| 4 | $1.00 | $21.25 | | $1.00 | $7.08 |
| 5 | $2.00 | $10.63 | | $2.00 | $3.54 |
| 6 | $1.00 | $21.25 | | $1.00 | $7.08 |
| 7 | $5.00 | $4.25 | | $5.00 | $1.42 |
| 8 | $3.00 | $7.08 | | $3.00 | $2.36 |
| total | $25.00 | | | $25.00 | |
| Win | | | | | |
| pool | $21.25 | | | $7.08 | (each) |

We can see that the place bet for horse #1 is paying $0.71. Prior to 2000, TAB Queensland had a rule that if less than 50% of the pool was bet on any one outcome and if, after totalisation, the dividend was less than $1, then it would be rounded up to $1. That is you at least get your money back if you get a place. This can greatly reduce the TAB's % profit as we can see in Table 3.

Table 3
*An example of a totalisation system with a minimum return equal to bet*

| Horse # | Win bets | Win pays | total pay | place bets | place pays | total pay |
|---|---|---|---|---|---|---|
| | | | | | | |
| 1 | $10.00 | $2.13 | $21.25 | $10.00 | $1.00 | $10.00 |
| 2 | $1.00 | $21.25 | | $1.00 | $7.08 | $7.08 |
| 3 | $2.00 | $21.25 | | $2.00 | $3.54 | $7.08 |
| 4 | $1.00 | $21.25 | | $1.00 | $7.08 | |
| 5 | $2.00 | $10.63 | | $2.00 | $3.54 | |
| 6 | $1.00 | $21.25 | | $1.00 | $7.08 | |
| 7 | $5.00 | $4.25 | | $5.00 | $1.42 | |
| 8 | $3.00 | $7.08 | | $3.00 | $2.36 | |
| total | $25.00 | | | $25.00 | | |
| Win | | | | | | |
| pool | $21.25 | | | $7.08 | (each) | |
| | | profit | $3.75 | | profit | $0.83 |
| | | % profit | **15.0%** | | % profit | **3.3%** |

In the above example $10 is bet on the horse #1 place outcome, resulting in a totalised return on this of $0.71. Since the $10 is less than 50% of the total bets ($25) the 0.71 is rounded up to $1 so that in the event of a place by #1, the $1 outlay is returned. Should #1 place, the TAB will pay out a total of $24.16 for a profit of only 3.3%. However, in reality this rarely happens because as betting proceeds and the payouts are displayed, few will place bets on #1 showing a return of $1 and other punters will come in and increases the bets on the other outcomes thus redistributing the various returns. Nevertheless, it was this regulation that enabled the syndicate to act in the case of the present study.

## Professional Gambling Syndicates

We have shown that all casino betting situations operate with a mathematical expectation for the punter ranging from about 85 % to 97.3 %.
Professional gamblers operate from two basic axioms:

- the mathematical expectation is greater that 100%
- short term losses can be covered

As such, professional gamblers do not bet at casinos (except for blackjack, which is discussed separately), but confine their betting to on track racing, sports gambling and lottery situations where jackpots can occur and greatly increase the expectation (See, Peard, 1998, for an analysis of one such situation). Professional syndicates also generally avoid totalisation betting. However the Brisbane Courier Mail, 3/10/00, reported "Betting sting on ice after TAB hit". The hit referred to was a set of bets placed by a Canadian syndicate on an obscure greyhound race that netted the syndicate a $170 000 profit. This was a classic case of an organised syndicate betting legally with a positive mathematical expectation. Just how this could happen is a mystery to most members of the public. Unfortunately, the reporting of events such as this tend to reinforce public misconceptions about "systems" of betting on situations rather than explaining how this can happen. Fortunately, the mathematical analysis of this situation is relatively simple and was examined in the unit.

## An Analysis of the "Sting"

In order to analyse how this occurred, we need to able to compute the mathematical expectation of the betting situation. If totalisation systems deduct their profit margin before distribution of returns, how could the syndicate bet with a positive expectation? The key to this is the rounding up regulation (Table 3).The syndicate was able to bet $730 000 in such a manner that there were three possible outcomes;
1. a win of about $175 000,
2. a win of about $ 30 000, or
3. a loss of about $ 115 000
If we can estimate the probabilities of each of these, we can compute the mathematical expectation.
In the situation reported, the syndicate reasoned as follows:
1. To eliminate the influence of other punters, choose an obscure event with a small place pool (about $2000) and "swamp" it with a huge bet ($730000). Use the greyhound races when the field is always restricted to 8 dogs.
2. Select a race with two short odds favourites and bet on both so that the less than 50% rounding regulation will apply.
3. Place enough on the two favourites ($350000) to ensure a small dividend to them (which will then be rounded up to $1) and large dividends to the other six. 4. Bet ($5000) on all the other six.
5. Select an event where it is highly unlikely that neither of the two favourites will not place (again, the greyhound races meet this criterion).
6. Place the bets as close as possible to the last moment so that other punters will not have time to place enough other bets to greatly affect the payouts (All bets were placed in the last 6 minutes of betting from locations throughout the country).
After placing these bets the TAB would show:

Table 4
*TAB after syndicate places bets*

| Outcome | place bets | place pays |
|---|---|---|
| | | |
| 1 | $350,000 | **$1** |
| 2 | $350,000 | $1 |
| 3 | $5,000 | $41 |
| 4 | $5,000 | $41 |
| 5 | $5,000 | $41 |
| 6 | $5,000 | $41 |
| 7 | $5,000 | $41 |
| 8 | $5,000 | $41 |
| total | $730,000 | |
| | | |
| place pool | $206,833 | (each) |

There are three possible outcomes:

(a) If both favourites place, the syndicate collects $350000 + $35000 + 5000x$41 = $905000 for a profit of $175000.

(b) If only one favourite places they collect $350000 + $205000 + $205000 = $760000 for a profit of $30000.

(c) If neither favourite places, they collect $205000 x 3 = $615000 for a loss of $115000

The influence of other punters will affect these figures slightly.

*The mathematical expectation of the syndicate in this situation.* The determination of this illustrated a number of basic probability principles that have been developed. Some assumptions and estimations need to be made in order to estimate the probabilities of each of these.

Starting with (c), we consider the frequentist approach to estimate the probability, p, that neither favourite will place. It is very rare that in a greyhound race neither of two short odds favourites will place. From an examination of previous races we see that this happens about one in twenty times and we can make a frequentist estimation of p as approximately 0.05.

We can use this to make an estimate of the probability of each favourite placing.
We will assume that these are equal, and independent. Although this is not necessarily true, it will not greatly affect our estimation.

$$1- \sqrt{(0.05)} = 0.78$$

Using these estimates we get:

$p(c) = 0.05$
$p(a) = 0.78 \times 0.78 = 0.6$
$p(b) = 1 - 0.65 = 0.35$

From this, the Expected Return = 0.6 x 905 000 + 0.35 x 760 000 - 0.05 x 615000
= $778250
This is on an outlay of $730 000, or about 15% or ER = 115 %

We can examine the effects of our frequentist estimate of p being incorrect and compute the expectation under other values of p.

Table 5
*Syndicate's Expectation under varying values of p*

| Initial estimate neither place p(c ) | At least one places | Both place p(a) | Exactly one places p(b) | Expected Return | ER as % of outlay |
|---|---|---|---|---|---|
| 0.05 | 0.78 | 0.60 | 0.35 | $840,154 | 115.1% |
| 0.10 | 0.68 | 0.47 | 0.43 | $813,294 | 111.4% |
| 0.20 | 0.55 | 0.31 | 0.49 | $775,308 | 106.2% |
| 0.03 | 0.84 | 0.71 | 0.27 | $859,147 | 117.7% |
| 0.02 | 0.86 | 0.74 | 0.24 | $863,988 | 118.4% |
| 0.01 | 0.90 | 0.81 | 0.18 | $876,000 | 120.0% |

We see that even with p as high as 0.2 the syndicate has a positive expectation of more than 6% which is more than adequate for a professional syndicate to operate.

In actual fact, the two favourites placed 1st and 3rd with a payout of $1 each, 2nd place paid $40, and the syndicate won $170 000.

After this happened the Queensland TAB changed their rules so that no payout can exceed what is in the pool for that event. This was already in effect in other States. It is interesting to note that the TAB obtained a court order to withhold the payment until a court could rule on the legality of the bet. Not surprisingly the court ruled that the bet was legal, it was not a "sting" as reported in the newspapers (there was a small probability that they could have lost about $130000) and the syndicate received their money.

## Conclusions

Many misconceptions in probability are reported in the literature particularly with respect to gambling (see, for example, Shaughnessy, 1992; Borovcnik & Peard, 1997). Newspaper reports such as the one quoted in this study tend to reinforce the public's misunderstandings of betting systems. However, the analysis of the present study does not require any complex mathematics and can be understood by anyone capable of understanding basic probability and the concept of mathematical expectation. Students of the unit can see in this example how the concept of mathematical expectation is used in a decision making process. Earlier research by the author (Peard, 2002) has shown that many students with relatively weak algebraic and overall mathematical abilities are nevertheless able to use reasonably sophisticated techniques including Binomial and Poisson probabilities in analyses requiring the computation of mathematical expectation. This paper provides further evidence of the importance of the study of probability and the inclusion of these topics.

## References


Australian Education Council. (1991). A *National Statement on Mathematics for Australian Schools.* Curriculum Corporation. Canberra. Australian Government Printer.

Borovcnik, M. & Peard, R. (1997). Probability. In J. Kilpatrick (Ed.) *International Handbook of Mathematics Education.* pp. 371-401. Dordrecht: Kluwer.

Carroll, J. (1994). Why do some primary teacher trainees hate mathematics? A case study. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.), *Challenges in mathematics education. Constraints on construction.* Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia (pp. 137-144). Lismore: Mathematics Education Research Group of Australasia.

Garfield, J. B., & Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: Implications for research. *Journal for Research in Mathematics Education*, 19(1), 44-59.

Peard, R. (2002). Decision making using intuitive, frequentist and axiomatic probability. In M. A. Clements, H. Dhindsa, I. Cheong & C. Tendencia (Eds.) *Energising Science, mathematics and technical education for all.* (pp. 227-237). University of Brunei: Brunei Darussalam.

Peard, R. (1998). The understanding of mathematical expectation in gambling situations. In W. Bloom, G. Booker, I. Huntley & P. Galbraith. *Mathematical Modelling: teaching and assessment in a technology rich world.* (pp. 149-159). Horwood. Sussex, England.

Peard, R. (1996a). Difficulties teaching probability. *Teaching Mathematics, 21*(1), 20-24.

Peard, R.(1996b). Problems with probability. In P. Clarkson, (Ed.) *Technology in mathematics education. Mathematics Education Research Group of Australasia 19th Conference Proceedings* (pp. 437-445). Melbourne: Mathematics Education Research Group of Australasia.

Peard, R. (1995). The effect of social background on the development of probabilistic concepts. In Bishop, A. (Ed.) *Regional collaboration in mathematics education.* (pp. 561-570). Melbourne: Monash University.

Relich, J. & Way, J. (1992). Pre-service primary teachers' attitudes to teaching mathematics. In B. Southwell, B. Perry & K. Owens (Eds.), *Space- the first and final frontier. Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australiasia* (pp. 470-477). Kingswood: Mathematics Education Research Group of Australasia.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: MacMillan.

Truran, K. (1997). Beliefs about teaching stochastics held by primary pre-service teaching students. In In F. Biddulph & K. Carr, (Eds.) *People in mathematics: Mathematics Education Research Group of Australasia 20th Conference Proceedings* (pp. 538-545). Waikato, New Zealand: Mathematics Education Research Group of Australasia



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# NUMERICAL OR SYMBOLIC: A CHOICE BETWEEN THE DEVIL AND THE DEEP BLUE SEA?

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For first and second year mathematics classes learning about limits, sequences, sums, integrals, functions, vectors, matrices, etc. etc., we have seen two approaches adopted – either the numerical approach using mostly Matlab or the symbolic approach using a CAS like Maple or Mathematica. The author has made use of both approaches and in this paper will demonstrate their strengths and weaknesses with many examples drawn from lecture material, tutorials and laboratory classes. It is vital that we, as educators, face the fact that there can be major problems with either approach and do not merely use one approach blindly without appreciating the possible drawbacks.

## Introduction

I have used Maple in teaching Science and Engineering students for ten years and Matlab for the last five years, for various first and second year courses.

When I refer to the "symbolic approach" I mean either using Maple or the symbolic toolbox in Matlab; if I refer to the "numerical approach" I mean either Matlab or using Maple or a pocket calculator in a computational way.

## Description of Maple and Matlab tutorials

I think the best way to ascertain the effect of the two approaches is to compare and contrast how we used these two approaches for the same courses – in this case, for a first year and second year Calculus and Linear Algebra course.

Specifically these were:-

**1(a) First year Maple.** Twelve tutorials, including introductions to arithmetic, algebra, limits, differentiation, integration, sequences, sums, plotting (simple, multiple, parametric and implicit), Taylor and Maclaurin series and their uses in approximating e and $\pi$ and numerical integration, vectors (addition and scalar multiplication, dot product, norm, orthogonality, linear independence), matrices (addition, multiplication, inverses) , Gaussian reduction, solving equations, determinants, Leslie and transition matrices, characteristic polynomials, eigenvalues and eigenvectors).

**1(b) First year Matlab.** Ten modules, including introductions to arithmetic, limits, numerical differentiation, plotting and visualising limits, graphical solutions, the Bisection and Newton's method, numerical integration, sequences and series, Taylor series, vectors and matrices as above.

**2(a) Second year Maple.** Tutorials in first and second order ODE's – exact solutions for separable, linear, Bernoulli, Exact, Homogeneous, Second Order with constant coefficients (homogeneous and non-homogeneous), resonance and practical resonance; plotting families of solutions and particular solutions; numerical solutions to non-linear ODE's using Maple's *dsolve numeric*. Tutorials in multiple integrals and plotting 3D surfaces; use in vector fields.

**2(b) Second year Matlab.** Confined to using *ODE 23* and *ODE 45* to find numerical solutions to non-linear ODE's and plotting solutions and phase plane.

For all first year laboratory classes both Maple and Matlab tutors introduce students to these packages with demonstrations for the first two or three sessions and there is a detailed workbook with references both to their textbook(s) (in first year Stewart's Calculus and in the second year also Kreyszig's Advanced Engineering Mathematics and the course website. Students have a laboratory class each week and have ten percent of their assessment from one or two computer assignments and a question on their final exam.

## General observations

The main difference between the two approaches is that Maple emphasizes *functionality* and can, of course, perform derivatives and integrals, whereas Matlab emphasizes the *computational* side of the material.

The Calculus Reform movement suggests that we should always try to emphasise the Rule of Four: Where appropriate, topics should be presented geometrically, numerically, analytically and verbally. We could say, then, that the Maple approach provides the analytic part, Matlab the numerical part, both provide the geometrical part and the facilitator the verbal part, but, of course, in many cases, they cross these naïve boundaries.

## Specific comparisons – first year courses

**Arithmetic**. Maple uses *integer* arithmetic, so that the student sees immediately that even complicated expressions like, for example, $\dfrac{2^{12}+4^{13}-5^{8}}{3^{15}+2^{11}+7^{8}}$, are returned as *exact* rationals and the students must convert them into decimals if they need them in such floating point form. We should add that Maple allows us to control the number of digits to any chosen number, so that, if we need a very good approximation, we could, for example, calculate √2 to a hundred decimal places.

Students are encouraged to keep all fractions as integer quotients so that all subsequent calculations are exact, removing all problems like 2/9 x 3/16 x 24 is not 1 but 1.00000001 or 0.99999999, and, by preserving √2 as an entity, instead of evaluating it, we have $(\sqrt{2})^2 = 2$ exactly and not 2.00000000001 or 1.999999999.

Matlab, of course, converts all expressions to decimals, and, unless the student remembers to change to the long format or use rat(expr) it will only exhibit four significant figures, and the student merely copies these down as a true representation of the answer. Again, Matlab has only 16 digits even in the long format and so high order estimations are impossible – for example, e or π, to 50 decimal places.

**Algebra**. One of the greatest aids in Maple is being able to simplify long algebraic expressions. Even something like a higher order binomial is difficult for students these days. For example, $(2a + 3b)^5$ ,or simplifying expressions like $(x + 1)^6 - (x - 1)^6$ or $\dfrac{x+1}{x-2}+\dfrac{x-3}{x-1}+\dfrac{x-2}{x}$. Of course, this is not available in the Matlab approach.

**Plotting**. Both Maple and Matlab have easy ways to plot functions, both in two and three dimensions. Matlab has *ezplot* with no domain needed and Maple has *plot* and *plot3d*. It is vital for students to be able to plot more complicated functions to be able to understand their behaviour, especially three dimensional surfaces.

**Functionality**. Maple has both *expressions* and *functions*. The student can use the former when no functional evaluation is required. The latter has good notation – a function is entered as f:= x-> *the function*, emphasizing the mapping quality of a function. In Matlab the student must set up their function in an M-file using the .operator – students find this confusing – just to enter the function $\dfrac{x^2}{1+x^2}$ requires x.^2./(1+x.^2). It is very easy to forget one of the dot operators.

**Limits**. This is a fascinating problem area and one where the student must beware. Maple has an analytic way of finding limits, both for finite and infinite values of the variable, so that limits like $\lim_{x \to 0} \dfrac{1-\cos(x)}{x^2}$ or $\lim_{x \to \infty} \dfrac{\ln(x)}{x^3}$ are no problem.

If we try the numerical approach by setting up either a suitable sequence which converges on the desired number ( 0 for our first case) or which "increases to infinity" (as for the second case) interesting things can happen. Try this on your calculator or any numerical calculator with a relatively small number of digits and you will get essentially this sequence for the first case: 0.49, 0.499, 0.4999, 0.49999, 0, 0, …. and the student might well conclude that the answer is 0! The reason, of course, is that the numerator behaves like $x^2/2$ for small x, so that once x is greater than $\sqrt{N}$ , where N is the critical size for the calculator, the numerator is zero before the division by $x^2$ can be completed – there are many such limits.

However, for limits like $\lim_{x \to 0} \frac{\sin(x)}{x}$ or $\lim_{x \to \infty} \frac{2x^3+1}{x^3+2}$, numerical methods work well. The student can see how the function approaches the limit, and, if they also plot the function, this certainly helps them visualise the concept of a limit.

**Infinite series.** Probably more alarming are problems associated with infinite series, the obvious being series like $\sum \frac{1}{n}$ or $\sum \frac{1}{n\ln(n)}$ or $\sum \frac{1}{n\ln(n)\ln\ln(n)}$. I gave the first one to my class to add up on their calculators over the weekend and the best calculator only reached about 11!

Even setting up a sequence on Matlab you could well only reach about 16, since $\sum_{n=1}^{N} \frac{1}{n} \sim \ln(N)$ and $\ln(10^{16}) \cong 37$ ! Students could well conclude that 37 is a small number and conclude that the sum is *finite*. Maple can easily set up a sequence of $10^n$ for n from 10 to 1000 to show how the partial sums are still increasing since it uses an analytic expression for these.

With the second and third series the effect is, of course, even more dramatic. Since $\sum_{n=10}^{N} \frac{1}{n\ln(n)} \sim \ln(\ln(N)) - \ln(\ln(10))$, after $10^{16}$ terms we have only reached about $\ln(\ln(10^{16}))$ – $\ln(\ln(10))$ or about 2.77 – that certainly does not convince the student that this series will diverge!

The numerical approach cannot, in general, cope with the convergence or divergence of infinite sums. Another example may illustrate the point more clearly. Take the two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$. By the p-series test the first *diverges*, but the second *converges*, but after 1000 terms there is virtually no difference between the two partial sums. We really need an analytic approach to such problems.

**Differentiation and Integration.** Maple can, of course, perform differentiation and evaluate the derivative at the required point. Matlab inevitably computes the derivative by numerical methods, but, where the limits work well, the student can visualise the slope of the chord approaching the slope of the required tangent.

One of the best features of Maple at this level is the ability of the student to discover if a certain function is integrable by asking Maple to integrate it and having either a function returned (in which case it can be integrated, though the student may well not recognize this new function) or a null return (in which case it possesses no integral in closed form). For numerical methods we can set up various ways of estimating areas – left and right boxes, midpoint sums, the Trapezoidal and Simpson's Rule and Matlab's inbuilt function *quad*. Again, if Maple return a null result, all we have to do to instigate quadrature is enter >evalf(%); and the result is returned. I guess the advantage of using the numerical integration is to see how the upper and lower estimates for the area (however chosen) converge to the required area as the number of partitions increase and the grid width decreases.

Students have much more difficulties with integration than differentiation and Maple can take them through integration by parts (you can choose which function is "u" and which is "dv/dx"), integration by substitution (you can see the effect of choosing a particular substitution on the integral), log integrals

(Maple can do partial fractions and then you can do the integrals individually afterwards).

**Taylor and Maclaurin series.** The hardest subject for students in first year seems to be Taylor series and they need the greatest amount of help to grasp the concepts. Maple allows the student to find n terms of the Taylor (or Maclaurin) series of any function around any point (if it exists!) very simply by *taylor(f,x=a,n)*. The student can now use the series to find closer and closer approximations to important constants like e or $\pi$ and estimate functions close to known values – for example sin(31 $^{o}$) or tan (46 $^{o}$). They can also set up plots of how the Taylor polynomials better approximate the chosen function by taking more and more terms of the series and ascertain the intervals in which any polynomial is a good approximation. Students report that this is probably the best way to understand how Taylor series work.

Matlab, of course, has no Taylor series, and if you want to use them, you have first to find the required Taylor series analytically and then enter it into Matlab. Since we might well want twenty terms for reasonable accuracy, this is a very lengthy task for the student. The compromise is usually to take only a few terms of the series and then the approximation is not good and the student cannot understand how the Taylor series converges to the function.

**Bisection and Newton's method.** Students can easily understand the Bisection Method to find approximations to the solutions of non-linear or transcendental equations. For either Matlab or Maple some code needs to be written and I usually present this as a module where all students have to do is plot f(x), where the equation is in the form of f(x) = 0, enter two suitable points where f(x) has opposite signs taken from the graph and observe what happens at each iteration. For Newton's method, if they use Maple, the derivative and its value are immediately available through f(a) and D(f)(a); for Matlab the student must first find the derivative of f(x) before proceeding. In each case a short *for* loop is all that is required, but a *good* initial guess is *vital* and students must plot f(x) first. If their first approximation is indeed good, convergence is usually very rapid. Maple can, of course, keep track of rational approximations as it proceeds, for example, for square or cube roots (where the convergence is very rapid).

**Further methods of finding solutions to equations.** Maple has *solve* and *fsolve*. The first allows us to solve algebraically for polynomials, inequalities, exponentials, logarithms. This is very useful, but the student may well get more than they bargain! For example, in solving $x^3 = 64$, Maple will return three roots, the real one they probably want and two complex ones as well. The second is a numerical solver and allows the student to specify the interval in which to search.

Matlab has no equivalent to *solve*, but has *fzero* which is like *fsolve*, but requires a good starting point. In either case, students are encouraged to plot the function *first* to ascertain where likely solutions (if any) may occur.

**Maxima and minima.** Whether we are looking for global or local maxima or minima, a graph is very important in our understanding of the of the behaviour of the function. As long as we can see the function, we call tell its likely behaviour. But, of course, computer graphs lie (depending on scale, discontinuities, etc.) and we can only be sure of the true behaviour by checking the derivative. Maple can do this very simply – enter the function, find its derivative and where this is zero (using *solve* or *fsolve*) or undefined. If there are no such points then the function is monotonic over the interval and the global maximum and minimum occur at the end of the interval. If there are solutions, then we can easily evaluate f(x) at each local maxima or minima. This requires, of course, that the function is *continuous* over the interval – Maple can easily check this for us first.

Matlab has no analytic differentiation nor way of finding discontinuities of the function or its derivative. The student must do this first and plot both the function and its derivative. Hopefully from these graphs it can be ascertained what the behaviour is and then either the endpoints be used or *fzero* to find the local maxima or minima.

**Vectors and Matrices.** Matlab was, of course, set up for matrices and here its notation and ease of entering matrices makes it easier for the student to use than Maple. The first drawback of Maple is that the student must load the *linalg* package and they seem alarmed by the associated screed – of course, you can suppress this output by using a colon instead of a semi-colon. Matlab has simple ways of entering row and column vectors and matrices. Maple has a more complicated way of entering a matrix via an array of vectors. For example, in Matlab A = [1,2,3;4,5,6;7,8,9], but in Maple A:=array([[1,2,3],[4,5,6],[7,8,9]]); Again, most operations on vectors or matrices are easily written in Matlab – for vectors b and c the transpose is b‘, the dot product is b.c, the cross product is bxc, whereas in Maple we need transpose(v), dotproduct(b,c), crossproduct(b,c). For matrices A and B addition, subtraction and multiplication are A+B, A-B, A*B, whereas Maple has evalm(A+B), evalm(A-B), evalm(A&*B). Other operations are easy in Matlab – the inverse and determinant of A are inv(A) and det(A) – Maple has inverse(A) and det(A). We can easily extract submatrices in Matlab – A(m,:) is the mth row and A(:,n) the nth column – for Maple we need row(A,m) and col(A,n) – not much difference, but for submatrices in Matlab we need A(2:4,1:4) for the second to fourth rows and first to fourth columns, whereas in Maple we need submatrix(A,2..4,1..4) – much longer.

For solving sets of linear equations Maple has gausselim(A) and rref(A) – Matlab has only rref(A). Both can solve Ax = b, Matlab by A\b and Maple by linsolve(A,b), but here Maple easily wins out as Matlab has no way of giving parametric solutions in the case of A being singular and r(A|b) = r(A), whereas Maple can return the parametric solutions. Here we should note that Matlab cannot deal with matrices with non-numeric entries, whereas Maple can return all the operations on matrices with any algebraic entries. An obvious use of this is finding for what values of a parameter the matrix is singular.

For special types of matrices Matlab is much easier – eye(n) produces the n x n identity matrix; zeros(n) produces the n x n matrix zero matrix; ones(n) produces an n x n matrix with all entries 1. The equivalent Maple commands are &*( ), array(sparse,1..n,1..n) and two *for* statements to set up the matrix with entries all 1.

For eigenvalues and eigenvectors Maple is slightly better. Having set up a matrix A, we can set up the characteristic polynomial by charpoly(A,x) (including parameters if needed) and use *linsolve* to find the eigenvalues, or directly eigenvals(A) (given in exact form) and eigenvects(A) (given in a neat form [[eigenvalue, multiplicity, {set of corresponding eigenvectors}]). In Matlab we enter [V,D]=eig(A) and are confronted with two matrices – the first contains the eigenvectors as columns (which confuses students) and the second is a diagonal matrix, which has the eigenvalues in order down its diagonal. Of course no parameters are allowed. For the characteristic polynomial we enter poly(A) and see only the coefficients of the characteristic polynomial – this certainly confuses students.

However, Matlab can easily augment matrices by [ A B] or [A B b'] or [A' B'] or [A;B] – the equivalent Maple is much more cumbersome augment(A,B) or augment(A,B, transpose(b)) or stack(A,B).

## Specific comparisons – second year courses

In this course we are mostly concerned with first and second order ODE's, described above in the Maple tutorials, which have *exact* solutions and plotting their families of solutions and particular solutions. Of course Matlab cannot be used here and Maple does a good job through its *dsolve* and its ability to set up sequences of solutions and plot these multiple solutions. The structure is not difficult and students find this very helpful, especially as they can identify the homogeneous and particular solutions.

In the case of resonance or practical resonance they can easily see which frequency gives rise to the greatest output.

**Non-linear ODE's.** For non-linear ODE's I teach Maple's *dsolve numeric* and Matlab's *ODE 23* and *ODE 45* to find numerical solutions. The advantage of Maple is that the student can use the same structure for the exact solution and merely add *numeric* – instead of dsolve({DE, initial conditions}, y(x)) they need dsolve({DE, initial conditions}, y(x),numeric) and Maple returns a table function using the Runge Kutta method. It is now easy to plot the solution, the velocity and the phase plane, all of which are important for the student.

Matlab has ODE 23 (the Heun Method) and ODE 45 (Runge Kutta), but first the student must convert the ODE into a system of first order ODE's before they can be entered into the M-file. This often causes problems and in a survey I carried out only one student out of about 600 students found Matlab easier to use that Maple.

**Multiple integrals.** Again, Matlab cannot perform multiple integrals, but Maple is invaluable. Especially with multiple integrals using cylindrical or spherical co-ordinates students can easily forget to include the Jacobians – Maple allows the student to transform to any co-ordinate system, automatically including the appropriate Jacobian. Typical examples of where Maple saves the day are problems of finding centres of mass of non-uniform bodies and moments of inertia, where I, personally, always get the often very complicated integrals wrong. I always tell the students to take great care setting up the multiple integral and then use Maple to evaluate it. Of course, parameters in the expressions present no problems.

**Vector fields.** Matlab has no way of dealing with general functions and so cannot do vector fields. Maple has all (and more!) that the student might need. If v is vector f a scalar function and F a vector function, then grad(f,v), diverge(F,v), curl(F,v) and laplacian(f,v) provide the three vector operators and the Laplacian. The Jacobian is jacobian(F,v). All these are easily entered by, for example, f:=x*y*z; v:=[x,y,z]; grad(f,v); or F:=[x*y, y*z, z*x]; curl(F,v);

These can all now be combined in Green's, Gauss' and Stoke's theorems with the multiple integrals that result.

**Linear Algebra 2.** In this section we teach LU and PLU decomposition, Vector Subspaces, Matrix norms, Condition numbers, Gerschgorin circles, the Power method and diagonalization of matrices. We use Matlab exclusively here due to the ease of writing commands in Matlab. Students are expected to write simple algorithms for LU, PLU and the Power method and this is easily done in Matlab – it's what Matlab was meant to be used for! Especially those students who are engineers need to be able to write short programs in Matlab and this further reinforces their competency.

## Student comments

**First year courses.** We have surveyed students each year as to their attitudes to using these packages in their courses. Admittedly, the results have been varied. Some students found Maple's syntax hard to use and this made any use of it difficult and frustrating. Others found Maple a great help, especially with the sections on integration, series and particularly Taylor series.

Many students found Matlab's numerical approach annoying and unconvincing and wanted an analytical approach. They also hated writing Matlab code. However, they found the use of Matalb for matrices helpful. The ability to graph functions in either Maple or Matlab was appreciated.

**Second year courses.** Maple was regarded by all as their saviour in the multiple integrals and vector fields section and useful to check their solutions in the ODE's. The ability to perform complicated triple integrals was a great asset.

For the Linear Algebra part students reactions were surprisingly divided. They liked Matlab's ability to find LU and PLU and do iterations for the Power Method, but they hated having to write any Matlab code to actually produce any matrix operations.

## Conclusion

Inevitably we all teach very general courses which include both calculus and linear algebra, ODE's and several other topics that can be squashed in. I believe, as hopefully this in-depth analysis has shown, that no one computer package really does it all for us. Each has drawbacks which often confuse or hinder the student.

The only solution that seems reasonable to me is to use Matlab for what it was intended – matrices and numerical calculations – and Maple for its strength in algebra, calculus and where functionality is required.

This would, of course, mean students learning *two* packages in first year. We are told this is not on – but, students doing CS type study learn Java, Spreadsheets, online skills and often more. Asking our engineering and science students to learn just two packages does not seem much in comparison.

The great advantage would be that they would use the best suited technology to each of their different topics and would realize that a modern engineer or scientist must be able to understand both the symbolic and numerical approach to problems. In fact, learning one approach without the other leaves the student in a poor position – they can either solve particular problems (the numerical approach) and not be able to see the general solution or solve general problems (the symbolic approach) and not be able to solve the numerical problem that inevitably ensues (unless, of course, there is a closed form for the general solution).

This all leads to the solution that several of us have put forward – divide all first and second year classes into specialist sections. Students take Calculus 1 and 2 ( 3 and 4?) where Maple is used; Linear Algebra 1 and 2 (and 3), where Matlab is used; ODE's 1 and 2 where Maple is used; Scientific Computation 1 and 2, where Matlab is used; Algebra 1, 2 and 3 and Number Theory 1 and 2, where Maple is used; Scientific Biomathematics, where Matlab is used; Combinatorics and Graph Theory, where Maple is used.

By third year all students would be very proficient in both packages, enabling both they and their facilitators to undertake much more interesting, challenging and importantly, rewarding work than they could have done otherwise.

If we only choose either the symbolic or the numerical approach we face a choice between the devil and the deep blue sea!

I have regrettably included no references as I could find no other parallel studies on this subject.

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# FOUNDATION PROGRAMMES IN MATHEMATICS: ANY WISDOM GAINED?

JAN PERSENS

Students' abilities to deal adequately with mathematical concepts and operations do not seem to care for any social criteria or measure used to distinguish among them. This fact is demonstrated by the achievements of students in international competitions. They often come from vastly different backgrounds and with severely different opportunities. Unfortunately, the same holds true for misconceptions and the lack of skills in problem-solving. The only positive aspect of the latter situation is the fact that mathematicians and mathematics educators can gather and reflect upon their concerns, which carry an unusual sameness with them. In this paper we reflect upon a number of practices, which obtain within the South African situation as ways of addressing such conceptual and other concerns at first year entry level. Some suggestions are then made towards an initial effort to define a programme aimed at establishing a coherent and, hopefully, effective approach to dealing with 'ill-prepared' students entering tertiary mathematics programmes.

## Introduction

As a way of removing any misunderstanding, let me state that this paper will concentrate on the concept of bridging into initial undergraduate mathematical sciences courses from a ('weak') school background. I believe that a kind of bridging process can also be introduced between under- and post-graduate level, especially into a programme of (applied) mathematical sciences from less purely mathematical backgrounds. A deliberation on such forms of bridging would be a topic for a different occasion. Such an approach can serve as an attractor into high(er) level mathematical sciences. This paper will also not address the issue of bridging courses "for adults who, having fled maths years earlier, suddenly find that they need it to undertake college or university courses in Engineering, Science, etc.". (Mittal (2003)) A very interesting view of bridging in mathematics is that espoused by the Bridging Mathematics Network, viz., "bridge conceptual and cultural as well as geographical differences" which was focused on at "Bridging the Distance" conference at The University of Southern Queensland, Toowoomba, Queensland, New Zealand, July 2-4, 1998. Culture and geography are not often enough considered as barriers hindering mastering of mathematical concepts. Yet there are crucial questions to be asked in this regard: Whose culture should dominate? Whose location – geographic and other wise-- determines the norm? Could ethno-mathematical constructs/artifacts serve as a mediator between concepts?

Those of us who grew up in the 1960s, or even early to mid-1970s, did not know about bridging programmes in mathematics. At least, that was the situation in South Africa. Given the abnormality in South Africa, one often wondered whether the 'swim or sink' attitude, of university faculty members towards their students, was meant to encourage them by realising that if others could do it, they can also succeed or simply letting them know that they do not have an aptitude for mathematics. The existence of separate ethnic institutions made it easier to link non-performance, especially in mathematics, with ethnicity, and hence, a view that some ethnic groups are just not cut out for mathematics. In addition, knowledge was generally viewed as neutral and absolute, something of which the content and methodology have already been decided and practised over many centuries. Undergraduate subject contents were fixed and of a certain pre-defined standard, perhaps by the unseen but omnipresent 'gods'. So, all syllabuses and examinations had to adhere to some international standard, the origin and specifications of which most of the implementers had no clue. In South Africa one often heard that we operate according to the British system no matter how much that system might have changed. In fact, often degrees obtained somewhere else, especially in the United States of America, were considered as sub-standard.

## High school results as indicator: some questions

Then in the late 1980s the University of the Western Cape started to challenge the validity of the various matriculation examinations as indicators for university preparedness. There were about 17 of them. Clear evidence emerged that students' performance at university often had no resemblance to their achievements at school. What was called a 'responsible' admissions policy – others called it an open admissions policy – was introduced at UWC. The essence of this policy was to accept the matriculation examination results as some indication of a student's ability but not as the only and definitive evidence. Since this approach did not mean that students would not need any 'strengthening' of their analytical skills, so-called academic support programmes were introduced. At UWC we realised that the majority of our students would need academic support in one form or another. This meant that separately run support programmes became unaffordable, especially if it meant that students could not enter any first year courses at their first registration at the University. This dilemma must also be linked to the state's refusal to fund any compensatory programmes. On the other hand, students were also not happy to be stigmatized or labeled as a 'short pants' university student. This perception was reinforced by the fact that certain tertiary institutions considered their bridging programmes as totally independent and separate from their 'usual' programmes. They were 'out-sourced' in a certain sense. We then introduced the concept 'academic development'. This basically meant that any 'support' activities were supposed to be infused in our ordinary lectures and tutorials, perhaps with greater emphasis on and a more articulated approach to pedagogy. After all, we were dealing with a majority phenomenon. In the mathematical sciences the situation was especially acute and visible. This new approach, therefore, needed be given special attention as far as mathematics was concerned.

## From Bridging to Foundation programmes: Some examples

At first year university level mathematics intervention has gone from bridging programmes, to academic support programmes to academic development programmes. Did this mean improvement, both conceptually and practically? Did it mean that students found the new situation more acceptable? After all, they were being academically developed!! For the sake of comparison, let us take a brief look at a few examples at South African institutions. First, there is the UNIFY (University First Year) programme at the University of the North, which was basically a pre-university bridging programme aimed at revising high school mathematics but with no credit for a first course or part-course. It was externally funded and a very expensive programme which even involved lecturers from the donor country. Secondly, there is, what could be considered a Science Bridging Programme at Wits which allows students to complete their first year, or part of it, over two years. Again, it was and, I believe still is largely, externally funded and can almost be viewed as an out-sourced model. Thirdly, let me relate some of the UWC experiences where the thinking was that letting students repeat school mathematics, was not very inspiring. Pushing post-matric mathematics down their throats, was also not a smart approach. Subsequently a modified course/programme which would lead to a first year credit was considered more appropriate. But how? The 'solution', primarily driven by Larry Kannemeyer (1996a, 1996b), can be summarized as follows: Students who wanted to do mathematics or who needed first year mathematics as a prerequisite for other subjects, were advised to drop one of their first year subjects in order to open up space for more in-depth mathematical interaction. Life skills such as time management and appropriate study methods were also included as required. Key mathematical concepts and constructs were introduced early and cyclically revisited and reinforced a few times. 'Other' mathematics, such as discrete mathematics was introduced; the Calculus Reform principles were employed; lecture-workshops partially replaced formal lectures; and the Realistic Mathematics approach initiated by Hans Freudenthal and further developed by his 'followers', served as a well-researched way of using real-world problems to motivate understanding.

Of course, as far as the total undergraduate curriculum was concerned, the situation became quite complex. It seems as if mathematics departments took a more anxious approach than, say, the humanities. Or, perhaps, judging by today's activities in those fields, there was a stronger denial that their students simply lacked the prerequisite analytical competencies or tools to engage meaningfully with the subject matter. The extent to which greater access to tertiary education was becoming one of the new South Africa government's political platforms for reform, the more we have seen the introduction of remedial or support programmes. However, financial support for these programmes remained a stumbling block. The cry for a mixed mode of three-year accelerated and four-year normal basic bachelors degree programmes became louder. [A number of four-year or longer professional degree programmes already existed.] For the past two years or so, the nomenclature has shifted to foundation programmes. Clearly they will add another year to a student's basic undergraduate life but they will be subsidized by government. So, they have become more acceptable. Developing curricula and content material for these programmes has become almost a profession.

So, now we have moved from *bridging* programmes, to *academic support* programmes, to *academic development* programmes to *foundation* programmes!!

## En route to the promised land?

But, what does this mean for mathematical sciences? Has a conceptual shift really taken place? I believe that some shift had occurred in the early 1990s. At least, those of us who believed that the 'short pants' university student approach created more emotional damage than resulting in improvement of critical skills in mathematics, got involved with the so-called Mathematics Reform Movement (MRM) or Calculus Reform Movement (CRM). We soon realised that alternative presentations of subject matter, meant meaningful epistemological access for more students. For example, the application of the so-called Rule of Three (RoT) – and later Four (RoF): analytic, graphical, numeric – and later-- verbal, allowed students to choose their 'access road' to certain mathematical concepts or test their conceptual understanding by comparing results flowing from the application of more than one approach to a problem. In fact, my hunch is that for some role players, the RoT or RoF became a kind of road map for measuring progress towards the 'promised mathematical land' in which we have ensured that students enter tertiary institutions well-prepared or in which mathematical scientists can once again return to their ivory towers to crunch out theorem after theorem. We have even read about, and sometimes, participated in the so-called 'holy' Calculus Wars. Some of us came close to eliminating our 'enemies'. But, is this 'promised land' metaphor a realistic view of the result of our various shifting efforts over the past thirty years or so? Is there any indication somewhere, especially where lots of money and incentives were thrown at the 'problem', that permanent positive impact was made? Do we see hands raised in confirming this paradise-like state? Or, do I hear the congregation says 'Amen, thank you MRM and CRM'? Oh! I see, the choir is not present today and the deacons are also on various outside duties!! *Commemorating* the tenth anniversary of the death of *Sister Bridging*, *reburying* the last remains of *Brother Academic Support*, *supporting* the bereaved relatives of *Sister Academic Development* and *praying* for the acceptance of *Brother Bridging* as the newly anointed … .

## Restarting the cycle or initiating drastically new ideas?

What are some of the responses one can aspect to our continued quest for ways and means to address the seemingly unsolvable problems that students have with mathematical sciences, and perhaps other related fields. Well, we could imagine a bridging programme into the foundation programmes!! And, I am not trying to amuse you. But as the situation in our schools deteriorates, one may experience a situation in which school mathematics is no longer viewed as preparation for university needs. Then there is the situation practised in some African countries: Simply raise the entrance barrier, not as a deliberate mechanism to deny access to potential students, but mainly as a practicality. There are simply not sufficient resources to reach out to the under-prepared.

If one were to reject the above ways of addressing the challenge, then what could be identified as the elements of this new approach? First steps towards a promised land!! Year after year, the results of mathematics competitions show that students, with vastly different backgrounds and contexts, seem to perform comparably well. Students' abilities to deal adequately with mathematical concepts and operations, do not seem to care for other criteria or measure used to distinguish among them. What is, perhaps, surprising is the fact that there is not necessarily a correlation between the mathematical achievements of teams of students, and the severe challenges faced by the economies of their countries. If any, there often seems to be an inverse correlation. It, therefore, seems as if mathematical talent serves as some sort of 'equalizer'. Unfortunately, the same holds true for misconceptions and the lack of skills in problem-solving. Thus, developing and developed countries find themselves very much in the same pool of frustration. The question is: How can we deal with this dilemma constructively? It may good to launch a concerted and well-organized mathematics war 'for' something, rather than 'against' something. There is no room for 'holier-than-thou' egos. Perhaps the following could be viewed as aspects of a programme to address our common dilemma:

- Create a database and disseminate the information of efforts developed and implemented thus far as widely as possible.
- Tabulate various measures of success or examples of best practice in order to avoid re-inventing the wheel or to derive basic principles on which successful interventions can be built..

- Pay special attention to methodology of both teaching and learning practices.
- Pool resources and ideas and develop ways of embracing mathematicians and mathematics educators as collaborators and co-developers of programmes, as a first priority.
- Academically charged and result-oriented joint workshops of mathematicians and mathematics educators, should become our hallmark for the search of a 'solution' to our challenge.
- Provide adequate space for students to share their successes and frustrations in understanding and mastering mathematical concepts and problem-solving.
- Form focus groups which do initial groundwork to well-defined topics or themes.
- Funds should be sought to ensure and encourage undivided attention to some agreed-upon aspects of the challenge.

## References

MacDonald, T.H. (2002). *Foundations of Mathematical Analysis: A voyage for discovery for people*, DK Publishers Distributors

Kannemeyer, L. (1996a). *An Examination of a Didactical Procedure to Engage First Year University Students in Meaningful Mathematics Activity.* Masters mini-thesis. Cape Town: University of the Western Cape

Kannemeyer, L. (1996b). *Report on the Mathematics 114/124 Course: 1993 to August 1996*. Technical Report. Cape Town: University of the Western Cape.

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# Explaining the Mystery of Statistics

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**Abstract**

Why use a divisor of $(n-1)$ when calculating the standard deviation of a sample? At first-year level, most teachers of statistics give a plausible explanation of why this is necessary, but some students are not totally convinced, and with some justification. Occasionally a divisor of $n$ is proposed in the literature as a simpler alternative, with arguments advanced in its favour. Only rarely is the "real" reason referred to, utilising the underlying mathematical theory, as a "teaser" for those students going on in statistics. At second-year level, the necessary linear algebra, or vector geometry, can be taught or revised in the space of one 50-minute lecture. The appropriate visual image is a right-angled triangle, with the squared length of one side being the "sum of squared deviations from the mean." This side of the triangle lies in an $(n-1)$-dimensional subspace. Developing this properly allows students to see that the divisor must be $(n-1)$. Similar geometric considerations apply when there are samples from two or more study populations, with data analysed by way of $t$-tests or analysis of variance, and for regression or analysis of covariance (in fact, throughout "linear models" generally). For example, in the case of analysis of variance, the "treatment" sum of squares and the "error" sum of squares add up to the "total" sum of squares by application of Pythagoras' Theorem to the squared lengths of the sides of an appropriate right-angled triangle. In this paper we describe how we used these geometric ideas to develop a new approach to teaching linear models in a second-year applied statistics course. This course was taught at the University of Canterbury for 15 years, and proved popular with the students. Similar results were obtained with a group of 80 graduate students in agriculture at the University of California at Davis.

## 1 Introduction

In 1984 the two authors of this paper developed a new method of teaching linear models as the main component of a second-year applied statistics course (STAT222) at the University of Canterbury. The idea was to base the teaching upon the mathematical basis for these models, the geometry of $n$-dimensional space. This geometry was at the genesis of the development of these models, being used by Sir Ronald Fisher to inspire the invention of new techniques such as analysis of variance and analysis of covariance and to derive proofs for existing methods such as the Student's $t$-test. Unfortunately, Fisher's geometric arguments were not universally understood by his peers because of his tendency to "let too much be clear or obvious," to quote the words of his colleague Gosset, who also used the pseudonym "Student" (Box 1978, page 122). Our aim in using geometry was to provide an elementary yet firm mathematical basis for statistical methods which are commonly explained by way of "cook book" formulae.

In Section 2 of this paper we further discuss the "mystery" of statistics. In Section 3 we give an example of our geometric approach for a simple case, a paired samples $t$-test with a sample size of three. In Section 4 we outline a second example involving an analysis of variance. In Section 5 we describe the course coverage, in Section 6 we discuss the link with computing, in Section 7 we discuss student reaction to the geometric approach, and in Section 8 we speculate upon reasons for its popularity, and mention its publication in two textbooks.

## 2 The Mystery of Statistics

When calculating the standard deviation of a sample of size $n$, why *square* the deviations from the sample mean before summation? Also, why use a divisor of $(n-1)$? The answers lie in the underlying mathematics, which is most elegantly and naturally expressed in terms of an appropriate right-angled triangle in $n$-dimensional space. The squared length of one side of this triangle is the "sum of squared deviations from the mean." This side of the triangle lies in an $(n-1)$-dimensional subspace of $n$-space. That is, the sum of squares and the "degrees of freedom $(n-1)$" both arise naturally when the statistical problem is viewed in its most natural mathematical setting. When viewed otherwise, the statistical definition of standard deviation is shrouded in mystery.

In a similar vein, in an analysis of variance the "treatment," "error" and "total" sums of squares are usually defined by way of algebraic formulae involving sums of squared deviations of treatment means about the overall mean, of observations about treatment means, and of observations about the overall mean. Mysteriously, the treatment and error sums of squares add to the total sum of squares, and the term "degrees of freedom" is again used, with the treatment and error degrees of freedom adding to the total degrees of freedom. Why are these things so? Again, the most convincing answers lie in the $n$-dimensional geometry. The three sums of squares are the three squared lengths of the sides of a right-angled triangle, so Pythagoras' Theorem can be applied. Also, the degrees of freedom are simply the dimensions of the subspaces within which each side must lie.

In the context of a regression analysis, the computing routine produces an analysis of variance table, with "regression," "error" and "total" sums of squares. Again, why do the first two of these quantities add up to the third? Again, the answer lies in a simple application of Pythagoras' Theorem to an appropriate right-angled triangle.

In fact, many of the most commonly used statistical methods (linear models based upon the normal distribution) can be derived as simple applications of elementary vector geometry.

Unfortunately, most teachers of statistics feel that such an approach would not be suitable for their students. We agree that if not properly developed, a geometric approach could serve to confuse more than to enlighten (e.g., if it is used as an interesting aside). In the approach that we have developed, we have used the geometry as the primary teaching device, have tried to reduce the geometry to its simplest terms, and have worked upon integrating theory, practice and computing in a "seamless" manner. The resulting approach has proved very popular with the students, as evidenced by the student reviews. The approach worked equally well with second-year university statistics students and with postgraduate agricultural research students. We do not claim that it would be suitable for introductory statistics courses at first-year university level.

In the next section, we briefly outline the approach for a simple example involving a paired samples $t$-test with a sample size of three.

## 3 Paired Samples Example

For our example we suppose that we have three sets of mixed-sex twins for which we are interested in the difference in height between male and female (thinking of the three differences in height as a random sample from some larger normally distributed population with mean difference in height $\mu$). We shall suppose that the differences in height between male and female are 15, 6 and 18 cm respectively. We wish to use our data to test the null hypothesis $\mu = 0$ against the alternative hypothesis $\mu \neq 0$. (This example is based upon a similar data set in Saville and Wood (1996).)

To set our problem in 3-dimensional space, we write our data as a vector, which we call the *data vector*,

$$\mathbf{y} = \begin{bmatrix} 15 \\ 6 \\ 18 \end{bmatrix}$$

## 3.1 Orthogonal Coordinate System

For our problem, an appropriate set of orthogonal (perpendicular) coordinate axes for 3-space is

$$\mathbf{U_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{U_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{U_3} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Here $\mathbf{U_1} = [1, 1, 1]^T/\sqrt{3}$ is the *direction associated with the parameter of interest* $\mu$. This direction is of special interest because if $\mu$ is large, the length of the projection of the data vector onto this direction will be large. The other two coordinate axes serve to complete the orthogonal coordinate system and allow estimation of the random error component.

## 3.2 Orthogonal Decomposition

The next step is to project the data vector onto each of these coordinate axes, yielding an "orthogonal decomposition" of the data vector into components in the special direction associated with $\mu$ and the two "error" directions:

$$\begin{bmatrix} 15 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \\ 13 \end{bmatrix} + \begin{bmatrix} 4.5 \\ -4.5 \\ 0 \end{bmatrix} + \begin{bmatrix} -2.5 \\ -2.5 \\ 5.0 \end{bmatrix}$$

This decomposition is shown in Figure 1.

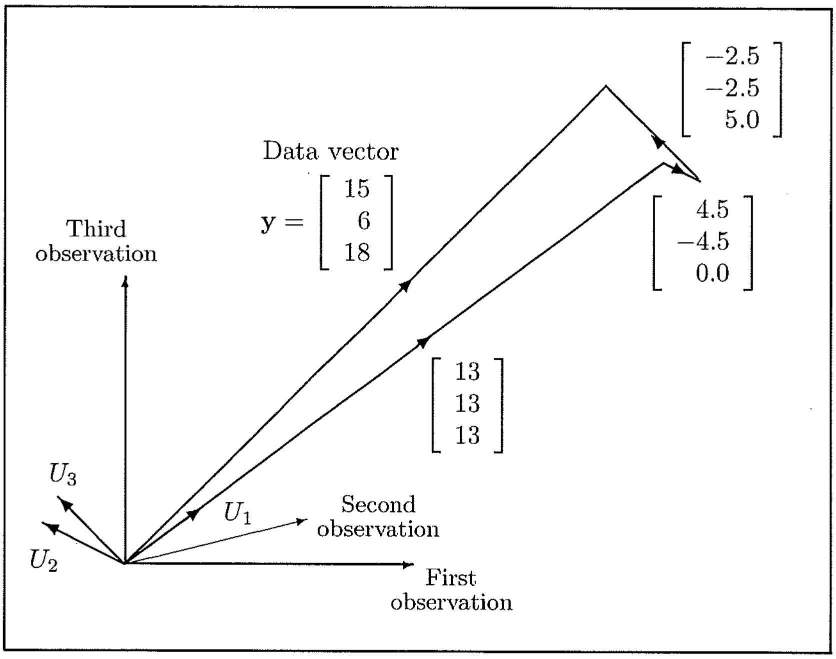


Figure 1: The orthogonal decomposition of the data vector in terms of three orthogonal projections.

As an aside, the orthogonal decomposition can also be written in the form

$$\begin{array}{ccccc} \begin{bmatrix} 15 \\ 6 \\ 18 \end{bmatrix} & = & \begin{bmatrix} 13 \\ 13 \\ 13 \end{bmatrix} & + & \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix} \\ \mathbf{y} & = & \bar{\mathbf{y}} & + & (\mathbf{y}-\bar{\mathbf{y}}) \\ \text{data} & & \text{mean} & & \text{error} \\ \text{vector} & & \text{vector} & & \text{vector} \end{array}$$

where $\bar{y} = 13$ cm is the best estimate of the mean difference in height between male and female twins ($\mu$). This decomposition is shown in Figure 2. (Note that we are using $(\mathbf{y} - \bar{\mathbf{y}})$ to refer to the vector of differences between the observations and the sample mean.)

### 3.3 Testing the Hypothesis

We now investigate the question: Is the true mean difference in height between male and female twins zero? For our hypothesis test, we check whether the squared projection length for our special direction $\mathbf{U}_1$ is greater than the average of the squared projection lengths for the two "error" directions.

The associated Pythagorean breakup is

$$585 = 507 + 40.5 + 37.5$$

where the terms are the squared lengths of the vectors shown in Figure 1.

The resulting test statistic is

$$F = \frac{507}{(40.5 + 37.5)/2} = \frac{507}{78/2} = \frac{507}{39} = 13.0$$

as illustrated in Figure 2.

Is this observed $F$ value large or small? Now if $\mu = 0$, it can be shown that all three projection lengths come from a normal distribution with mean zero and variance $\sigma^2$, where the latter is the variance of the differences in height in the study population. In this case, therefore, the observed $F$ value comes from the $F_{1,2}$ distribution, which has 90 and 95 percentiles of 8.5 and 18.5 respectively. The result is that our observed value of 13 is somewhat unusual, since it is greater than the 90 percentile of 8.5. Hence we conclude that we have found only weak (10% significant) evidence to support the idea that there is, on average, a difference in height between the male and female twins in our study population.

### 3.4 Paired Samples $t$ Test

Note that our test statistic $F = 13.0$ can be transformed to a $t$ test statistic by taking the square root, yielding

$$t = \frac{\sqrt{507}}{\sqrt{(40.5 + 37.5)/2}} = \frac{\sqrt{507}}{\sqrt{78}/\sqrt{2}} = 3.606$$

with reference distribution $t_2$.

## 4 Analysis of Variance Example

For an analysis of variance example, we analyse data from Exercise 4.5 of Saville and Wood (1996), using data from just two replicate plots per treatment to keep the description brief. In this example, an experiment was carried out to determine the best method of applying a fertiliser mix to corn. Three treatments, given below, were assigned in a completely random manner to the field plots. The weight of corn, in kilograms, yielded by each plot was as follows:

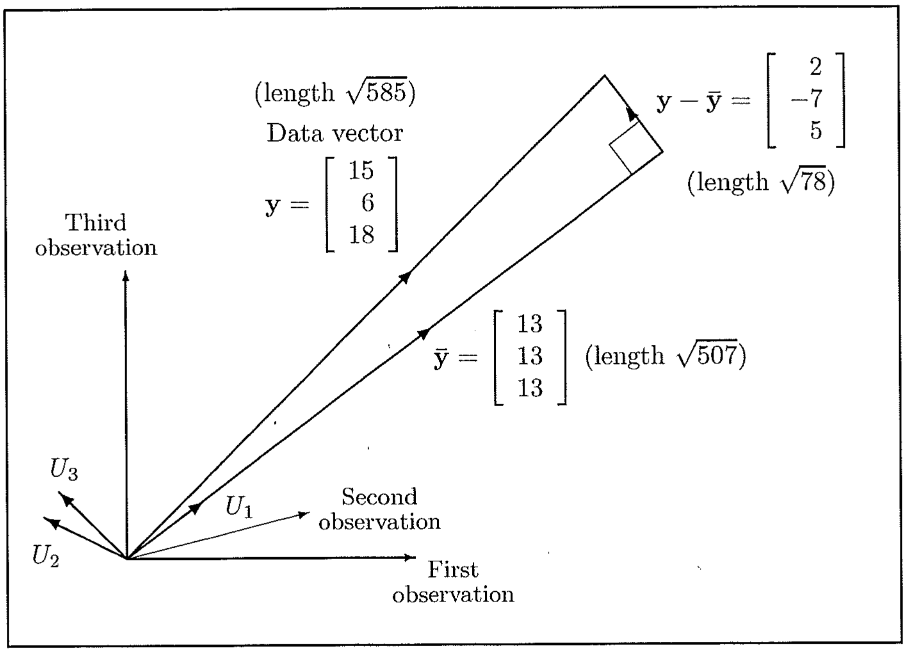


Figure 2: The orthogonal decomposition of the data vector, $\mathbf{y}$, in terms of the mean vector, $\bar{\mathbf{y}}$, plus error vector, $\mathbf{y} - \bar{\mathbf{y}}$.

| | | |
|---|---|---|
| 1. Control (no fertiliser) | 45.1 | 46.7 |
| 2. 300 kg/ha ploughed under | 56.7 | 57.3 |
| 3. 300 kg/ha broadcast | 53.3 | 55.0 |

The two questions of interest are:

- Did fertiliser increase corn yield?
- Was there a difference due to method of application?

The corresponding orthogonal contrasts are:

$(\mu_2 + \mu_3)/2 - \mu_1$

$\mu_2 - \mu_3$

The data vector, and an appropriate orthogonal coordinate system for 6-space which includes two unit vectors ($\mathbf{U_2}$ and $\mathbf{U_3}$) associated with these contrasts, are:

$$
\begin{array}{ccccccc}
\mathbf{y} & \mathbf{U_1} & \mathbf{U_2} & \mathbf{U_3} & \mathbf{U_4} & \mathbf{U_5} & \mathbf{U_6} \\
\begin{bmatrix} 45.1 \\ 46.7 \\ 56.7 \\ 57.3 \\ 53.3 \\ 55.0 \end{bmatrix} &
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Big/ \sqrt{6} &
\begin{bmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Big/ \sqrt{12} &
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \Big/ \sqrt{4} &
\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Big/ \sqrt{2} &
\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Big/ \sqrt{2} &
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \Big/ \sqrt{2}
\end{array}
$$

Here the unit vectors $\mathbf{U_4}$, $\mathbf{U_5}$ and $\mathbf{U_6}$ are "error" directions.

The corresponding orthogonal decomposition of the data vector $\mathbf{y}$ is

$$\begin{bmatrix} 45.1 \\ 46.7 \\ 56.7 \\ 57.3 \\ 53.3 \\ 55.0 \end{bmatrix} = \begin{bmatrix} 52.35 \\ 52.35 \\ 52.35 \\ 52.35 \\ 52.35 \\ 52.35 \end{bmatrix} + \begin{bmatrix} -6.45 \\ -6.45 \\ 3.225 \\ 3.225 \\ 3.225 \\ 3.225 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.425 \\ 1.425 \\ -1.425 \\ -1.425 \end{bmatrix} + \begin{bmatrix} -0.8 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.3 \\ 0.3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.85 \\ 0.85 \end{bmatrix}$$

As an aside, the second and third projection vectors sum to the vector of treatment effects, while the last three projection vectors sum to the "error vector." If we also take the first vector over to the left-hand side of the equation, we obtain the decomposition shown in Figure 3.

With regard to our original, full decomposition, the resulting Pythagorean breakup is

$$16578.97 = 16443.135 + 124.8075 + 8.1225 + 1.28 + 0.18 + 1.445$$

The first test hypothesis is "fertiliser does not increase corn yield." The corresponding test statistic is

$$F = \frac{124.8075}{(1.28 + 0.18 + 1.445)/3} = \frac{124.8075}{0.96833} = 128.89$$

This is very "large" in comparison to the percentiles of the $F_{1,3}$ distribution. We therefore reject the hypothesis, concluding that there is very strong evidence that fertiliser increased corn yield.

The second test hypothesis is "there is no difference in corn yield due to method of application." The corresponding test statistic is

$$F = \frac{8.1225}{(1.28 + 0.18 + 1.445)/3} = \frac{8.1225}{0.96833} = 8.39$$

This is larger than the 90 percentile of the $F_{1,3}$ distribution (5.54), but smaller than the 95 percentile (10.13). We therefore conclude that there is just weak evidence of a difference between the two methods of fertiliser application.

# 5 Course Coverage

The topics covered by the geometric approach in the courses at both the Universities of Canterbury and California at Davis are as follows:

1. Paired samples (Single population).
2. Independent samples (Two populations).
3. Analysis of variance, including:
   (a) orthogonal contrasts classified into four types (class comparisons, factorial contrasts, polynomial contrasts and pairwise comparisons),
   (b) various blocking structures including completely randomised, randomised complete block, latin square and split plot,
   (c) high order $2^n$ factorial designs, including fractional replication and aliasing.
4. Simple and polynomial regression.
5. Analysis of covariance.

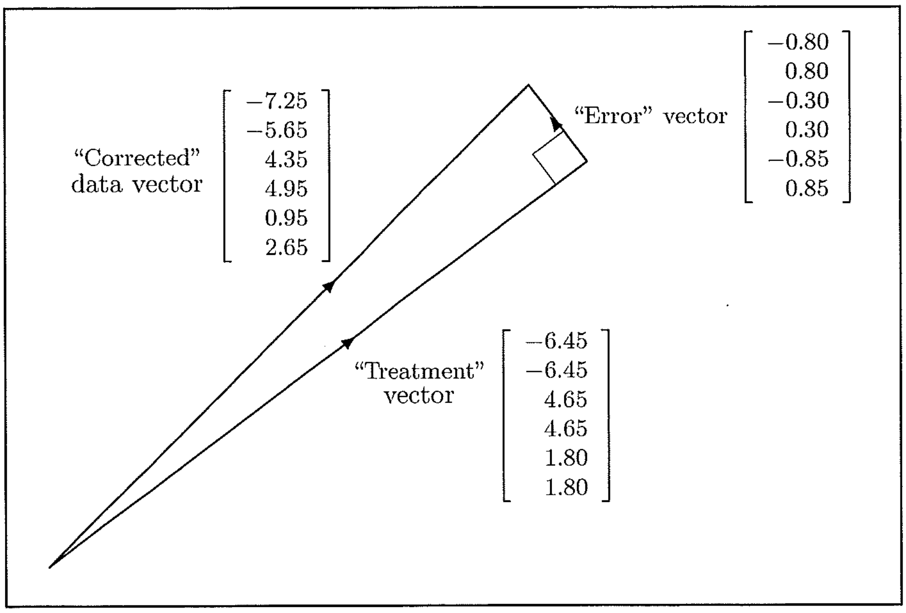


Figure 3: The orthogonal decomposition of the data vector ("corrected" for the mean) in terms of a vector of treatment effects and a vector of errors. Note that the treatment vector is in turn a sum of two "contrast" vectors.

# 6 Computing

The approach to computing is simple. A multiple regression routine suffices for the computing, since "regressing" can be thought of as synonymous with "projecting." The method is to set up the data vector $\mathbf{y}$ as a variable to be used as the dependent or response variable, and then to set up the second and subsequent unit vectors in the "model space" ($\mathbf{U_2}$ and $\mathbf{U_3}$ in the analysis of variance example) as independent variables. Then, regressing $\mathbf{y}$ onto $\mathbf{U_2}$ and $\mathbf{U_3}$ will yield the two projection lengths as the two regression coefficients (e.g., $b_1$ and $b_2$). The error mean square ($s^2$) will also be calculated by the regression routine. The required $F$ tests can then be calculated as $b_1^2/s^2$ and $b_2^2/s^2$.

Note that the projection onto $\mathbf{U_1}$, which we normally ignore, is by default automatically calculated by the regression routine as the intercept term, and printed in the form $\bar{y}$. In the paired samples case, when this is the projection length of primary interest, the regression needs to be fitted with "no intercept" (or "no constant") so that the required projection length and the correct error mean square are printed.

The regression routine included in any statistics package is sufficient for the job. We have used Minitab in our courses, but Genstat, SyStat, Statistica, SAS or any other reputable package could also be used. The spreadsheet Excel also includes a regression routine which is normally adequate.

# 7 Student Reaction

At the University of Canterbury, where the course (STAT222) ran from 1984 till 1998, student reviews were excellent thoughout the 15-year period, and numbers of students attending the course grew from 20 in the first year to 55-75 in later years. The course was jointly taught by the two authors for the first year (1984), then by Graham on his own for several years, then by Dr Murray Smith for several years, and finally by Dr Peter Heffernan for several years. The latter two lecturers were equally enthusiastic about the geometric approach.

The course and teacher evaluation surveys by the university Education Research and Advisory Unit in the last year of the course are presented in some detail in Saville (2001). The number enrolled in the course was 59, and the numbers responding to the two surveys were 33 and 35 respectively. Ratings of 1-5 were used in each survey. For the questions for which 5 was the most favourable rating, the mean ratings were all between 4.0 and 4.8. For the other two questions, concerning workload and level of difficulty, for which the most favourable rating was a 3 (= "reasonable"), the mean ratings were both 3.2, suggesting that students did not find the course overly difficult.

The course at the University of California at Davis was run by the first author in 1985. It was course "AGRONOMY 205A" on the design, analysis and interpretation of experiments in the Department of Agronomy and Range Science, and was attended by about 80 agricultural research graduate students. The students learnt the necessary geometric tools in the space of one 50-minute lecture, and mostly got 100% in the subsequent homework ("just a matter of following the rules," they explained when queried). Overall, the students found the approach very rewarding - one said, "it's like a box, a bit hard to get into, but once inside, there are lots of goodies." The final course evaluations were favourable, and the students suggested that we needed to publish our lecture notes in book form. This led to our 1991 book (Saville and Wood 1991).

# 8 Discussion

The popularity of the geometric approach in our courses is perhaps due to two simplifications that we decided upon. The first is the idea of setting up *unit* vectors to serve as coordinate axes in the "model space." Specifying directions by vectors of length *one* meant that the length of the projection of the data vector onto a direction corresponding to a test parameter could be calculated simply as the "dot product" of the data vector with the unit vector (i.e., simply by cross multiplying individual vector components, then summing). This was conceptually very simple, and it also allowed simplicity of computing using a regression routine (each projection length was computed simply as a regression coefficient).

The other simplification was to always insist upon orthogonality of the coordinate axes corresponding to different parameters (i.e., we specified parameters independently, and did not allow over-parametrisation). This meant that tools of matrix algebra were not required, with the basic requirement being simply the ability to project one vector onto one other vector (typically, the data vector onto a unit vector).

The approach is fully described in Saville and Wood (1991). This book served as the textbook for STAT222 at the University of Canterbury from 1991 to 1998, and was well liked by the students.

In an attempt to introduce the geometric approach to a wider audience, a simpler, introductory book was published in 1996 (Saville and Wood 1996). This book focuses on four key case studies.

After the 1991 book was published, we received a letter from Philip A. Carusi asking how our geometric approach related to the geometry of the $p$-value. We were unaware of this relationship, but have since worked upon this subject. This material is the topic of a workshop at this conference, is included in an appendix of Saville and Wood (1996), and is the subject of two papers (Wood and Saville 2002 and "pending final acceptance"). We mention it here since for the interested reader, it adds further depth to our explanation of the mystery of statistics.

# References

[1] J.F. Box, *R.A. Fisher, The Life of a Scientist*, Wiley, New York, 1978.

[2] D. J. Saville, *A hands-on, interactive method of teaching statistics to agricultural researchers*, In: C. Batanero (Ed.), Training Researchers in the Use of Statistics (pp 197-213). International

Association for Statistical Education and International Statistical Institute, Granada, Spain, 2001.

[3] D. J. Saville and G.R. Wood, *Statistical Methods: The Geometric Approach*, Springer-Verlag, New York, 1991.

[4] D. J. Saville and G.R. Wood, *Statistical Methods: A Geometric Primer*, Springer-Verlag, New York, 1996.

[5] G.R. Wood and D. J. Saville, *A new angle on the t-test*, The Statistician **51** (2002), 99–104.

[6] G.R. Wood and D. J. Saville, *The ubiquitous angle*, The Statistician (final acceptance pending, following revisions as requested).

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# Integration (multi-variable included) from First Principles

Walter Spunde

**Abstract**

Integration has traditionally been *so* closely linked to the interpretation as an area and to the techniques of anti-differentiation as to appear inseparable from them. While largely a consequence of the fact that, in pre-Personal-Computer times, anti-differentiation was the key to effective integration and that line and surface integrals were generally intractable using that technique, the advent of computer algebra systems and easy large scale numerical computation seems not to have had much effect on the way integration is presented in standard texts. On the one hand, it is not clear what sorts of skills are required for a novice to handle computer algebra effectively and on the other hand most available software does not provide the data structures and tools for dealing conveniently with numerical integration from first principles in the general case. We focus on the latter.

In this article we examine an approach to the principles of integration based on computer manipulation of multi-dimensional arrays for the coordinate grids, referring to the area interpretation as only one among several possibilities and presenting integration as the solution to non-trivial anti-differentiation problems. Underlying the implementation of the approach is the mathematical notation of Iversons **J**, an array-processing, functional, computer language. We suggest that the mathematical foundations of the topic – existence of, and convergence to, the limit – should be postponed till after students can effectively compute and manipulate the approximations that are used to define integrals from first principles.

## Introduction

It is often suggested (for example, Rodin [3]), that the quickest way for students to tackle an interesting mathematical problem is to compute the area under a curve such as a parabola. While areas inside closed curves may well be interesting, the artificiality of the formulation of an *area under a curve* seems to be a distortion invented by teachers of the calculus as the only way of introducing integration. The length of a curve is just as interesting a topic, but only calculus teachers can explain why it is not studied first (or ever interpreted as an area). The amount of gold in a necklace or pollution in a lake might also introduce integration. They don't immediately evoke an image of area. We will argue that the reasons for not introducing integration through problems of computing lengths, surface areas and volumes, besides *areas under a curve* are no longer convincing in the face of readily available and easily implemented computing power.

If we think of integration problems as ones whose solutions can be obtained by summing up the contributions of small elements, then their formulation as

$$T = C(R) = Sum\,(C(dE))$$

is apparently recursive. The summation is over all of the elements $dE$ that partition a region $R$ $(= \bigcup dE)$ and $C$ is an unknown function that returns the contribution, $C(S)$, of sub-region $S$ to the measure of some quantity, which, over all of $R$ has a value of $T$. Escape from the recursive loop comes from the possibility that, for sufficiently small elements, we may be able to approximate the values of $C(dE)$ and to do it in such a way that, as the size of these elements is decreased indefinitely and we add together an increasingly large number of increasingly small contributions, the overall approximation continues to improve and approaches a limiting value

$$T = C(R) = \lim_{dE \to \emptyset} Sum\,(Ca(dE))$$

where $Ca$ is some approximating, or averaging, function.

For a mathematician it is apparent that the technique will be fraught with potential subtle pitfalls. Nevertheless, it is probably a mistake to embark on any discussion of what is involved in the formal justification of these limiting procedures with anyone who has not already acquired a certain level of mathematical maturity. Computing approximating sums, $Sum\,(Ca(dE))$, however, is very easy to understand and to do *if* one has good tools to hand. By *poor* tools we mean ones that impose extraneous demands such as special syntax, declarations, and extensive unrelated learning. But, good tools are not necessarily those that attempt to slavishly imitate classical notation ([8]).

# 1 Single Variable Sample Calculus

Surely the most basic of integration problems is to find the circumference of a circle, or $\pi$. To approximate the circumference we divide it up into small elements each of which we assume is nearly identical to a straight line. Since we can get the lengths of these straight lines using Pythagoras's Theorem the problem is simply one of adding up the pieces. Archimedes had an elegant way of solving this problem but we will take a direct and simple minded approach based on Cartesian geometry.

To find the coordinates $(x, y)$ of the endpoints of the elements on a circle of diameter 1, we need to find a list of $x$ values, ranging from 0.5 to $-0.5$ and back again and then compute $y$ values from $\sqrt{0.5^2 - x^2}$ and $-\sqrt{0.5^2 - x^2}$.

Using the functions presented in [6] and implemented in various computing environments in [5], the instructions in **J** for the computation might be:

```
n =: 100                              |  x  =: xf, xb
xf =: n sample 0.5,  _0.5             |  y  =: yf, yb
xb =: behead |. x                     |  dx =: d x
yf =: sqrt (0.5^2) - xf^2             |  dy =: d y
yb =: - sqrt (0.5^2) - xb^2           |  dl =: sqrt (dx^2)+(dy^2)
                                      |  L  =: Sum dl
```

The result 3.14076 is correct to two decimal places. With a thousand elements, (`n=:1000`) we get four decimal places and with a million elements, nine places. No improper integral is involved and a limiting value is clearly being approached.

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

It may not look as pretty or as familiar as the classical integral but the computer evaluation is quite transparent.

Changing the radius it is easy to construct tables and graphs that show that the result is proportional to the diameter. More interestingly, if we kept cumulative (or partial) sums, with a function which, following Leibniz, we represent by `S`, and executed `L = S dl` then plotting `x` against `L` and `y` against `L`, would show graphs of approximations to the cosine and sine functions.

The arclength between any two points on a curve for which we have a parameterization (in two or three dimensions) can be approximated with equal ease and the coordinates, as functions of arclength, can be plotted without any explicit formulae.

### Line Integrals

The introduction of a line density of some quantity distributed along a curve, leading to the summation `S f dl`, raises the question of what value of `f` to use for an approximation to the amount of the quantity present along a small element of the curve. We could take the value at the midpoint of the element, the average of the two values at the ends of the element, the average of the values of a parabola through three consecutive points in the element or indeed the average of a large number of randomly chosen values of `f` over the element, depending on what information was available. The only reason one might take values at left and right endpoints would be to obtain bounds.

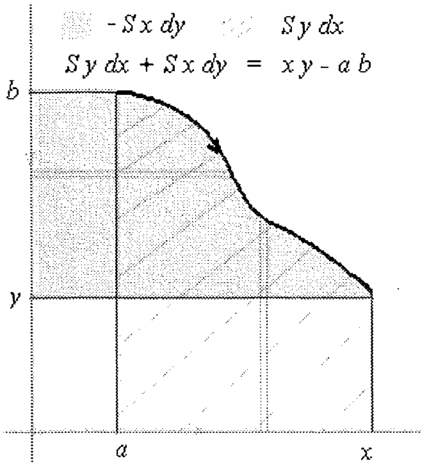

Figure 1: Integration by Parts

Clearly the area inside the circle of diameter 1 is given by `-Sum y̅ * d x`, where `y̅` is the average value of `y` over each element and the area can be seen as made up of the difference between two *areas under the curve.* (Alternatively, `Sum x̅ * d y` for the same result.) Approximating the average with `mids y` (the average of the two values at the ends of each element) and a million elements gave a value for $\pi$ correct to 6 decimal places (3.141592644). Similarly we could find the area bounded by any closed planar curve.
Areas *under the curve* comes up naturally here and in a discussion of the difference between `S y̅ * d x` and `S x̅ * d y` as we traverse a curve, keeping the accumulated sums along the way.

## 2 A Good Notation facilitates Thought

Single variable calculus is not too hard to handle even with relatively poor tools and it is amazing how far one can go without getting bogged down in the need for good algebraic skills. Few computer notations make it as easy to write natural mathematical expressions as Iverson's **J**. With strict right to left evaluation and vectorized operators, the executable statement `S y * d x` where `x` and `y` are lists of related values reflects Leibniz's original insights exactly ([4]). When it comes to calculus in several variables, most computer languages start to become unwieldy.

The way in which the demands of a computing environment can destroy the immediacy of an illustrative computation can be seen in the Waltons' [9] suggestions for evaluating double integrals from first principles. They produced 36 lines of code in BASIC to approximate $\iint_R 4-(x^2/9+y^2/16)\;dA$ $(=21.5)$ where $R$ was the rectangular region between the origin and (3,2). Here is an improved version in **J**.

```
dx =. dy =. 1
x  =. sample 0,3,dx
y  =. sample 0,2,dy
dA =. dx * dy
xm =. mids x
ym =. mids y
```

```
   x grid y
+---+---+---+---+
|0 0|1 0|2 0|3 0|
+---+---+---+---+
|0 1|1 1|2 1|3 1|
+---+---+---+---+
|0 2|1 2|2 2|3 2|
+---+---+---+---+
```

```
]rm =. xm grid ym
 +-------+-------+-------+
 |0.5 0.5|1.5 0.5|2.5 0.5|
 +-------+-------+-------+
 |0.5 1.5|1.5 1.5|2.5 1.5|
 +-------+-------+-------+
```

A **J** function `F` was defined to compute the function values at the coordinates within each box and then a grid of midpoints was created from the two lists by computing `rm =. xm grid ym`. The double integral was then obtained from[1]

```
Sum (F rm) *each dA.
```

yielding 21.5868 and with a $300\times200$ grid, 21.5000. The instruction remains unchanged for different functions `F` or for non-uniform grids for which `dA` is a matrix of the same size as `rm`, the array of representative coordinates in each element.

[1] `F =.(4: - [: +/ [: % & 9 16 *:) &.>`   `Sum =. [: +/ [: , >`

```
grid3d(<y),(<2*y),(<3*y)
+-----+-----+-----+
|0 0 0|1 0 0|2 0 0|
+-----+-----+-----+
|0 2 0|1 2 0|2 2 0|
+-----+-----+-----+
|0 4 0|1 4 0|2 4 0|
+-----+-----+-----+

+-----+-----+-----+
|0 0 3|1 0 3|2 0 3|
+-----+-----+-----+
|0 2 3|1 2 3|2 2 3|
+-----+-----+-----+
|0 4 3|1 4 3|2 4 3|
+-----+-----+-----+

+-----+-----+-----+
|0 0 6|1 0 6|2 0 6|
+-----+-----+-----+
|0 2 6|1 2 6|2 2 6|
+-----+-----+-----+
|0 4 6|1 4 6|2 4 6|
+-----+-----+-----+
```

A 3x3x3 boxed array.

If line density is used as a model to introduce line integrals then surface and volume densities can motivate the introduction of surface and volume integrals. The integration problem consists of partitioning the region into small elements, computing an approximation to the average density in an element, computing the size (surface area or volume) of the elements and finally summing up the products. Probably it is this understanding that, for engineering and other students of electricity and magnetism, is even more important than the ability to iterate and evaluate such integrals as can be completed in closed form [1].

It is not important how the primitives of a computer language are combined to produce a function such as `grid3d` or the earlier `grid` and `sample`. Their effect is easily understood and a student need only learn the proper syntax for calling each function. `grid` and `grid3d` provide arrays of coordinates in two and three dimensions – a facility that few other programming languages offer. To form a grid over a surface where `z` ← `G(x,y)`, the function `G` is applied to each cell of the grid formed by `r =. x grid y` with `Surface =. r ,each (G each r).`

```
   Surface
+-----+------+------+------+
|0 0 4|1 0 3 |2 0 0 |3 0 _5|
+-----+------+------+------+
|0 1 3|1 1 2 |2 1 _1|3 1 _6|
+-----+------+------+------+
|0 2 0|1 2 _1|2 2 _4|3 2 _9|
+-----+------+------+------+
   d1 Surface
+------+------+------+
|1 0 _1|1 0 _3|1 0 _5|
+------+------+------+
|1 0 _1|1 0 _3|1 0 _5|
+------+------+------+
|1 0 _1|1 0 _3|1 0 _5|
+------+------+------+
   d2 Surface
+------+------+------+------+
|0 1 _1|0 1 _1|0 1 _1|0 1 _1|
+------+------+------+------+
|0 1 _3|0 1 _3|0 1 _3|0 1 _3|
+------+------+------+------+
   ]a=. curtail d1 Surface
+------+------+------+
|1 0 _1|1 0 _3|1 0 _5|
+------+------+------+
|1 0 _1|1 0 _3|1 0 _5|
+------+------+------+
   ]b=. curtail"1 d2 Surface
+------+------+------+
|0 1 _1|0 1 _1|0 1 _1|
+------+------+------+
|0 1 _3|0 1 _3|0 1 _3|
+------+------+------+
   ]N=. a cross each b
+-----+-----+-----+
|1 1 1|3 1 1|5 1 1|
+-----+-----+-----+
|1 3 1|3 3 1|5 3 1|
+-----+-----+-----+
   ]dA=. mod each N
+-------+-------+-------+
|1.73205|3.31662|5.19615|
+-------+-------+-------+
|3.31662|4.3589 |5.91608|
+-------+-------+-------+
   Sum dA
23.8364
```

Computing the average density over each element can be done in various ways – averaging over the values of the density at the corners of the region (cf. trapezoidal rule), taking the value of the function at the "centre" of the element (cf. midpoint rule), taking the average of many randomly chosen points in the element (cf. Monte Carlo) or applying some more sophisticated averaging method (cf Simpson's rule). Again the details of how the software does these easily understood tasks is not really important. The important thing is that students can see that they must compute either the function values at representative points or average selected function values over an element so that each element is associated with an approximate average value. Next we need functions that find the vectors joining adjacent grid points in the $x$, $y$ and $z$ directions. Again it is not terribly important to know how the software achieves this, but having obtained the joining vectors, the student should be able to work out which pairs or triples are to be combined in cross or scalar triple products to obtain estimates of the size of the elements – and whether or not some averaging over different choices might not be appropriate.

The display at left shows the computation of the surface areas of the portion of the surface $z \leftarrow G(x, y) = 4-x^2-y^2$ with the same large elements considered earlier, using the approximating parallelograms obtained by completion from the vectors joining adjacent grid points from the lower left vertex. The surface area in the limit can be obtained by explicit integration as 24.1792, the same answer being obtained with a $300 \times 200$ grid.

A student who has mastered elementary vector algebra should be able to sort out this computation and similar volume integral computations. Basically, we need three functions – `d` to estimates the size of elements, `m` to find representative value for each element and `Sum` for `Sum (F m grid) * d grid`.

With most other mathematical interpreters, the simplicity of the computation is lost in a tangle of nested loops and subscripted variables and is quite impossible to incorporate into a mathematics class. The data structure in **J** and a display that evokes the geometry is what makes managing these computations possible.

# 3 Differentiation – before Integration?

The above discussion should suggest that there is a great deal of integration that can be done before any discussion of rates of change come to mind at all. The shear size of the grids required to achieve any sort of accuracy quickly leads to the realization that one has to be smart about the way one adds up things, about the way one divides up things. Being able to reduce a computation to a sequence of one-dimensional integrals is highly desirable.

Trying to be clever about variables leads to the question of how they change relative to one another. This is much more sophisticated than simple adding up and is reflected in the difference between using `Sum` and `S`. The former simply returns a number, the latter follows the changing values as we pass through intermediate points. (It is a key step in understanding what differentiating an integral is all about, but overlooked in most calculus texts.)

For example we might suspect that the volume of a sphere will, for a small increase in radius, increase in proportion to its surface area (`d V = A * d r`) or the area of a circle in proportion to its circumference (`d A = C * d r`).

In general, if we can discover these rates of change, or density, functions for our quantities then we can compute the quantities by integration, since

$$\mathtt{S\ d\ V} = (\mathtt{last} - \mathtt{first})\,\mathtt{V}$$

Thus finding approximate values for the relative changes

$$\frac{\mathtt{d}\,V}{\mathtt{d}\,r} \quad (= \mathtt{A})$$

and graphing variable `V` against `r` by computing the values of the integral `S A d r` seems to be the natural order of things: find a differential equation relating the variables and hence their relationship through integration.

In fact, all our elementary functions might be seen as the solutions of a differential equation that gives the function's derivative in terms of arithmetic combinations of elementary functions of the two variables, with the constant zero-valued function being the base on which we build. When we have established a collection of elementary functions we may be able to discover anti-derivatives (`V`) for the density functions (`A`) by means other than integration but there is a great deal we can do before that with functions constructed piece-wise from even the simplest elementary functions ([7]. Parrott's ([2]) strategy of teaching integration first has merit but more than just a change of order is required.

# 4 Conclusion

It is not just the ability to be able to compute integrals that is important but also the ability to do it conveniently without the baggage of computer languages. A computer implementable language like J has certain features that make for natural mathematical expressions. It allows dyadic, or in-fix, syntax for defined functions, as in `a cross b`. It allows functions to be defined in terms of other functions in various natural compositions such as `first - last`. It allows boxed arrays so that multi-dimensional grids, whose boxes contain coordinate strings, are easy to create and manipulate. It allows functions to be modified so as to apply, for example, to the contents of boxes as in `F each gridpoint`. These features make it feasible to set up and evaluate line, surface and volume integrals in a relatively natural notation so that definite and indefinite integrals be can computed not only without the need for proficiency in symbolic manipulation but also with a compelling demonstration that symbolic analysis and transformation are invaluable for achieving accuracy. The instructor may then be able to make a strong case for describing integration as a method by means of which we can solve differential equations. The fact that sometimes integration problems can be solved by anti-differentiation will then be seen as a wonderful bonus.

## References

[1] E. J. Burge,*Mastering the Integrals of Basic Electricity and Magnetism*, Physics Education **22(6)** (1987), 375–380.

[2] David Parrott, *Integration First?*, The Challenge of Diversity: Proc. $\Delta'99$ Symp. Undergrad. Math., The $\Delta'99$ Committee, Toowoomba, 1999, pp. 155–159 (`http://www.sci.usq.edu.au/staff/spunde/delta99/Papers/parrott.pdf`)

[3] B. Rodin, *Calculus with Analytic Geometry*, Prentice Hall, New Jersey, 1970.

[4] W.G.Spunde, *What Leibniz might have done... (to introduce calculus with a computer)*, Proceedings 1st Asian Technology Conference in Mathematics, Association of Mathematics Educators, Singapore, 1995, pp. 511–520.

[5] Walter G. Spunde & Richard D. Neidinger, *The Sample Calculus Website*: `http://www.sci.usq.edu.au/staff/spunde/samplcal/`

[6] Walter G. Spunde & Richard D. Neidinger, *Sample Calculus*, The Mathematics Magazine, 72(3), 1999, pp. 171–182.

[7] Gilbert Strang, *Sums and Differences vs. Integrals and Derivatives*, College Mathematics Journal, **21(1)**, 1990, pp. 20–27.

[8] Hugh Thurston, *Can We Improve the Teaching of Calculus*, College Mathematics Journal, **31(4)**, 2000, pp. 262–267.

[9] Karen D. Walton, & Zachary D. Walton, *Computer Techniques for Evaluating the Double Integral*, J. Computers in Mathematics and Science Teaching, **11(3-4)**,1992, pp. 393–401.

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# AN UNDERGRADUATE COURSE FOR TALENTED SENIOR HIGH SCHOOL STUDENTS

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In most large university cities there are a significant number of high schools which have a problem challenging their high-achieving final year students. In this paper we describe the experience of our Department in developing and delivering a course to meet this need. We have used the University's Provisional Entrance procedures to allow such students to register for this course while in their final year (Year 13) of high school. They are able to credit this course to their subsequent degree studies. We discuss such things as the history, the curriculum, some of the outcomes, and the future of this course.

## Introduction

A number of factors have converged in time to support the setting up at The University of Auckland of a Mathematics course for talented Year 13 high school students. Such a course has been provided in each of the last three years. It is on an evolutionary path, partly because of changes in the New Zealand secondary sector.

Over the past decade the demographic characteristics of the typical New Zealand high school have changed. One major feature has been the much larger percentage of the age cohort which has continued to the end of high school — Year 13 (seventh form). Teachers are required to target the average student, and it can be argued that curriculum and assessment changes have exacerbated that trend. One outcome is that there is less challenge for the talented Year 13 students than there once was. Several heads of high school Mathematics Departments have suggested that The University of Auckland consider the provision of a course designed for such students.

In the 1990s The University of Auckland steadfastly refused to permit the concurrent enrolment of high school students. However in semester 1 of the 2000 academic year the Department of English trialled the participation of about twenty Year 13 students in one of their courses. It was an outstanding success from all points of view. The authors learnt about this English experience and persuaded the Department of Mathematics to provide a special course for appropriate students in the 2001 academic year. Section 2 of this paper is extracted from reviews of the experience and outcomes of the first years of this course. The course acquired the nickname "MAX" – Mathematics Acceleration and eXtension. It developed as a hybrid version of the union of the two usual first year University of Auckland courses for students who have taken Mathematics successfully to the end of high school. In 2001, these two courses were MATHS 151 and MATHS 152. In section 4 we describe future directions for MAX, especially our efforts to collaborate with our colleagues in the Department of Engineering Science of the University's School of Engineering.

## Review of experience and outcomes

### *MAX in 2001*

MAX started the year with 30 students. The target group was students with 75% or better in Bursary Calculus or a 1 in SFC Mathematics, the latter intended only for exceptional students.

Two students withdrew early because of new sporting commitments and five (all without Bursary Calculus) withdrew a little later after finding the course rather more difficult than was comfortable for them. Sixteen of the students, when offered the option in early September, chose to sit the second semester MATHS 151 exam, where ten gained A grades (five with A+). Two MAX students gained the top two marks in the 151SC exam. Many of these students were capable of doing well in 152, but found themselves under pressure from Bursary expectations towards the end of the year. The remaining seven sat the MATHS 152 exam, and six achieved A+.

These results show that the course is practicable. Student evaluations were positive about the course, in spite of the difficulties encountered in several areas. Most students said that it provided them with challenge and interesting mathematics, while some of the more able students said it had been essential in keeping them interested in mathematics as they were finding it tedious repeating Bursary Calculus at school.

MAX staff learnt a great deal during the course about the problems of teaching year 13 students, and planned changes intended to make the 2002 course experience more satisfactory for both staff and students.

*Workload problems*

1. The course compressed 96 standard lectures into 48, which proved a little tight (compare this with MATHS 130, the accelerant option, which does not cover all of the MATHS 151 and MATHS 152 material).
2. Many students missed a significant number of lectures due to school and sporting camps, school exams, and to family holidays during university semester time.
3. In spite of staff efforts to organise the syllabus delivery so that the workload was minimised towards the end of the year, many students found themselves under pressure and chose to sit the 151SC examination, thus avoiding finishing and revising the 152 syllabus before their Bursary examinations.
4. The very long break between semesters meant that the first few weeks of the second semester were a struggle for staff and students, as the latter had forgotten a great deal.
5. Half hour tutorials were held weekly. In contrast to 151 and 152 students, MAX students did not find these valuable. Half an hour did not appear long enough for the students to become involved in discussing and solving the problems offered. Meeting and discussing mathematics with other high achieving students should be a very important part of the course, and tutorials provide the only such opportunity.

*Solutions to workload problems*

We decided to hold lectures during school term instead of during university semesters. By starting in the second week of the school year we were able to offer approximately 32 weeks of lectures in place of 24, with slightly shorter lectures and longer tutorials. We offered MATH 151FC through MAX lectures: that is, students who found the time commitment heavier than was comfortable in the second semester could enrol in MATHS 151 in February or change their enrolment in mid March, and finish their university studies in June.

We stressed to potential students and to their teachers that the course is specifically for those who enjoy mathematics and can handle a solid workload. There appears to be some "snob" value in enrolling in a university course early, and we hoped to better discourage unsuitable candidates.

*Communication problems*

1. MAX staff were accessible to students outside lectures via telephone, email and initially also office hours. Of these three options students used only email (and not many used that). In addition a website was provided, and most (but not all) students used it. We did not have satisfactory communication with students between lectures (to announce errors in assignments or for tutoring purposes), or with students who missed lectures.
2. For students who missed lectures the MAX website could cover the gap but only if the week's assignments, tutorials and handouts were on the site shortly after the lecture. In this case the handouts needed to show expected reading, exercises and usually to review parts of the text. In addition the short lecture time available meant that the handouts could usefully contain important examples in some detail and occasionally proofs and overviews. Creating these handouts in a form suitable for the website made a higher workload for staff than was always comfortable.

### *Enrolment problems*

These were a major factor in 2001. Much of the problem was student generated. Year 13 students are not surrounded by friends familiar with university procedures and nor did many MAX students understand they should take these procedures seriously. MAX staff had to track down three students who considered their absence from lectures sufficient notice of their decision to withdraw from the course. Other students failed again and again to provide their SFC results. A plan aimed at improving enrolment procedures for Year 13 students was drawn up in conjunction with the University's Year 13 committee.

### *Fees*

The course in 2001 cost each student $735. After representations made by Ivan Reilly, in 2002 the Provisional Entrance fee of $60 was waived for year 13 students, and the Student Services fee was reduced from $167 to $25 per semester. This reduced the cost of MATHS 153 to $558 and made the option of offering MATHS 151 ($533) as a stand-alone paper viable. Some high schools used the Ministry of Education's STAR scheme to subsidise these fees for their students.

### *MAX in 2002*

The new schedule – having classes in school term and starting in early February – gave us 14 weeks for MATHS 151 content, and 17 for MATHS 152 content. After some experimentation we used this for only slightly more lecture time (including an extra 6 for revision), with a 55 minute lecture followed by a 45 minute tutorial, and then a 40 - 45 minute lecture each week.

This has been a very successful change: students mainly missed lectures only when school exams were being held, and the tutorial experience was much improved and much more appreciated. The students had enough time to get to know each other, to do some serious work and to enjoy doing mathematics together.

Starting lectures in February meant that students had a good understanding of the workload and were be able to make an informed decision about which course suited them best before the last date for making course changes, in mid March.

We introduced an application form requiring schools to approve enrolment and to estimate the results expected in Bursary or SFC to better ensure our students belong to the target group. The form also provides staff with full contact details for all students right from the start. We were able to communicate between lectures by email and /or telephone where necessary. Better communication with both the university administration and with the students also meant that we were able to provide help with enrolment problems.

Much more use was made of handouts containing essentially the lecture material, mainly freeing the students from writing, and more time was spent having the class contribute during the lectures. We continued the practice of short weekly assignments and the Max website continued to be valuable. Persuading students to seek tutoring from staff was still a challenge, and only a few students had teachers at their schools who provided useful help. All students kept in touch via email, generated by the fact that staff could initiate email contact, but not many of these emails requested mathematical advice. Half the students came to the university for extra tutoring during the year, usually after finding the scheduled revision sessions useful.

The 153 A&B course evaluations (taken in late August, see Section 3) were very good.
The students without Bursary Calculus were much happier than were those in 2001, and all students would recommend the course to others with a similar background to their own.

### *Target group*

This was students with a 1 in SFC Mathematics, or 70% or better in Bursary Calculus, slightly down from our first year as the option to stop with MATHS 151 makes the programme viable for students who would otherwise be under too much pressure at the end of the year. The students without Bursary Calculus had some difficulties in the first semester, but did not appear disadvantaged thereafter. The lowest Bursary Calculus mark amongst MAX students in 2002 was 71, this was from a highly motivated student who got an A+ in MATHS 151.

*MAX in 2003*

The class of 2003 is currently progressing very successfully on the same basis as for 2002.

We find every year that quite a few students come to the first two or three classes before discovering that a university course is actually rather more work than they wish to be involved in just yet. A few students withdraw a little later after finding that their school or sporting activities do not leave them enough time for MAX. Those remaining are highly motivated students who thoroughly enjoy both the mathematics and the contact with each other. Most of them do not plan a career in mathematics, but are enjoying the challenge provided by the course.

We began to use an internet forum this year. That has been very successful with the class, although it does not contain a lot of mathematical discussion. Around 20% of the class regularly receives tutoring via email, and about 30% of the class has so far come to the University for face to face tutoring. Most students back themselves to sort out their problems without extra help, and generally do so.

Over the two and a half years of the existence of MAX, 55% of the students have gained an A+ in their course, 77% gained an A grade, and B is the lowest grade achieved. The corresponding figures for the last year and a half are 69% and 92%, reflecting mainly the fact that the course is more manageable for students when held in school term time rather than in university semester time.

## Student views

Below is the 2002 MAX student course evaluation, taken in late August.

***Please respond to the following statements on the basis of your experience in MATHS 153 this year. Consider each item and indicate the extent to which you either agree or disagree with a statement, using the following scale:***

**1 = Strongly Disagree, 2 = Disagree, 3 = Neutral, 4 = Agree, 5 = Strongly Agree**

| | **Class average response** |
|---|---|
| 1. I usually have a clear idea where I am going and what is expected of me in this course..... | 4 |
| 2. The lecture handouts are very useful for me.................................................................... | 4.7 |
| 3. The tutorials are valuable for me...................................................................................... | 4.5 |
| 4. The course is overly theoretical and abstract..................................................................... | 2.8 |
| 5. The lecturers work hard to make their lectures interesting............................................. | 4.5 |
| 6. I am generally given enough time to understand the things I have to learn...................... | 3.2 |
| 7. My lecturers are extremely good at explaining things..................................................... | 4.7 |
| 8. This course puts a lot of pressure on me as a student.................................................... | 3.3 |
| 9. The Study Guide is very useful for me............................................................................ | 4.5 |
| 10. As a result of this course I feel confident about tackling unfamiliar problems.................... | 4.3 |
| 11. The lecturers make it clear right from the start what they expect from students.............. | 4.5 |
| 12. Overall I am satisfied with the quality of this course...................................................... | 4.5 |
| 13. I would recommend this course to students with a similar background to my own......... | 4.5 |

**Amongst replies to the question "What have you liked about this course?" were the following:**

"This course is very challenging and fun to do, being able to do these problems makes me feel very excited and questions at a lower level seem much easier...."

"Challenging myself ... its just fun to try something hard and know that you can do it. Also helps with school calculus... going to uni is fun."

"Able to meet other people who are also into maths...."

"Some topics have really helped me in my 7th form work (I didn't do 7th form calculus in 6th form) ... The class has been friendly and helpful. The lecturers and tutors are very patient, helpful and approachable."

"I have really enjoyed the extra challenge and rigour provided... I also like the fact that the course materials have been available on the web in case of missed lectures etc."

## Plans for 2004

In 2004 we will be joining with the Department of Engineering Science, who have been providing ENGSCI 111 extramurally to high school students for the last two years, to offer a joint first semester paper available both internally and extramurally. This coincides and works well with changes in both the standard stage 1 entry courses and in sixth and seventh form qualifications. Students accepted into the MAX programme for 2004 will enrol in MATHS 153, a first semester 2 point course which will be academically equivalent to the first year engineering mathematics course ENGSCI 111 (Mathematical Modelling 1) and to the new first year mathematics course MATHS 150 (Advancing Mathematics 1, replacing MATHS 151 at a slightly higher level). It is also an excellent alternative to MATHS 108 (General Mathematics 1, currently called Mathematics for Commerce and Technology). Students who pass MATHS 153 will be eligible to enrol in any of ENGSCI 211, MATHS 250 or MATHS 208 when they become full time students.

Admission to the MAX programme requires the consent of the Department of Mathematics. Consent will be given to any student currently at high school who:

- has a mark of 70% or better in Bursary Mathematics with Calculus; or
- has achieved excellence in some of the Mathematics Achievement Standards 2.1, 2.2, 2.3, 2.4 and 2.8 or 2.9 and at least merit in the others, and is studying Level 3 Calculus in 2004;
- and has a strong recommendation from their school's Head of Mathematics. Intending distance students must be able to study well on their own and from written material.
- the University's global requirements must also be met.

The course will be delivered, for extramural students, via a compact disc containing the lectures mostly in power point format, with, as far as possible, the added comments and explanations that would be made during a standard lecture. Along with this is a booklet containing the lectures for easier reference, and there will be a required text. We will have a class forum, and will add an FAQ (Frequently Asked Question) page to the web page and we expect that students will make good use of all these and will also use email for tutoring. It is hoped that monthly tutorial sessions will be held in some centres so that distance students can meet the lecturers and fellow students and review the work already covered.

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# Practical Number Theory in a Discrete Mathematics Class: The RSA Cryptosystem in Microsoft Excel and on the Web

Stephen J Sugden

**Abstract**

Digital cryptography is an increasingly important topic in modern discrete mathematics and CS/IT classrooms [2]. The Rivest-Shamir-Adleman (RSA) cryptosystem [11] has been around for almost 30 years. Despite many intensive efforts to crack it, these have met with only limited success [12]. The system remains a very important, widely-used public key system, and it is certainly worthwhile to present the underlying mathematics to CS/IT majors. However, it is a formidable challenge to teach such material while coping with the great diversity of student backgrounds. This paper discusses one approach.

## 1 Background

Bond University is a private university in Queensland, Australia. As noted in some recent papers and presentations [7], [8] the diversity in student background and skills at Bond, certainly where mathematical fundamentals are concerned, is very great indeed. Many students have excellent grounding in the traditional areas of high-school mathematics, such as algebra and calculus, and, not-surprisingly, with well-developed problem-solving skills. By contrast, a typical tertiary class in Discrete Mathematics also contains significant numbers of students whose mathematics background is very modest indeed. Such students struggle with basic algebra. Why are these students in the class anyway? Short answer: the unit is a compulsory one for IT students. For the Bachelor of Information Technology (BIT). although many majors exist, including a mathematics/statistical modelling major (Forecasting and Modelling), very few students indeed choose this.

Even though a vast body of mathematics underpins most or all of the BIT majors at Bond, the 24-unit degree contains just two compulsory mathematics/statistics units: *Discrete Mathematics* and *Statistics for Computing.* A recent survey [13] by the author indicates that two compulsory units of mathematics in a Bachelor of IT is the norm for Australian universities, with discrete mathematics being required in almost all cases. Given this situation, in many respects not unlike many other universities, it becomes a real challenge for the teacher of compulsory mathematics units to cater for the broad range of student skills and backgrounds, while still managing to cover the advertised syllabus, and making the material clearly relevant to discipline majors. Many authors have created or adapted some very innovative techniques for handling this problem, and their efforts are well-documented; for example, [9]. Whatever approaches are used for presenting essential mathematics for IT students, the teacher must surely strive to make the material appear relevant, with clear examples of applications of mathematical principles to the appropriate IT sub-disciplines.

## 2 Possible assignment topics

Topics in the IT mathematics subjects at Bond for which the author has set assignments include automata, recurrences (Catalan binary tree counting), design of combinatorial logic circuits using

Boolean Algebra and implementation by way of a software simulator (Capilano's LogicWorks [3]), basic propositional and predicate calculi, and elementary combinatorics and discrete probability. The author has done a considerable volume of consulting work on KENO for Jupiter's Casino in Australia [14], and the some of more elementary mathematical principles behind this game are suitable for investigation by undergraduate students. The students definitely appreciate practical real-world examples. A gambling example such as this one also enables students to gain some appreciation of expected return on investment if they choose to participate in such forms of recreation. Basic principles of probability and expectation are employed to design a payscale for the KENO game, and the students quickly see how much they stand to lose if the game is played over a period of time. For many of the assignments and laboratory work, *Microsoft Excel* is the modelling vehicle used.

## 3 Algorithms and programming

One of the ways in which the Discrete Mathematics topics are made to appear relevant is to look at the elementary algorithms associated with basic mathematical concepts and then ask the students to implement these in a language such as Visual BASIC, Delphi or Java. However, Discrete Mathematics is a first-year subject with no prerequisite. Thus, it cannot be assumed that the students have done any computer programming at all. However, it is usually the case that a good percentage (more than 50%) have done at least one programming subject, either at school or at university. Interestingly, in the author's recent experience, *it has nowadays become quite common to encounter IT students who have quite a strong aversion to computer programming!* At first it was thought that this may be a local phenomenon. However, discussion with colleagues at other institutions points to the conclusion that it appears to be quite a widespread phenomenon. Because of this, and a desire to expose the mathematical principles as clearly as possible without computer code getting in the way, and finally, a personal distaste for the Java language, the author has, in recent years, sought to do as much modelling in Microsoft Excel as possible. In the vast majority of instances, mathematical fundamentals are illustrated with no coding whatsoever. With the exception of one student, who was reluctant to try Excel, since she was used to her graphics calculator, in the author's experience, *there has not been a single negative comment about the use of Excel for mathematical investigations in about seven years of using this approach.* Another benefit of using Microsoft Excel is its ubiquity. Nowadays, students tend to already have some reasonable familiarity with this very common package before they embark on tertiary studies. Generic skills gained in using for the relatively abstract mathematical modelling of Discrete Mathematics are transferable to other application areas. For further discussion of such "spin-off" benefits, see for example [7].

## 4 RSA and number theory

In the Bond September trimester 2002, the RSA Cryptosystem (see Appendix) was introduced as an application of basic number theory and arithmetic of congruences. Apart from a brief mention of the quotient-remainder theorem, and $\mathrm{mod}(x, n)$ or $x \bmod n$ as a two-place function (remainder after division), the course until recently had no material on modular arithmetic. In order to support the RSA investigations, one or two lectures were introduced (with supporting workshop and tutorial) on modular arithmetic and simultaneous linear congruences (SLCs). Room for this material, including sections on solving modular equations, was made by deleting a section on recurrence relations, which the students have always found very difficult (it contained significant amounts of algebra).

Since the solution of SLCs may be regarded as the intersection of several arithmetic sequences, solutions may be illustrated very neatly in Excel by making use of a feature known as *conditional* *formatting.* The students are also required to demonstrate mastery of an algebraic approach to solving SLCs.

Basic concepts of primality, the Fundamental Theorem of Arithmetic, and elementary notions of proof are introduced in the lectures. Extra material, including elementary functions of number theory and their application to RSA are introduced in several workshop sessions. Most of the modelling work is done in Microsoft Excel, with a very small amount of Microsoft's Visual Basic for Applications (VBA), so that, for example, one may compute Euler's totient $\varphi(n)$, and the number of divisors function, $\tau(n)$, as defined by eqs 1, 2.

$$\varphi(n) = \sum_{\substack{\gcd(n,k)=1 \\ k\leq n}} 1 \tag{1}$$

$$\tau(n) = \sum_{\substack{n \bmod k=0 \\ k\leq n}} 1 \tag{2}$$

As noted, I generally try to avoid a lot of coding in my efforts to use the computer to illustrate fundamentals of Discrete Mathematics. For the RSA venture, I was forced to depart from this rule slightly, but was still able to reduce the requirement to just three or four VBA functions, each of which was only a few lines of code. For the RSA assignment, students were required to write two of these: modular exponential and modular inverse. For example, assuming that an inverse exists, some naive, but correct, code for modular inverse in VBA is as follows:

```
Function modinv(t As Long, m As Long) As Long
    modinv = 0
    Do
        modinv = modinv + 1
    Loop Until (modinv*t) Mod m = 1
End Function
```

# 5 Scientific Workplace with Maple V Release 5.1 or MuPAD 2.0

The document processor on which this paper was prepared is Scientific Workplace (SWP) 4.0 [10]. It embodies a LaTeX processor with essentially WYSIWYG support for writing mathematical documents and also for *doing mathematics*, and only rarely does one need to resort to hacking out the TeX code. To support the *doing mathematics* capabilities, the software incorporates a Maple V Release 5.1 and MuPAD 2.0 computer algebra and numerical computation engines, so that a wide variety of mathematical operations can be rapidly invoked. While browsing through the many supplied sample documents containing examples of both mathematical typesetting and computation, I happened to find one on the RSA Public Key Cryptosystem [11]. It was little more than half a page, and contained a full example with primes of moderate size (11 decimal digits).

The SWP software, with Maple or MuPAD, allows the large-integer computations (hundreds of decimal digits) necessary for RSA. The students were not able to use this facility, but it did allow preparation of significantly larger examples (for lectures) than were possible in Excel. However, students *were* able to access the ACME website [4] for large examples (see Appendix). In any case, it was clear from the SWP documentation, the basic, but sometimes deceptive, simplicity of the RSA system could be presented to the students: software tools had matured to the point where the mathematics of this very common cryptosystem could be realistically illustrated to a discrete mathematics class without a lot of "lead-up" material. This is described in the next section.

# 6 Strategy for teaching and assessing RSA

1. Prepare a few lectures on modular arithmetic and solution of simultaneous linear congruences. This material included discussion of the basic fact that modular arithmetic is nothing strange or new: all digital computers use it for integer calculations! To not know this is to be essentially ignorant of phenomena such as overflow due to finite word size, and the options one has in dealing with such inevitabilities. Modular multiplication tables were constructed effortlessly in Excel, and the concept of modular inverse easily defined by references to such tables. [Aside: wherever possible, I use references to "obvious" computing concepts such as "table lookup" or "array searching", and find that these tend to remind the class that we are not looking at just mathematical arcana, but concepts with immediate application to computing.]
2. Use Microsoft Excel to illustrate the basic computations, with very small integers, and a few simple VBA functions.
3. Prepare a few examples, each using larger integers than its predecessor, with the necessary computations done in SWP/Maple.
4. The students were then given unique public and private keys, created by me in Excel, and some ciphertext to decrypt.
5. In order to do this, they were asked to write the necessary VBA functions, and use Excel to perform the decryption.
6. The website [4] was then used by the students to verify their Excel/VBA computations.
7. Now familiar with the website, they were then required to choose their own pairs of 128-bit primes, generate public and private keys, and encrypt and decrypt a given piece of text.
8. The final task was a small research essay on how RSA achieves both secrecy and authenticity by use of super-encryption.

# 7 Conclusion

The students performed rather well on these tasks, and it was very pleasing to see that they were able to complete the assignment without a great deal of help from me. Some implemented more efficient versions of the VBA functions, and it was clear that they appreciated the practical nature of the assignment. Several unsolicited comments were received about how interesting and relevant the assignment was. Naturally, this was very encouraging, and now that I have a fair idea of its difficulty, I plan to extend and improve the assignment for next year.

# 8 Appendix The RSA Assignment, September 2002

## 8.1 Introduction

For this assignment you have to investigate the Rivest-Shamir-Adleman (RSA) security system. The RSA public key cryptosystem is based on the computational difficulty of factoring large integers. For each user of the system, a key issuing authority allocates a pair of keys: a public key and a private key. A public key is just like a telephone number, and may be listed in a directory accessible to anyone. The private key, as the name suggests, is known only to the owner, and is like a password.

For this assignment, each student is given a pair of keys, $e$ (public) and $d$ (private), plus the modulus, $n$, and some ciphertext, $y$, to decipher. For yours, check the subject website.

## 8.2 Brief Summary of RSA Theory

We first generate a suitable pair of keys, one public and one private. This process is described, followed by the encryption and decryption formulas.

1. Generate two large prime numbers, $p$ and $q$. In a real implementation, these values are kept secret, and each is typically a 512-bit integer.

2. Compute $n = pq$. The value of $n$ is published. By the Fundamental Theorem of Arithmetic, its only factors are $p$ and $q$, and it may be up to 1024-bits in size. This is a large number! Such numbers are difficult to factorize in a reasonable time, even with supercomputers. The strength of the cipher relies on this basic fact.

3. Compute the Euler *totient* function $\varphi(n)$. This is the number of positive integers $\leq n$ and relatively prime to $n$. Since $p$ and $q$ are *relatively prime* and totient is *multiplicative*, we have $\varphi(n) = (p-1)(q-1)$.

4. Let $x$ be our original message, called *plaintext*, represented as a positive integer. Long messages must be broken up into smaller pieces so that for each piece $x$ it is true that $x < n$.

5. Choose $e$ to be a moderately large positive integer with the property $\gcd(e, \varphi(n)) = 1$. The public key is then given by the pair $(n, e)$.

6. Compute $d = e^{-1} \bmod \varphi(n)$. The private key is the given by the pair $(n, d)$.

7. Encryption. To encrypt the plaintext $x$ to get the *ciphertext* $y$, we use the formula $y = x^e \bmod n$.

8. Decryption. To decrypt the ciphertext $y$ to recover the plaintext $x$, we use the formula $x = y^d \bmod n$, where $d$ is obtained from step 6.

## 8.3 Example

1. Select $p = 31$ and $q = 19$.

2. Then $n = pq = 589$.

3. Totient $\varphi = (p-1)(q-1) = 540$.

4. Let our plaintext be $x = 229$.

5. Choose $e = 119$, so that $\gcd(e, \varphi) = \gcd(540, 119) = 1$.

6. Compute $d = e^{-1} \bmod \varphi(n) = 119^{-1} \bmod 540 = 59$.

7. Encryption. Ciphertext $y = x^e \bmod n = 229^{119} \bmod 589 = 571$.

8. Decryption. Recover plaintext $x$ as $x = y^d \bmod n = 571^{59} \bmod 589 = 229$.

## 8.4 Problems

You must solve the following problems. Your submission must include all Excel or other models, the relevant equations, references and URLs.

1. Develop an Excel model with the following columns: $n, y, d, x$. These are respectively the modulus for the cipher, the given ciphertext, your private key, and the decoded plaintext. To complete this model, you will need some special functions (see next question). (1 mark)

2. Write the functions **modexp** and **modinv** in VBA so that they may be used in your Excel model. The definitions of these functions follows.

$$\text{modexp}(x, n, m) = x^n \pmod m$$

$$z = \text{modinv}(x, m) \text{ is the (unique) solution of the equation } zx = 1 \pmod m$$

Note that you must code your functions in such a way as to handle very large integers. Note also that the function may assume $\gcd(x, m) = 1$. A more efficient solution here means more marks. You may need to do some research. (4 marks)

3. Decode the ciphertext given to you personally using your private key, and submit the decoded text as part of your solution to this assignment. See Figure 1. (1 mark)

4. Go to the site http://www.acme.com/software/bigint/ and check your solutions from above by using the mathematical functions available there. (1 mark)

5. At the site http://www.acme.com/software/bigint/, demonstrate how RSA works by creating public and private keys of at least 128 bits each. Encode the message "CRYPTOGRAPHY" using your public key, and demonstrate correct decoding of it using your private key. (2 marks)

6. Investigate the cryptographic concept of *authenticity*, and describe how a message may be encrypted by person A and sent to person B using the RSA system while achieving both security and authenticity. Your answer must carefully describe the method and give appropriate citations to relevant literature, at least one of which must not be a URL. (1 mark)

# References


[1] Rosen, K.H., (1995) *Discrete Mathematics and its Applications*, McGraw-Hill.

[2] Baliga, A., Boztas, S., Cryptography in the classroom using Maple, *Proceedings of the Sixth Asian Technology Conference in Mathematics, ATCM 2001*, Edited by W. Yang, S. Chu, Z. Karian, G. Fitz-Gerald. Dec 15 - 19, 2001, pp.343-350.

[3] Katz, R.H., (1994) *Contemporary Logic Design with Logicworks,* Benjamin-Cummings Publishing Company, ISBN: 0805327126.

[4] ___, Acme software website, URL: http://www.acme.com/software/bigint/

[5] Schneier, B., (1996) *Applied cryptography: protocols, algorithms, and source code in C, second edition*, New York, Wiley.

[6] Denning, D.J. (1982) *Cryptography and Data Security*, Reading, Mass, Addison-Wesley.

[7] Sugden, S. J. (2001). Bits, Binary, Binomials and Recursion: Helping IT Students Understand Mathematical Induction. *Quaestiones Mathematicae, Journal of The South African Mathematical Society*, Supplement #1.

[8] de Mestre, N. and S. J. Sugden (1999). Designing a Successful General Elective Mathematics Course. *Delta '99 Symposium of Undergraduate Mathematics, the Challenge of Diversity*, Laguna Quays, Queensland.

[9] Spunde, W., Cretchley, P., Hubbard, R. (eds), *Proceedings of Delta '99: Symposium of Undergraduate Mathematics, the Challenge of Diversity*, Laguna Quays, Queensland.

[10] Hunter, R., Bagby, S., (2002) *Creating Documents with Scientific Workplace and Scientific Word Version 4.0,* MacKichan Software.

[11] Rivest, R., Shamir, A. and Adleman, L. (1978) A method for obtaining digital signatures and public key cryptosystems. *Communications of the ACM*, 21(2):120–126.

[12] Cavallar, S. *et al.* (2000), Factorization of a 512-Bit RSA Modulus, in B. Preneel (Ed.): *EUROCRYPT 2000*, LNCS 1807, pp. 1-18, 2000, Springer-Verlag Berlin Heidelberg.

[13] Sugden, S. J. (2000). *Should IT Students Study Mathematics?* QANZIAM Conference Presentation, Noosa Heads, Queensland, Australia.

[14] Noble, C. and S. J. Sugden (2002). Stochastic Recurrences of Jackpot KENO. *Computational Statistics & Data Analysis* **40**: 189-205.

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# A GRAPHICAL ILLUSTRATION OF THE LIMIT OF A COMPOSITE FUNCTION

KOK-MING TEO

In first-year undergraduate calculus course, students learn the concept of the limit of a function and the various limit laws for calculating the limits of functions. In this paper, we look at how a graphical method for composing functions can be used to give a geometric argument of the following theorem on limit of composite function:

If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$.

This serves to convince students who do not learn the $\varepsilon - \delta$ definition of limit why the theorem is intuitively true. We also give a graphical illustration of why the theorem is false if the hypothesis that $f$ is continuous at $b$ is not included.

## Introduction

In a typical first-year undergraduate calculus course, students learn the concept of the limit of a function, and the different techniques of evaluating the limits of functions. In order to calculate the limits of different functions, they have to know the various limit laws. For example, they learn how to evaluate the limit of a function formed by the sum, difference, product, quotient or composition of two functions by applying the corresponding limit law. It is not difficult for the students to believe that the limit of the sum of two functions is the sum of the limits of the respective functions; and likewise for the limits of the difference, product and quotient of two functions. However, the following theorem (see, e.g., [2], section 2.4 Theorem 8), which allows one to compute the limit of a function formed by the composition of two functions, is not intuitively clear to the students.

## Theorem

If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b).$$

As a lecturer of a calculus course, I was not satisfied with simply asking my students to memorize the theorem and use it to calculate the limits of some composite functions; I wanted to convince them that the theorem is intuitively true. Since my students did not learn the epsilon-delta definition of limit, I could not show them the rigorous proof. I had come across the paper [1] on composing functions graphically, and also a problem from [2] Exercises 1.3 Question 59 which has a similar geometric construction of function composition. It struck me later that I could actually use the graphical method for function composition to give a geometric argument of why the theorem is true.

In this paper, we shall give a graphical illustration of the above-mentioned theorem. We shall also show graphically why the theorem is false if the hypothesis that $f$ is continuous at $b$ is not included. Lastly, we give an example to illustrate the pitfall of the graphical method, which serves to warn the students that it is not a substitute for a rigorous proof.

## Graphical Illustration

We assume that the reader is familiar with the graphical method for function composition (see [1]). We draw the graphs of the functions $f$ and $g$ on the same coordinate axes. Note that we choose a function $g$ that is not continuous at $a$ to illustrate the point that $g$ does not have to be continuous at $a$. Following the graphical method of function composition, we construct two sets of horizontal and vertical lines, as shown in Figure 1.

One set begins with a vertical line drawn from the point $(a,0)$ on the $x$-axis to the point $(a, b)$, where $b = \lim_{x\to a} g(x)$, and ends with a horizontal line drawn from the point $(b, f(b))$ on the graph of $y = f(x)$ to the point $(0, f(b))$ on the $y$-axis. The other set begins with a vertical line drawn from an arbitrary point $(x, 0)$, which is close to $(a, 0)$ on the $x$-axis and ends with a horizontal line drawn from $(g(x), f(g(x)))$ on the graph of $y = f(x)$ to the point $(0, f(g(x)))$ on the $y$-axis. Now, the students can visualize that as the vertical line from $(x,0)$ to $(x, g(x))$ approaches the vertical line through $(a,0)$, the corresponding horizontal line from $(g(x), f(g(x)))$ to $(0, f(g(x)))$ approaches the horizontal line through $(0, f(b))$, that is, $f(g(x))$ gets closer and closer to $f(b)$ as $x$ approaches $a$. In this way, we can illustrate to the students graphically why the theorem is intuitively true.

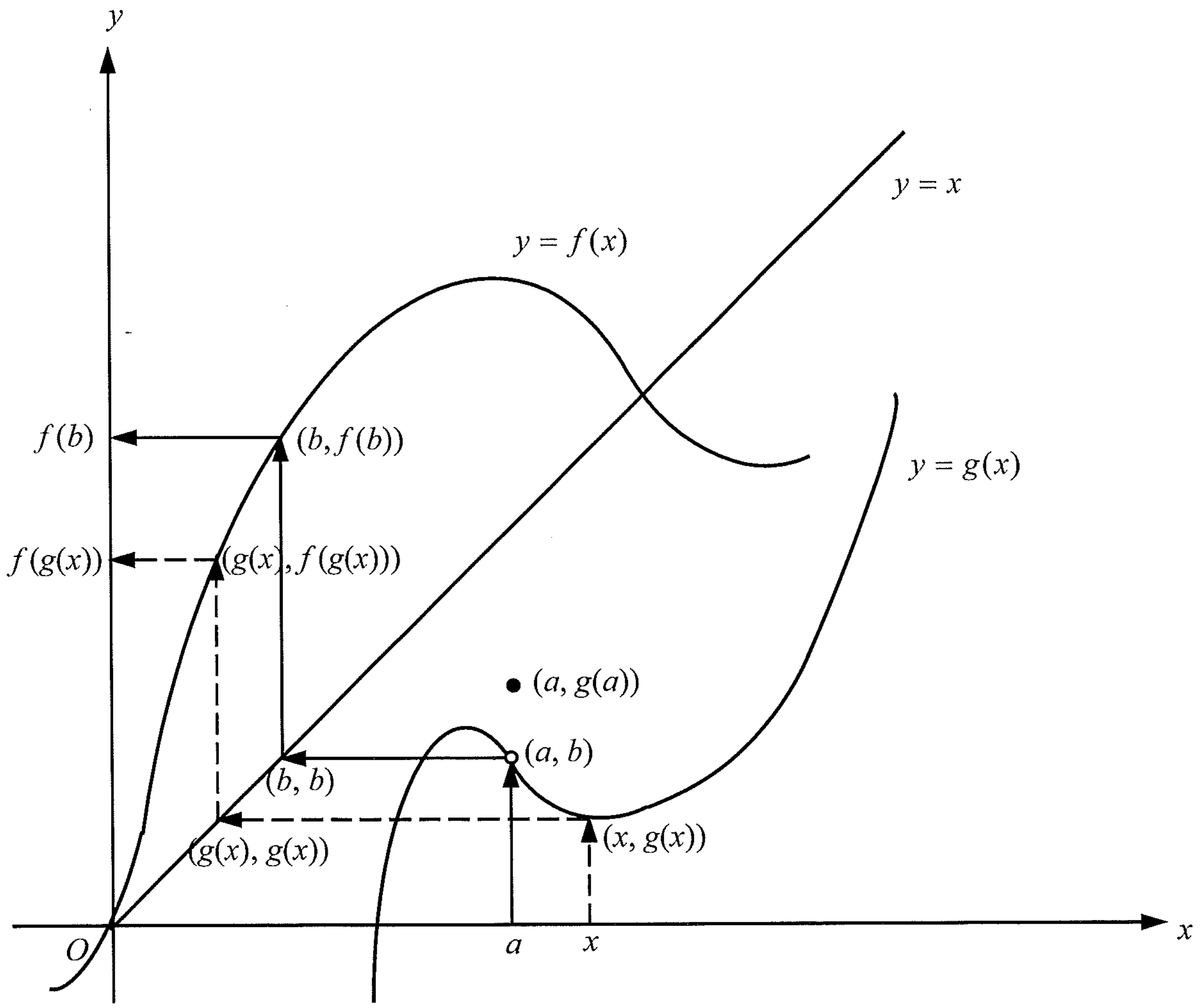


*FIGURE 1.*

## Continuity of *f* is necessary

Using the graphical method, we can also show the students visually why the theorem is false if the hypothesis that *f* is continuous at $b$ is not included. In Figure 2, we have the graph of a function *f* that is defined but not continuous at $b$. Note that the function $g$ is continuous at $a$. Using the graphs, we are able to demonstrate to the students that as $x$ approaches $a$, $f(g(x))$ does not get as close as we want to $f(b)$. In fact, from the graph, students can see that $f(g(x))$ is getting closer and closer to a number $L$, not $f(b)$, as $x$ approaches $a$.

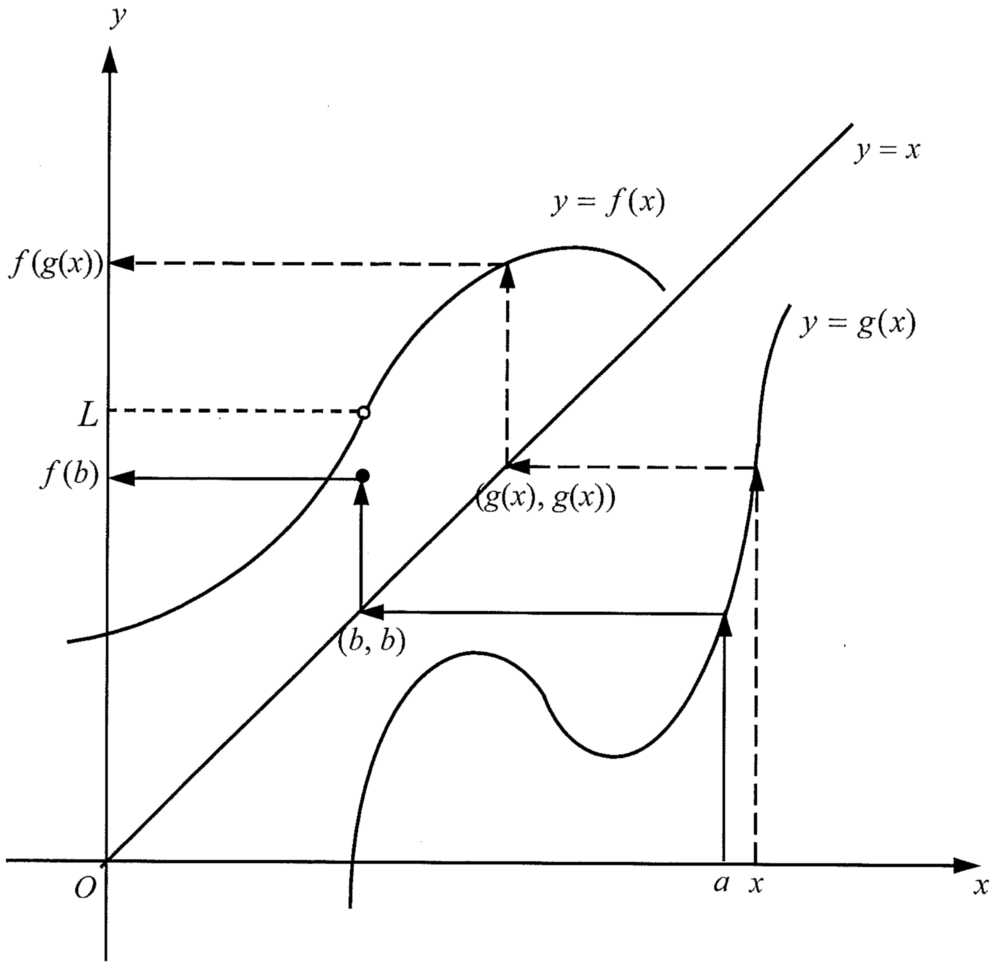


*FIGURE 2.*

In Figure 2, it is clear that $L$ is in fact $\lim_{x \to b} f(x)$. Therefore, it seems plausible from Figure 2 that the theorem might be generalized to the following:

If $\lim_{x \to a} g(x) = b$ and $\lim_{x \to b} f(x) = L$, then $\lim_{x \to a} f(g(x)) = L$.

However, we can see from the graphs in Figure 3 that as $x$ approaches $a$ from the left, $f(g(x))$ is always equal to $f(b)$, so $\lim_{x \to a^-} f(g(x)) = f(b) \neq L$, and hence the generalization above is in fact false.

Therefore, even if the graphical method is appealing to the students, we should caution them against relying on it to deduce any mathematical statement, unless it is justified by a rigorous mathematical proof.

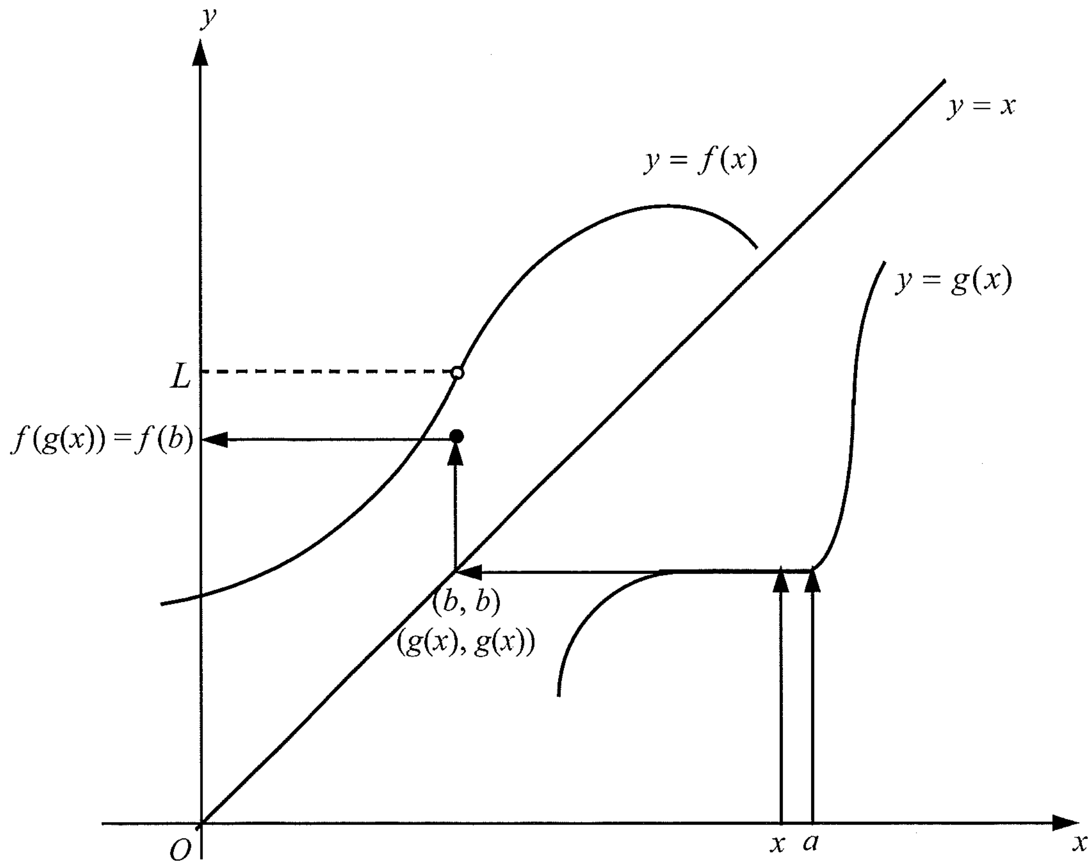


*FIGURE 3.*

## Conclusion

Although the example above shows the pitfall of the graphical method, I believe that it has one advantage over the rigorous proof; namely, it allows the students to visualize the theorem, and to see clearly why the hypothesis that $f$ is continuous is necessary. For students who have seen the epsilon-delta proof, the geometric argument will provide them with a geometric insight that will enhance their understanding of the theorem.

I have used the graphical method to illustrate the above theorem to students in my calculus course. In general, my students had a better understanding of the theorem; however, a number of students did not understand the geometric argument. I found out that their difficulty stemmed mainly from the fact that they had not really understood the graphical method for function composition, and the graphical interpretation of the limit of a function.

In conclusion, for students to benefit from the graphical illustration of the limit of composite function, instructors need to ensure that their students have fully understood the graphical method for function composition, as well as the graphical interpretation of the limit of a function. Finally, the above geometric arguments can be very well illustrated by using some available dynamic geometrical software, such as Autograph and The Geometer's Sketchpad. I believe that the use of these software tools could give a better visual effect and impression on the students than static graphs.

*Acknowledgments.* This paper, with some minor changes, first appeared in the journal Teaching Mathematics and its Applications, 21(4) (2002).

## References

1. G. J. Davis, *A graphical method for function composition,* Teaching Mathematics and its Applications, **19(4)** (2000), 154-157.
2. J. Stewart, *Calculus, Concepts and Contexts (2nd edition),* Brooks/Cole, 2001.

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# USING DYNAMIC GEOMETRY SOFTWARE IN TEACHER EDUCATION

ANDREAS ULOVEC

In recent years, dynamic geometry software (DGS) like Euklid DynaGeo, Cabri or Cinderella is being used in schools more and more frequently. However, the teacher education at the universities is currently not taking this into consideration and does not prepare future teachers for using this new tool (moreover, it does not even let them know of their existence). This article has two purposes: To present the possibilities of dynamic geometry software for use in mathematics teaching, and to present the results and experiences of a first university class held at the University of Vienna, featuring an introduction in dynamic geometry software and its classroom application.

## What is dynamic geometry software?

Dynamic geometry software - contrary to regular drawing software - keeps the relations between individual geometric objects, even if some or all of these objects change their positions (e.g. let M be the center of a line AB. If one changes the positions of A and/or B, the position of M changes accordingly, so that M stays the center of AB). This helps students to grasp concepts like "[a certain relationship] holds for all triangles", because they not only see the theoretic proof, but can also try the relationship on a large number of different triangles by just changing the position of one or more vertices (by "drag and drop") and thereby experience that it holds for "all" triangles. A research study regarding students' interpretations of geometrical objects in connection with DGS has been published in [1].

There are several dynamic geometry software products available (e.g. Cinderella, Cabri, DynaGeo). For our studies, we used DynaGeo, a product copyrighted by Roland Mechling, a German mathematics teacher.

Usual DGS products are capable of regular "ruler and compasses" constructions (e.g. draw lines between two points, construct the center of a line, construct the perpendicular line through a given point, ...), as well as measurement and transformation (e.g. reflection, translation, rotation). Some pure drawing features are also included, like changing color and style of lines, labeling objects, etc.

## Possibilities of using DGS in mathematics teaching

As mentioned above, DGS allows pupils to better understand the term "this holds for all [objects]", as well as to discover certain geometric relationships by themselves. They also allow teachers to prepare complex geometric constructions at the computer before the actual lecture instead of having the need to prepare them on the blackboard or the overhead projector during the lecture (with all the difficulties this includes). A description of different examples for DynaGeo can be found in [2].

### *Some examples of using DGS in mathematics teaching*

To demonstrate the usage of a DGS, we will show a step-by-step construction of a typical example of undergraduate geometry: The circumference of a triangle. We start by drawing the triangle itself (by just using the menu item "draw triangle" and clicking at the three positions of the vertices).

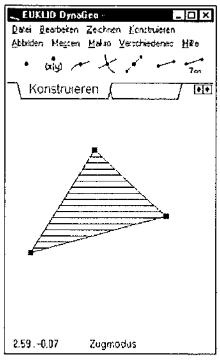


*FIGURE 1*

Next, we construct the perpendicular lines on the edges:

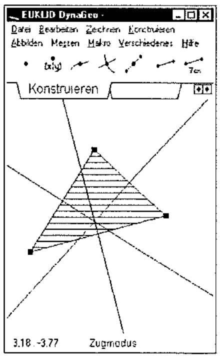


*FIGURE 2*

Finally, we construct the intersection of two perpendiculars (the circumcenter) and draw the circle:

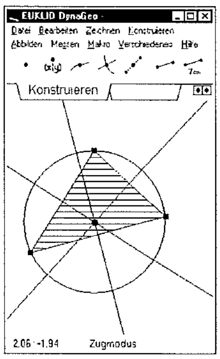


*FIGURE 3*

So, with three easy steps one can construct the circumference of a triangle. Certainly this would have worked just the same way with a regular drawing program. But now we can use the dynamic property of the drawing to construct a large number of similar drawings by just shifting the position of one or more vertices. This shows that the constructed circle is in all cases the circumference of the triangle:

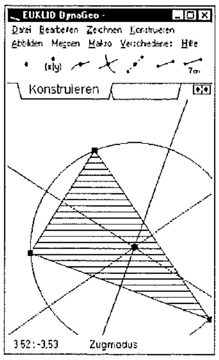


*FIGURE 4*

Another impressive example is the construction of Euler's line (the line that connects the barycenter, orthocenter, and circumcenter of any given triangle, showing that these three points are collinear). As most teachers have experienced, if one constructs this at the blackboard or in the notebook, due to small construction inaccuracies, the three points almost never line up. In a DGS they always do, and again one can show that they line up in "each and every possible" triangle. We will not show all of the construction steps here, but just the result:

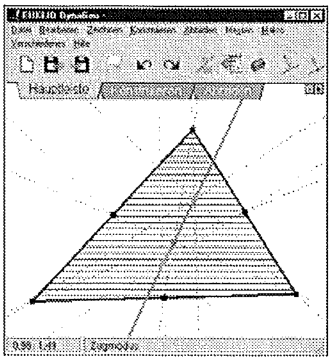

*FIGURE 5*

As a third example, we want to show how DGS can be used to let pupils discover the theorem of Thales by using the "follow the path of a given point" function. We simply construct a rectangular triangle and let the pupils guess what will happen if we move the "upper" vertex (i.e. the vertex with the right angle). It can soon be discovered that it moves along a semi-circle. Now, as all the above mentioned examples, this is no formal proof, but gives the pupils a feeling for what actually happens in connection with Thales' theorem:

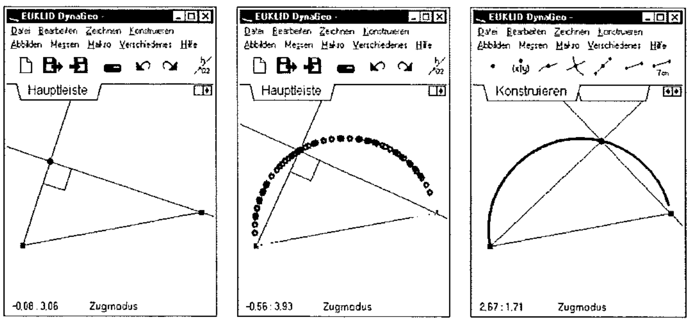


*FIGURE 6*

## What else is possible?

Some other features of DGS have not yet been mentioned. There is a "flashback" option that allows one to review the whole construction process. This gives the teacher the possibility to prepare the construction before the actual lecture (particularly useful with more complex constructions) and only use the flashback to explain it in classroom. A macro option allows one to combine given constructions to a more complex construction. For instance, this would allow one to create a macro "circumference of triangle", that from this time on could be used just like a pre-built construction, i.e. one could mark three points, execute the macro, and get the circumference in one single step.

Of course it is also possible to work with coordinate systems to do constructions with fixed coordinates. Usually the (hardcopy) printed constructions are 1:1 scaled, i.e. 1 cm in the coordinate system corresponds to 1 cm at the hardcopy printout.

Finally, there is an "animation" item, that allows one to create "living" constructions, i.e. constructions where a point moves along a given path, or a numeric parameter changes constantly. This could be useful to do certain simulations (e.g. [3]) or to direct the attention of pupils to a certain parameter without having the need of moving a point by mouse movements by themselves.

Further material about the use of DGS can be found in [4].

## DGS in teacher education

DGS has not yet been taught during mathematics teacher education at the University of Vienna. We initiated and held a first lecture featuring DynaGeo (amongst other things in connection of computer application in mathematics teaching). We introduced DynaGeo and gave explanations as to its advantages and disadvantages, as well as a number of examples about using it in teaching. As this was a hands-on lecture, the students were asked to solve examples by themselves and to create further possible examples for use in different grades in school. They were specifically asked to solve a number of examples by means of paper-and-pencil as well as with DGS, and to compare the processes of solution, applying the following criteria:

- time
- convenience
- understandability of the individual steps of construction
- (optical) quality of the result

This was done at "static" examples only (e.g. construct the barycenter of a given triangle) and was not used to discover certain relationships, give proof, or any other dynamic application.

As we expected, the results here were not very convincing. Students noticed virtually no difference in time and understandability between paper-and-pencil and DGS, and only very small advantages of DGS in the convenience (particularly at complex examples) and optical quality items.

In a second survey with (mathematics education) students we used "dynamic" examples, like proofing Thales' theorem or the sum of angles in a triangle and asked the students again to compare using the given criteria. Not surprisingly, the time, convenience, and result quality of the DGS solution were rated significantly higher as in the paper-and-pencil constructions, while there was no significant difference in the understandability of the individual steps of construction.

The second survey also included an open question: "Describe the advantages and disadvantages of using a DGS in comparison with paper-and-pencil construction." The two answers given most frequently were "I can try [a particular construction] on many different objects without much effort," and "it is easier to understand the proof/construction with DGS." These points have been noted by about 50 % of the students. The two most frequently named disadvantages were "I had to learn a lot of program details before I could start working," and "some constructions are more complex with the DGS." Both points has been mentioned by about 15 % of the students, while 60 % did not give any disadvantage.

The survey ended with the question "Do you recommend DGS for teaching in schools?" 81 % of all students answered the question in the affirmative. Overall we can say that students do see significant advantages in using DGS, particularly in tasks where dynamic aspects are involved. Application in school is definitely recommended by future teachers.

## DGS in school

The research study in school included four sixth-grade mathematics courses in triangular geometry (age of pupils: 12-13, number of pupils: 23-25 per course). Two courses were held strictly paper-and-pencil, two courses were held with explanations given both in paper-and-pencil and in DGS. All four courses introduced standard triangular geometry tasks, like barycenter, orthocenter, circumcenter, and incenter, as well as paradigmatic proofs to certain theorems.

At the end of the courses, a test and a survey with some examples (some static, some dynamic) had been carried out. In the DGS courses 7 % failed the test, in the paper-and-pencil classes 16 % failed. The survey showed an almost equal rate in static examples, a significantly higher rate of DGS in dynamic examples, and particularly a significantly better understanding of constructions and proofs in the DGS courses.

Of the pupils in the DGS courses, 63 % marked that they would prefer a combined DGS and paper-and-pencil course (like the one they just went through) over a strict paper-and-pencil course, 22 % would prefer a pure DGS course, and 15 % voted for a pure paper-and-pencil course.

## Conclusions

DGS can help pupils to understand certain constructions and tasks significantly better than with a classic paper-and-pencil course. Nevertheless, it is recommended to use *both* DGS and paper-and-pencil in instructional courses, as some explanations are easier done on real paper, and some pupils might find it hard to learn a new software *and* new geometric concepts in one single course.

We also recommend the use of DGS in teacher instruction courses at the universities and other teacher education facilities, to allow future teachers to get to know such software and the possibilities of its use in classroom teaching.

## References


1. K. Jones, *Providing a Foundation for Deductive Reasoning,* Ed. Stud. in Math, **44(1-2)** (2000), 55-85.
2. H.-J. Elschenbroich, *Geometrie beweglich mit EUKLID*, Dümmler, Bonn, 1996.
3. M. J. Gonzalez-Lopez, *Using Dynamic Geometry Software To Simulate Physical Motion*, International Journal of Computers for Mathematical Learning **6(2)** (2001), 127-142.
4. J. R. King, *Geometry turned on!*, MAA notes 41, Washington, DC, 1996.



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# WHAT STUDENTS SAY THEY LEARNED – A CASE STUDY IN FIRST YEAR CALCULUS

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The role of writing in the teaching and learning of undergraduate mathematics has attracted significant interest over the last fifteen year, particularly in the United States. As students write in mathematics, they are encouraged to think through their problem solving processes and their struggles, and to express and reflect upon their feelings and beliefs about mathematics. On the other hand, teachers benefit by receiving information about their students an their course.

This paper reports on the experimental use of writing in a first year calculus course as an end-of-task reflective instrument. It explores the different ways students express in writing what they have learned by completing a mathematics assignment. It also explores the relationship between student responses and their performance in the subject.

## Introduction

A review of the mathematics education literature shows that over the last fifteen years there has been a significant interest in incorporating writing to the teaching and learning of mathematics at undergraduate level. Most of the research in this area stems from two movements originated in the United States: *Calculus Reform* (see for example, Smith (1994) or Tucker & James (1995)), and *Writing-to-Learn* which involves the use of writing as a tool for learning course content (see for example, Miller (1991), Price, (1989), and Rishel (1991)). There are hundreds of articles reporting different ways in which writing was integrated to the learning activities and assessment; a good starting point is Sterret (1992) and Powell (1997). There are also several books advocating the benefits of writing in mathematics and providing guidance to teachers; see for example, Meier & Rishel (1998).

Practitioners and researchers agree that as students write in mathematics, they are encouraged to think through their problem solving processes and their struggles, and to express and reflect upon their feelings and beliefs about mathematics. On the other hand, teachers benefit by receiving information about their students an their course. Student writing is the "cleanest window" teachers have to the mind of their mathematics students, and "students often reveal more in their writing than they think" (Smith 1996, p. 33).

This paper focuses on what students say, in writing, about what they learned, and how could the teacher interpret their reflections. It reports on the experimental use of writing in a first year calculus course as an end-of-task reflective instrument. It explores the different ways students expressed their learning outcomes after completing a mathematics assignment. This study was conducted as the first step in an investigation of the relationship between what students said they learned and their performance in the subject, and the extent to which student reflections can serve as a reliable indicator of effective learning.

## Method

The study involved all 47 students undertaking the first level calculus subject in semester two 2002. Even though the unit is available to all science and information technology students, this cohort of students did not include any mathematics majors candidates; all students were doing this subject as a core requirement for an information technology course, or a prerequisite for a science study. The cohort of students exhibited variation in their affective reactions and preferences relative to mathematics study, and in their learning processes employed; this was determined by administering the *Experiences of Studying Mathematics* and the *Mathematics Studies Processes* inventories (Eley & Meyer, 2000).

The subject was taught using the *concepts-first* approach, that is, concepts were first introduced informally through examination of numerical and graphical representations of examples before they were developed in a precise mathematical form. The subject was assessed with three assignments, two tests, and a final examination. The aim of the five semester assessment components (three assignments and two tests) was to help students develop regular learning habits and to receive continuous feedback throughout the semester.

Each of the 3 assignments included the same task: *Explain briefly, in a paragraph or two, what you have learned by doing this assignment.* The task was the last task in a numbered list; the numbering of the task was used to indicate to students that this task was as important as all other tasks and that students were expected to complete it. The focus of the three assignments was on applications; however, they also included some tasks of a more theoretical nature. The assignments did not include exercises that only required the application of a particular technique.

## Responses and their categorisation

The analysis of students' responses revealed several different ways in which students expressed what they have learned. Their responses were divided into statements; the number of statements per response ranged between one and four. Statements were classified into different categories, and these categories were grouped under *related to subject learning objectives* or *not related to subject learning objectives.* These are shown in Table 1. Students were classified in groups according to their overall performance in the subject: *high achieving* (more than 80%), *middle-band* (between 60% and 70%), *pass* (between 50% and 59%), and *failed* (less than 50%). Tables 2 and 3 provide a summary of the number of times each category was represented in each of the groups of students.

On average, students expressed what they learned using statements belonging to at least two different categories; however, students with a performance between 60% and 79% tended to provide three different statements. The most common statement included a list of isolated topics, procedures, or both, with no connections between the topics listed. Some of these statements consisted strictly of a list of topics that could have been taken from the table of contents of a textbook (category LT), such as the one provided by Student #8 (high achiever):

> This assignment has taught me antiderivative graphs, points of inflection and differential equations.

Others were more specific on the procedure they learned (category LP); for example,

> I have learned how to solve logarithmic functions, use a transformation to get a new graph, solve trigonometric functions .... I have learned how to find a limit using local linearisation, how to determine where the function is increasing or decreasing.
>
> Student #1 (high achiever)

TABLE 1
*Categories of statements*

| Related to subject learning objectives | Not related to subject learning objectives |
|---|---|
| LT = list of topics | STR = identifies learning strategies |
| LP = list of procedures/techniques | DIFF = expresses difficulties |
| RE = revised known material | IW = identifies weaknesses |
| EXT = built upon existing knowledge | CON = expresses confidence (or lack of) in mastering of material |
| APP = applications and relevance of topics already known or learned in lectures | CE = conveys expectations |
| SN = *Scientific Notebook* | |
| MP = mathematical representation and terminology | |

TABLE 2

*Frequencies of categories related to subject learning objectives grouped by student overall performance in the subject.*

| Performance | n | LT | LP | RE | EXT | APP | SN | MP | Total |
|---|---|---|---|---|---|---|---|---|---|
| high achieving | 13 | 8 | 4 | 5 | 2 | 2 | 7 | 1 | 29 |
| middle band | 17 | 6 | 4 | 8 | 6 | 5 | 10 | 1 | 40 |
| pass | 9 | 5 | | 2 | 2 | 3 | 4 | | 16 |
| failed | 8 | 1 | | 3 | | 2 | 3 | | 9 |
| Total | 47 | 20 | 8 | 18 | 10 | 12 | 24 | 2 | 94 |

Some students went a little bit further, and aligned the learning of the procedure to each of the assignment questions; for example:

> In question 1 I have learnt how to calculate unknown constants in an equation, given points on the graph and how to explain various transformations, and how they were derived. In question 2 I learnt how to graph trigonometric functions, calculate periods, amplitudes, shifts and also maximum and minimum points. In question 3 ....
>
> Student #3 (high achiever)

while some provided more detail on the procedures:

> ... the first derivative shows the increasing or decreasing intervals of the function. The second derivative shows whether the graph of function is concave up or concave down over certain interval ....
>
> Student #39 (middle band)

All but one of the high performing students included a statement belonging to the LP or LT categories, while just over half of the middle-band and pass students, and only one of those who failed did so. LP statements appeared only in assignments submitted by students with a performance higher than 60%, and were used more often by high achievers.

The next more frequent category relates to revision (category RE); 18 students indicated that the assignments helped to revise or reinforce material already known prior to undertaking the subject. This kind of statements were made more often by the middle-band and fail students. It is surprising to note that 3 of the 8 students who failed described their learning as revision.

About 20% of the students phrased their learning in terms of building upon previous knowledge (category EXT). For example:

> I have rediscovered some concepts of methods from year 12 and applied it as well as some new concepts learnt in lectures, particularly background information about why formulas work and concepts behind exponentials ...
>
> Student #4 (middle band)

Students who touched on category EXT are spread over the three satisfactory performance levels.

Even though the focus of the assignments was on applications, only 12 of the 47 students, equally spread around the performance levels, mentioned this as a learning outcome (category APP). Most of these statements were very general; for example,

> Through thinking and applying the theorems etc, I have learnt how to use calculus in more practical situations.
>
> Student #23 (middle band)

or

> ...This assignment has taught me how to relate mathematics problem solving to real-life situations – a task which is often overlooked when teaching such material.
>
> Student #2 (middle band)

In this last instance the student also hinted that her mathematics experience was lacking on applications. Only one of the applications statements was somewhat more specific:

> ... I have also learned the application of piece-wise functions in realistic situations.
>
> Student #31 (middle band)

The statements related to the learning of the computer algebra system *Scientific Notebook* were placed in a separate category (category SN). Most students if not all in that group did not know *Scientific Notebook* before doing the first assignment of this subject. However, only half of the students reported learning this software. Moreover, those who did mention *Scientific Notebook* as a learning outcome, tended to place a great emphasis on this outcome. For example:

> By completing this project I have learnt that Scientific Notebook is easily downloaded [from a website] and that it does make it easier to do simple things such as graphs, which makes the presentation of the work easier to follow, however there are limitations within the program which can at time make it longer to present the material electronically.
>
> Student #15 (middle band)

Finally, 2 students made a reference to the use of mathematical terminology (category MP); for example:

> I have also learnt to be more mathematically correct in the language I use to explain the concept being tested ....
>
> Student #15 (middle band)

All the statements presented above have a direct connection to the subject learning objectives. The learning outcomes identified by the students coincided with the subject intended learning outcomes. There were however, other statements that did not relate directly to the intended learning outcomes of the subject; these were of a more personal nature: management techniques, difficulties with mathematics, and strategies used to cope with the subject. About 22% of the statements belongs to this group and these were largely made by the weaker students.

TABLE 3
*Frequencies of categories not related to subject learning objectives grouped by student overall performance in the subject.*

| Performance | n | DIFF | STR | IW | CON | SE | Total |
|---|---|---|---|---|---|---|---|
| high achieving | 13 | 1 | 1 | | | | 2 |
| middle band | 16 | 4 | | | 2 | 1 | 7 |
| pass | 9 | 2 | 2 | | | | 4 |
| failed | 8 | 3 | 2 | 3 | | | 8 |
| Total | 46 | 10 | 5 | 3 | 2 | 1 | 21 |

A few students reported having difficulties in completing the assignment(s) (category DIFF); some giving a reason,

> ... I found this assignment more difficult as it required more in-depth thinking in terms of the calculus and its applications taught so far this semester.
>
> Student #23 (middle band)

others not specifying the nature of their difficulties,

> From this assignment I really had a hard time. It is the hardest assignment I [have] ever done. I really need to do more revision before the exam. It is hard as I say and even skip question 5 [sic].
>
> Student #47 (fail)

Some students identified strategies they used or will follow to improve their performance (category STR). The strategies included getting help from others,

> ... I learned to collaborate with other people ...
>
> Student #21 (pass)

time management,

> [I learned that I should] Do it [the assignment] ahead of time, not the night before.
>
> Student #34 (pass)

and use of resources,

> By doing this assignment I need to find some reference from the lecture notes, so I can have a chance for revision. Also, I need to use Scientific Notebook to do this assignment so I am more familiar to use this software now [sic].
>
> Student #16 (high achiever)

More than a third of the failed students indicated that the assignment helped them to identify weaknesses in their knowledge (category IW); for example:

> Working on this assignment has made me realise that my command of the few chapters covered in this assignment weren't very strong and that I have to work on my problems to improve my grasp of maths.
>
> Student #42 (fail)

but none of the students in the other performance levels said so.

Two other students wrote about the confidence with which he/she addressed the assignment tasks (category CON); for example:

> ... I am not sure about my answer to question 5 but it was the best I could do. Other than that I am fairly confident about this assignment....
>
> Student #13 (middle band)

A student expressed the usefulness of the assignment as an indicator of what is expected from the learners of the subject (category CE):

> ... Although I did find this task rather difficult, I nonetheless appreciated doing it as a measure of the work I will need to put into the exam and my areas of weaknesses....
>
> Student #2 (middle band)

Overall, the elaboration and confidence with which the statements were made were usually aligned with the performance level of the students. Reflections from mathematically weaker students were generally vague, with no specifics, and very short. For example:

> In this experiment I have learnt a greater understanding of the concepts that were used in this experiment and also more knowledge on how to apply the concepts and use them in most of the ways listed in the experiment.
>
> Student #11 (pass)

Two higher achievers also presented very short reflections, but these however, were coherent and specific. One of them said:

> This assignment has taught me antiderivative graphs, points of inflection and differential equations.
>
> Student # 8 (high achiever)

On the other hand, there were a couple of poor performers whose statements would indicate good command of the subject material. For example:

> I can't honestly say that I have learned an awful lot of new things in doing this assignment. About the only newish material covered here is the Scientific Notebook, which I have found to be immensely useful tool and shall use it in the future. Other than that it is simply reinforcement of my ability to do with manipulating functions primarily and has also exposed a few loopholes and weak points in my ability.
>
> Student #7 (fail)

## Discussion

This preliminary study provides some insight into what first year university students say they learned by undertaking a mathematics assignment. Responses from students were rather superficial and did not indicate as deep a reflection a teacher would wish to read. Students' responses might be interpreted as to be in agreement with the findings by Crawford, Gordon et al., that the majority of first year mathematics students conceive mathematics as a fragmented body of knowledge (1998, p. 457). However, the cohort of students exhibited variation in their affective reactions and preferences relative to mathematics study, and in their learning processes employed, as determined by their responses to the *Experiences of Studying Mathematics* and the *Mathematics Studies Processes* inventories. This cohort of students included both deep and surface learners of mathematics. This might indicate that the lack of depth in students' responses is not always due to their surface approach to learning mathematics, but rather to their lack of skill in articulating their mathematics learning outcomes. The relationship between students' approaches to learning mathematics and their expression in writing of what they learned will need to be investigated further.

Despite the largely superficial responses, clear styles emerged. Students who succeeded in completing the subject were more likely to express their learning by providing a list of topics. At the other end of the scale, students who did not perform so well, made vaguer statements; they also tended to depart from saying what they learned, and to write instead about their time management problems, the strategies they followed, or the difficulties they encountered.

In this study, the only stimulus students received was a question in the assignment, which asked them to reflect upon what they learned; no further prompts, feedback, or stimuli were provided. It is widely agreed that reflection can foster a deeper and more complete understanding of mathematical content. Likewise, there is general agreement that the teachers can facilitate reflection through appropriate teaching strategies; by "including reflection in the mathematics classroom, one not only broadens and deepens mathematical knowledge but gives valuable impulses to their general education" (Neubrand 2000, p. 266). The next stage of the study will add the teacher intervention to the reflective process, and will explore whether there are significant differences in how students express what they learned, and in the relationship between the style of the students reflections and their overall performance in the subject.

## Conclusion

This case study shows that although students expressed their learning outcomes superficially, the different styles students used correlate with their overall performance in the subject. Students who succeeded in completing the subject were more likely to express their learning by providing a list of topics. On the other hand, students who did not perform well, made vaguer statements; they also tended to depart from saying what they learned, and write instead about their time management problems, the strategies they followed, or the difficulties they encountered.

Given the scope of the study, its results cannot be generalised in any way. However, they provide some insight into students' reflections about their mathematics learning which lead to several research questions. Is there a relationship between the students' learning approaches to mathematics when completing an assignment and the way they report their learning outcomes? How can mathematics teachers help undergraduate students to reflect about their mathematics learning? What effect could such teacher intervention have on expressing in writing their reflection?

## References


1. Crawford, K. Gordon, S. Nicholas, J. & Prosser, M. (1998). Qualitatively Different Experiences of Learning Mathematics at University. *Learning and Instruction.* 8 (5). 455–468.
2. Eley, M. G & Meyer, J. H. F. (2000). In C. Rust (Ed) *Student Learning 7– Improving Students Learning through the Disciplines.* OCSLD, Oxford, Brooks University.
3. Meier, J & Rishel, T. (1998). *Writing in the Teaching and Learning of Mathematics.* MAA.
4. Miller, L. D. (1991). Writing to Learn Mathematics. *Mathematics Teacher*, 84. 516–521.
5. Neubrand, M. (2000). Reflecting as a Didaktik Construction: Speaking About Mathematics in the Mathematics Classroom, in I. Westbury, S. Hopmann & K. Riquarts (Eds.) *Teaching as Reflective Practice – The German Didaktik Tradition.* LEA, London. Chapter 14.
6. Powell, A.B. (1997). Capturing, examining, and responding to mathematical thinking through writing. *Clearning House* 71 (1), 21–25.
7. Price, J.J. (1989). Learning Mathematics Through Writing: Some Guidelines. *College Mathemaitcs Journal,* 20 (5), 393–401.
8. Rishel, T. (1991). The Geometric Metaphor: Writing Mathematics in the Classroom. *PRIMUS,* 1, 113–128.
9. Smith, D.A. (1996). Thinking About Learning – Learning About Thinking in R. W. Robers (Ed.), *Calculus: The Dynamics of Change.* MAA Notes no. 39, Mathematical Association of America, 31–37.
10. Smith, D. (1994). Trends in Calculus Reform in A. Solow (Ed.), *Preparing for a New Calculus*, MAA Notes no. 36. Mathematical Association of America, 3–13
11. Sterret, A. (Ed) (1992). *Using Writing to Teach Mathematics.* MAA Notes no. 16.
12. Tucker, A.C. & James R. C. Leitzel (Eds) (1995). *Assessing Calculus Reform Efforts.* Mathematical Association of America Report.



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# MATHEMATICAL SKILLS FOR MODELLING

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The boundaries between applied mathematics, numerical analysis and computation are becoming increasingly blurred as these activities become part of more and more disciplines and are supported by ever more sophisticated software, both purpose made and general purpose. This creates both the need and the opportunities for non-mathematicians to acquire skills that complement rather than compete with the strengths of computer-aided mathematical technology. Examples from a recently completed project on modelling solute transport in natural porous media are presented to demonstrate some of these skills: the fight against complexity; the power of experimenting with a good guess; cycling between algebra, numerics and graphics; coaxing obstinate software; and thinking strategically rather than tactically. Even though modern mathematics textbooks incorporate the use of symbolic software, it is argued that these skills do not receive enough attention. Typical textbook problems are just too simple to give exposure to this inventive rather than mechanistic approach to mathematics. Traditional teaching of concepts, logic and techniques should be complemented by more substantial projects that focus on problems rather than techniques and leave room for intuition. From a teaching perspective, the main problem is to find suitable projects at an appropriate level and an exchange of tested examples between both experts and users of mathematics would benefit all parties.

## Introduction

Before the advent of numerical computing in the 1960's, mathematical modelling focussed on representing the essentials of a real world system in terms of equations that were *simple* enough to solve analytically. In addition to insight into the modelling domain (typically theoretical approaches to physical sciences, engineering and economics) the sophistication of models and the success that could be achieved depended largely on the mathematical skills and knowledge of the modeller. As a result the appropriate training for theoretical physicists or engineers largely coincided with that for future mathematicians.

With the increasing power of numerical computing in the 1970's and 1980's, modelling horizons broadened progressively and the focus shifted into producing models that are as *realistic* as possible while relying on numerical software to do calculations that were not possible analytically. Correspondingly, numerical analysis gained prominence as a discipline and was often taught in parallel with more formal mathematics, which itself was not largely affected.

The 1990's brought a new revolution in the application of computing technology to mathematical modelling, when symbolic algebra and later integrated packages such as *Mathematica* [1] and *Maple* [2] that combine analytical, numeric and graphic capabilities, became generally available. This time the teaching of mathematics was also profoundly affected. For example, a whole new generation of mathematics textbooks have been appearing over the last decade or more, that revise traditional texts by incorporating the use of software. In this way mathematical concepts can be more vividly demonstrated by sophisticated 3-dimensional plotting and animation, and even at relatively elementary level students can experience numeric methods and attempt more demanding analytic manipulation by using software support.

Mathematical teaching has certainly been greatly enhanced by this and has made mathematical modelling more accessible to a wider range of people who are not in the first instance specialising in mathematical methods.

However, this paper addresses the question whether the full implications of this revolution, as experienced in the actual practice of modern mathematical modelling, have penetrated into the way that mathematics is taught. When a new tool is invented, its first use is often to do the same things more efficiently. However, the more profound effect of a powerful new tool is to change the way we think about things. It liberates our thoughts from the (sometimes unconscious) confines imposed by old methods, and hence to make entirely different attitudes viable and new designs possible.

These ideas are demonstrated by presenting a few examples from an actual recent research project. The subject matter may appear mundane: the modelling of contaminant transport in a porous medium such as an underground aquifer. A stochastic partial differential equation was used to model this system, but the examples below are restricted to straightforward algebraic and calculus problems that arise from it.

**Examples**

It turns out that in a certain situation, the concentration of contaminant is given by the following equations:

$$c(x,t) = N\int_{-\infty}^{\infty} dy \exp[-F(y)] \qquad (1.1)$$

$$\begin{aligned} F(y) = y^2 - 2(w_1^2\phi_1 - w_2^2\phi_2) + (w_1^2\phi_1^2 - w_2^2\phi_2^2)\frac{2[y^2+(z+w_2\phi_2)^2]}{y \pm y\sqrt{y^2+4zw_2\phi_2} + 2w_2\phi_2(z+w_2\phi_2)} + \\ +(w_1^2 - w_2^2)\frac{y \pm y\sqrt{y^2+4zw_2\phi_2} + 2w_2(z+w_2\phi_2)}{2[y^2+(z+w_2\phi_2)^2]} \end{aligned} \qquad (1.2)$$

Here, the $w,\phi$ and $z$ parameters are all known functions of $x$ and $t$ containing some further parameters.. Performing this integral analytically does not seem an option, but for each choice of a set of 8 input parameters it can be done numerically. Typical results turn out to resemble a Gaussian peak, which is encouraging because in some simplified limits, such as in the absence of stochastic contributions, the integral can be done analytically and gives a Gaussian peak centred at a time-dependent value

$$Z(t) = w_2(t)\left(1-\phi_1(t)\right) \qquad (1.3)$$

To interpret the results, rather than just produce numbers, it is necessary to know how they depend on the individual input parameters. One could go through tedious numerical calculations to produce tables that represent those dependencies, but carrying these forward into subsequent calculations are bound to make interpretations increasingly opaque. The only satisfactory way to proceed is to find an analytical approximation of the integral.

The strategy that has been found successful to accomplish that, follows the following line of reasoning. First, the speculation: "*Just imagine how nice it would be if the full integral is still a peak located at the value given by equation (1.3)!*" This idea can be tested by numeric calculation of specific cases, and is confirmed to computational accuracy of $10^{-13}$. This leads on to a more ambitious speculation: "*Just imagine how nice it would be if the peak is still Gaussian!*" Out of these, the idea arises to make a Taylor expansion of how $F(y)$ depends on $z$, around the value $Z(t)$ given by (1.3).

To actually do this, it becomes necessary to solve for the zero of F(y). That looks like a problem tailor-made for symbolic algebra software, and the result found by *Mathematica* [1] is displayed in figure 1. In fact, the figure shows only about 3% of the expression for the first of six solutions, each of which occupies about 200 Kb of computer memory.

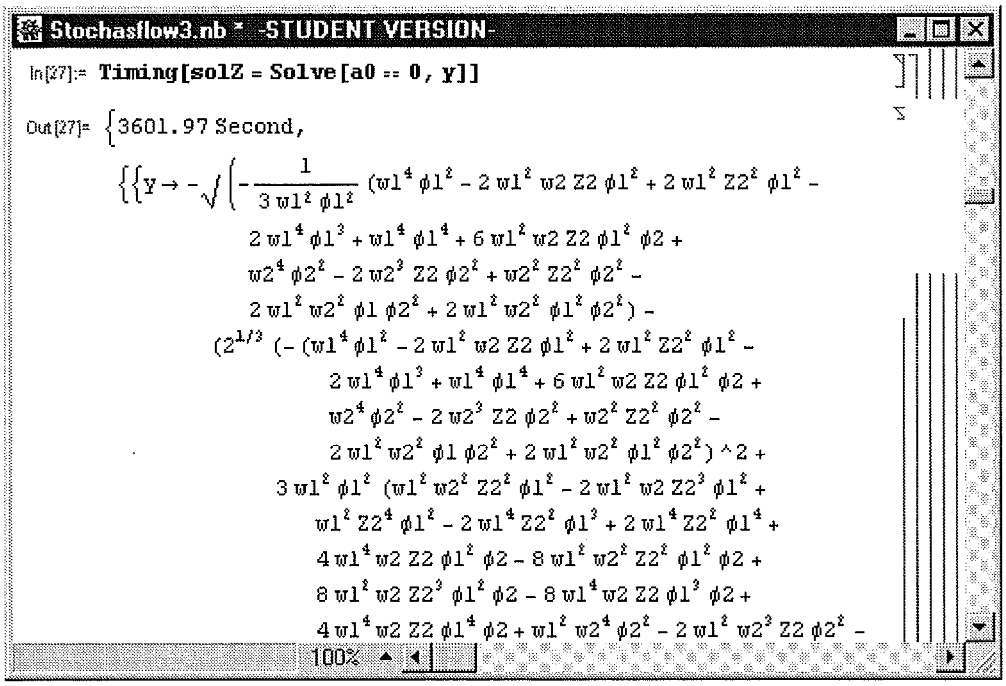


*FIGURE 1.* Part of symbolic solution for zeroes of $F(y)$.

This shows a typical problem arising from applying symbolic software to a "real" problem: the enormous explosion of complexity, that makes an explicit solution as bad or worse than a numeric one for the purpose of retaining an oversight.

The way out again relies on a speculative idea. For given parameter values, numerical plots of the expression are just smooth curves or surfaces, suggesting that "*Surely such simple behaviour cannot need such a complicated expression to represent it?*" . Based on this, the software was instructed to make series expansions of the large expression in terms of various parameters, around chosen values, and by comparing leading terms in the expansions the following function was *guessed* as one that would give the same leading terms in its own expansions:

$$\tilde{y} = \frac{(v_2 - v_1)\theta_1}{\gamma\sqrt{2(\theta_1 - t_c)}}\sqrt{\frac{t-\theta_1}{t-t_c}} \qquad (1.4)$$

Before arriving at this guess, experimentation with various choices of parameters and expansion points were needed. But amazingly, it turns out that this expression coincides with the full 200 Kb one within a tolerance of $10^{-6}$ for hundreds of numerical combinations tried. And yet, it is not exact – in some limits, derivatives deviate from those of the full expression.

To show that this is not entirely a matter of luck, another example from the same project is the expression arrived shown in figure 2 below. In this case the expression fills "only" a single screen, but has to be used as the multiplicand in a product expression that runs over the argument $m$ from 1 to many thousands.

Stochasflow9.nb * -STUDENT VERSION-

Am[λ, Δ, m]

$$
\begin{aligned}
&\big((1+\Delta)^2 (1+3m\lambda)(1+\lambda+3m\lambda)(1+\Delta-2\lambda+3m\lambda-3\Delta\lambda+3m\Delta\lambda)^2 \\
&\quad (1+\Delta-\lambda+3m\lambda-3\Delta\lambda+3m\Delta\lambda)(-1+\Delta+\lambda-3m\lambda-\Delta\lambda+3m\Delta\lambda)^2 \\
&\quad (1+4\Delta+3\Delta^2+3m\lambda-\Delta\lambda+12m\Delta\lambda-3\Delta^2\lambda+9m\Delta^2\lambda)^2\big)/ \\
&\big((1-\Delta-\Delta^2+\Delta^3-5\lambda+9m\lambda+5\Delta\lambda-9m\Delta\lambda+\Delta^2\lambda-9m\Delta^2\lambda-9\Delta^3\lambda+9m\Delta^3\lambda+ \\
&\quad 8\lambda^2-30m\lambda^2+27m^2\lambda^2-10\Delta\lambda^2+30m\Delta\lambda^2-27m^2\Delta\lambda^2+9\Delta^2\lambda^2+6m\Delta^2\lambda^2- \\
&\quad 27m^2\Delta^2\lambda^2+27\Delta^3\lambda^2-54m\Delta^3\lambda^2+27m^2\Delta^3\lambda^2-4\lambda^3+24m\lambda^3-45m^2\lambda^3+ \\
&\quad 27m^3\lambda^3+8\Delta\lambda^3-30m\Delta\lambda^3+45m^2\Delta\lambda^3-27m^3\Delta\lambda^3-13\Delta^2\lambda^3+27m\Delta^2\lambda^3+ \\
&\quad 9m^2\Delta^2\lambda^3-27m^3\Delta^2\lambda^3-27\Delta^3\lambda^3+81m\Delta^3\lambda^3-81m^2\Delta^3\lambda^3+27m^3\Delta^3\lambda^3) \\
&\quad (1+4\Delta+6\Delta^2+4\Delta^3+\Delta^4+\lambda+9m\lambda+6\Delta\lambda+36m\Delta\lambda+8\Delta^2\lambda+54m\Delta^2\lambda- \\
&\quad 6\Delta^3\lambda+36m\Delta^3\lambda-9\Delta^4\lambda+9m\Delta^4\lambda+6m\lambda^2+27m^2\lambda^2-2\Delta\lambda^2+ \\
&\quad 36m\Delta\lambda^2+108m^2\Delta\lambda^2-13\Delta^2\lambda^2+48m\Delta^2\lambda^2+162m^2\Delta^2\lambda^2-24\Delta^3\lambda^2- \\
&\quad 36m\Delta^3\lambda^2+108m^2\Delta^3\lambda^2-9\Delta^4\lambda^2-54m\Delta^4\lambda^2+27m^2\Delta^4\lambda^2+9m^2\lambda^3+ \\
&\quad 27m^3\lambda^3-6m\Delta\lambda^3+54m^2\Delta\lambda^3+108m^3\Delta\lambda^3+\Delta^2\lambda^3-39m\Delta^2\lambda^3+ \\
&\quad 72m^2\Delta^2\lambda^3+162m^3\Delta^2\lambda^3+6\Delta^3\lambda^3-72m\Delta^3\lambda^3-54m^2\Delta^3\lambda^3+ \\
&\quad 108m^3\Delta^3\lambda^3+9\Delta^4\lambda^3-27m\Delta^4\lambda^3-81m^2\Delta^4\lambda^3+27m^3\Delta^4\lambda^3) \\
&\quad (1+8\Delta+22\Delta^2+24\Delta^3+9\Delta^4-2\lambda+9m\lambda-24\Delta\lambda+72m\Delta\lambda-116\Delta^2\lambda+ \\
&\quad 198m\Delta^2\lambda-168\Delta^3\lambda+216m\Delta^3\lambda-74\Delta^4\lambda+81m\Delta^4\lambda+\lambda^2- \\
&\quad 12m\lambda^2+27m^2\lambda^2+24\Delta\lambda^2-144m\Delta\lambda^2+216m^2\Delta\lambda^2+166\Delta^2\lambda^2- \\
&\quad 696m\Delta^2\lambda^2+594m^2\Delta^2\lambda^2+360\Delta^3\lambda^2-1008m\Delta^3\lambda^2+648m^2\Delta^3\lambda^2+ \\
&\quad 217\Delta^4\lambda^2-444m\Delta^4\lambda^2+243m^2\Delta^4\lambda^2+3m\lambda^3-18m^2\lambda^3+27m^3\lambda^3- \\
&\quad 8\Delta\lambda^3+72m\Delta\lambda^3-216m^2\Delta\lambda^3+216m^3\Delta\lambda^3-72\Delta^2\lambda^3+498m\Delta^2\lambda^3- \\
&\quad 1044m^2\Delta^2\lambda^3+594m^3\Delta^2\lambda^3-216\Delta^3\lambda^3+1080m\Delta^3\lambda^3-1512m^2\Delta^3\lambda^3+ \\
&\quad 648m^3\Delta^3\lambda^3-216\Delta^4\lambda^3+651m\Delta^4\lambda^3-666m^2\Delta^4\lambda^3+243m^3\Delta^4\lambda^3)\big)
\end{aligned}
$$

100%

*FIGURE 2.* An example expression to be used as multiplicand in a large dimension product.

Once more, the first move is to plot the behaviour of the function as a function of $m$. The result is shown in figure 3, and again the striking feature is the rather simple, smooth behaviour. In this case, it was found that by making three series expansions – a power series in $(1/m)$ around $m = \infty$, in powers of $\lambda$ around $\lambda = 0$, and a double expansion in $\lambda$ and $m$ , around ( $m = 1/2\lambda$, $\lambda = 0$ ) gives after a somewhat tedious comparison of terms and guessing of functional forms the simple approximation

$$
\begin{aligned}
A(\lambda,\Delta,m) &\approx 1+\frac{3Q(\Delta)}{1+3m\lambda} \\
Q(\Delta) &= \frac{8\Delta^2(3+6\Delta+16\Delta^2+18\Delta^3+5\Delta^3+16\Delta^5)}{3(1-\Delta)^2(1+\Delta)^3(1+3\Delta)^2}
\end{aligned}
\qquad (1.5)
$$

This approximation holds to within a factor of $10^{-4}$ for physically plausible ranges of $\lambda$ and $\Delta$, and even beyond those gives a good qualitative representation. Using the simplified form it becomes possible to find an analytical expression for the product in terms of Gamma functions and hence to calculate the required large product range.

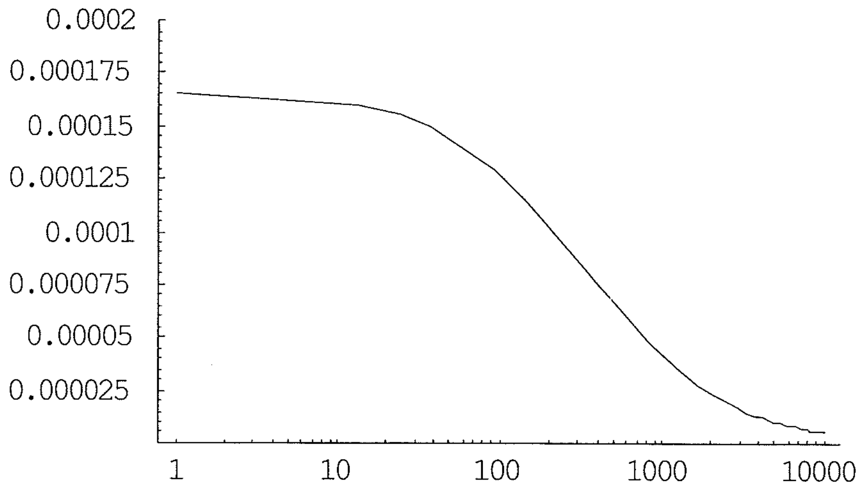


*FIGURE 3.* Plot of function $A(\lambda, \Delta, m)$ as function of $m$, for plausibly chosen values of $\lambda$ and $\Delta$.

## Discussion

The examples shown above are not exceptions; several more could have been quoted from the same modelling project. However, they should be sufficient to demonstrate the issues that are the subject of this paper.

The first of these is the *fight against complexity*. The power of symbolic software, which might have been expected to alleviate the need for proficiency in algebraic technique, has instead merely shifted the ground. The limiting factor is not so much our ability to manipulate mathematical expressions any more, but our ability to comprehend the results. And from the user's point of view part of the problem is that this is not experienced as a gradual increase, but often as an alarmingly sudden transition – operations which seem to be only moderately complex, can lead to results with overwhelming complexity.

The situation has some similarity with the early days of numeric computing, when the user was confronted with hundreds of kilobytes of numeric data. In the meantime various ways have been found to deal with that problem, through the use of graphic representations, databases, statistical analysis, curve fitting, etc. However when confronted with an analytical expression that runs into hundreds of kilobytes, there does not seem to be such obvious ways of dealing with the situation and it is easy to give it up as hopeless.

This reaction is unfortunately encouraged to some extent by the way that mathematics is taught. For teaching purposes, problems are selected to have straightforward solutions, and that is appropriate when the emphasis is on mastering a new technique. That seems to remain true even in new textbooks that incorporate mathematical software. However, this is not representative of the experience of actually using mathematics to model real systems.

In early versions of symbolic software, the complexity of results was often only apparent, and could be removed by skilful use of special operations such as factorisation, collecting of terms, etc. However, modern packages such as *Maple* and *Mathematica* have implemented powerful "Simplify" instructions that automatically and systematically try a larger variety of algebraic transformations than a user is likely to be able to. No doubt in the case of a 200 Kb expression the abilities of such automatic simplification might also be exceeded, but then a human user is even less likely to be able to pinpoint the crucial simplifying operations.

One message contained in this paper, is that surprisingly often it is nevertheless possible to win the fight against complexity by finding analytic approximations. This was demonstrated by the examples above. But by only showing the final successful results, they gloss over the fact that in fact these were in fact arrived at after many rounds of trial and error.

The second point to be made is that this can only be accomplished by relaxing the rigid mode of thought that is often associated with mathematics. Unlike in high school algebra, one cannot be led by only tactical thinking that takes the current equation or expression in front of you as the guideline of what to do next. Instead, one needs to adopt a much more *speculative, strategic approach* of considering where you would like to end.

There is of course, a danger that such speculations might be mere wishful thinking – but by using numerical and graphical experiments as guidelines, one can attempt to avoid that pitfall. Nevertheless, one has to be prepared to be proven wrong many times before achieving success.

In this process, the most important skill that is needed is the ability to come up with a continuous stream of *new ideas* that can be tried in order to achieve the overall goal. Without computational support, it is necessary to concentrate on finding the one foolproof idea that will solve a problem because too much time is wasted on trials of lesser ones. However, computational support makes the cost and effort of a trial much cheaper and so it becomes a waste of time to perfect an idea before trying it. Indeed, it is the lessons learned in unsuccessful trials that gradually guide one towards success.

In my experience students who have received the customary training in mathematics find it very hard to adopt this approach. Because the most efficient way to transmit mathematical knowledge is to present a linear argument where every step logically leads to the next, students are not exposed to the more complex, and partly speculative thought processes that are needed to make progress in the situations described above.

It is clearly not easy to incorporate this kind of experience in teaching. The lecturer needs to find problems which are sufficiently complex that there are many different approaches to try. Students need to find the time to sufficiently immerse themselves in such a problem that they can anticipate the answers and devise and try different strategies. Both parties need to resist the temptation of spoiling the experience of discovery by detailed guidance by the lecturer.

Nevertheless I believe it is a worthwhile goal and achieving it would better prepare students for applying mathematics in modelling the real world.

## References


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# CROSS-DISCIPLINARY TEACHING OF MATHEMATICS

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In this paper we describe findings from the literature on cross-disciplinary teaching of mathematics to undergraduates, as part of a review of cross-faculty teaching of science subjects at the University of Technology, Sydney. Disciplinary differences in teaching styles can inform difficulties faced by engineering and business students. The necessity for varied teaching approaches was illuminated by differences in learning styles among students. For mathematics, a common theme is optimising presentation to engineering students, with general agreement on the need for relevance; a concept which is also raised regarding teaching statistics to business students. Innovations include the development of courses based on interdisciplinary teaching approaches, with discussions focusing on course revisions.

## Introduction

Teaching across disciplines can offer many benefits as well as impediments. For staff and students, there is the opportunity to experience new ways of thinking and for the transfer of knowledge, but conversely a lack of enthusiasm or cooperation can raise barriers to the delivery of good quality teaching. This paper is centred on investigations into cross-disciplinary teaching in general, and such teaching in the framework of providing mathematics and statistics for engineering and business students.

There is a range of scenarios for cross-disciplinary teaching. At one extreme, it could be seen as mathematicians, statisticians, engineers and business experts together cooking up spicy dishes with ingredients from their different cultures. They serve the banquet to engineering and business students within an open plan kitchen/dining/living room. They all savour each dish, and discuss the components that have been blended together and analyse the cooking techniques. The other extreme involves each specialist coming out of their individual, sequestered rooms to produce their own single course in a designated kitchen after limited prior discussion with other staff. The meal is served in a formal dining room and the students gulp the food down silently, then sit passively waiting for the next specialist to come out of their room to dish up the next course.

Our recent research into cross-disciplinary teaching arose out of concerns about the effectiveness of the teaching of science subjects at UTS to students of other faculties. It involved a number of strands, including interviews with students and academics (reported on at the EMAC conference earlier this year [13, 25]). We also undertook a literature review to inform the research.

We have focused here on a selection of themes from the literature, with an emphasis on teaching mathematics to engineering students and statistics to business students. Writings on disciplinary differences provide some clues about the cultures of teaching staff in particular. Most deal with science teaching approaches and styles, versus those for the social arena, but it is clear students face differing teaching styles which are mirrored by their own variations in learning styles. We looked at literature on engineering mathematics and discussions concerning statistics for business students, and reviewed reports on innovative course design based on interdisciplinary collaboration.

## Disciplinary differences

There seems to be wide acceptance of the notion that differences exist between disciplines in their broad cultural characteristics, teaching (or cooking) approaches and styles, alongside differences in the ingredients that each discipline adds to the mix. The primary focus of research has been on differentiating between broad divisions, with some forays into categorising individual disciplines. Some of the literature is of limited utility here, since there is often little distinction made between theoretical sciences (such as mathematics) and vocational sciences (such as engineering and business). Nonetheless, it is clear that students taking cross-disciplinary subjects or team-taught subjects will probably be exposed to differing disciplinary approaches and cultures, which may pose obstacles for their learning experience.

Bradbeer [6] presented an excellent exposition of the concepts in summing up the arguments of many authors and comparing their conclusions. He started with the theories of Kolb [14; cited in 6], probably the best known of the researchers exploring the concept of disciplinary differences, and ranged through to Nulty and Barrett, Becher and others. Kolb [14] began his explorations in terms of his notion of experiential learning, which essentially asserts that learning depends on information transfer and transformation. He suggested that learning styles can be broken down into four basic types, centred on people's approach to dealing with information, and these form the foundation of his categorisation of disciplines. The categories were linked to psychological types, using the Myers-Briggs personality type theory. Bradbeer [6] noted that, "Personality types and learning styles do seem to be related in complex ways...[and] we can conclude that different disciplines both process and structure knowledge in different and distinctive ways." (p. 384)

In terms of groupings, Bradbeer presented a scheme based on Nulty and Barrett's approaches [19], Becher's concept of "academic tribes" (first presented in 1989 – see also [3]) and the classification systems of other theorists. When they are all overlaid, there is congruence in the categorisation of mathematics, described as hard/pure, abstract/reflective; and of engineering, which is regarded as a science-based profession (hard/applied, abstract/active). In these schema, statistics is placed under mathematics. Business is grouped with engineering as both being science professions (however, it should be noted that some business courses may tend more towards the social professions and therefore might differ markedly).

On the subject of disciplinary culture, Bradbeer [6] summed up Becher's theory of academic tribes as being in alignment with other theorists, founded on viewing, "... academic disciplines as simultaneously structures and organisations with their own very distinctive cultures; sufficiently distinctive to be plausibly regarded as tribes ..." [6, p. 388] Becher [3] stated that, "The cultural aspects of disciplines and their cognitive aspects are inseparably intertwined. ... disciplinary practices can be closely matched with the relevant characteristics of their associated domains of enquiry." (p. 153)

Where there is some dispute within the literature is on the extent to which these disciplinary differences affect student learning – of fundamental importance since it is they who have to consume the results. There is a general underlying assumption that the student who chooses a particular career is already formed in that mould; that is, an engineering student is already an engineer. Bradbeer [6], for instance, stated that, "Students learn in different ways and many have unconsciously chosen disciplines that are more accommodative of their personal learning styles" (p. 382). This seems of dubious reliability without more research – it may be true of some students, but relies on their having a sophisticated understanding of the chosen profession.

With time, the student may grow into the discipline, and Nulty and Barrett [19] investigated this possibility through a comparative study of Australian first and final year students. They found that first year students had similar learning styles irrespective of their discipline, and they also found little evidence of disciplinary styles in third year students. To explain this result, they postulated that most courses involve a range of subjects from different disciplines and so there is in fact little opportunity for the disciplinary culture of the major to be instilled in the students. They pointed out that one of the implications from these results is that teachers will need to accommodate the fact that their students will have a range of learning styles and approaches.

In addition, the existence of differences does not necessarily mean the outcomes are fixed, since students can adapt their learning approaches to circumstances. Despite the identification of disciplinary groupings of learning styles, Kolb, "... did not claim that particular learning styles and specific disciplines were necessarily linked, but rather that some learning styles were more adapted to some disciplinary knowledge structures than others. So it would be possible to learn something ... of a discipline using a learning style that did not exactly match the discipline's knowledge structure." [6, p. 392]

## Teaching mathematics to engineers

There was broad agreement that mathematics is an essential ingredient in engineering courses. O'Kane [20] presented a provocative view of the importance of mathematics for engineers:

> ... during the past three hundred years a handful of mathematicians of the highest stature, not engineers, progressively created and refined the fundamental concepts of engineering discourse, and these concepts have entered the engineering curriculum with a delay of roughly one hundred years. This promethean achievement was not a response to immediate engineering demand, and is the ultimate reason, now largely forgotten, for the presence of mathematics in our engineering schools. (pp. 362-363)

The mathematicians consistently maintained that mathematics should be taught by mathematicians. According to Varsavsky [23]: "There appears to be general agreement that mathematics is best taught by mathematicians and in many institutions mathematicians have the professional responsibility to develop and deliver mathematics subjects for engineers" (p. 344). Others also iterated the point (such as [2, 7, 10).

Kümmerer [14] presented a general discussion on the range of approaches used to teach mathematics to engineers. In some courses, mathematics is taught as a kind of machine to produce the answer to a problem. Another style is to make things 'user friendly' so that only the essentials are presented, which can lead to oversimplification. Alternatively, mathematics may be viewed as simply something to be endured before the student starts physics or engineering. Kümmerer proposed instead that understanding is crucial (his emphasis): "The most efficient way to learn mathematics *is to understand mathematics*. ... Therefore, as a primary goal a course on higher mathematics should enable students to handle mathematics on their own. ... I want to emphasize that even when there is limited time and differing needs, mathematics is best taught with understanding." [14, p. 324]

Bajpai *et al.* in 1975 [1] suggested a range of improvements to the then current teaching methods, most of which are still relevant thirty years later. They include: using a modelling approach; specialist engineering mathematics lecturers (whilst being based in the mathematics department); motivation, that is, clarifying for students the important role played by mathematics in engineering; an integrated approach towards mathematics techniques; presenting a comparison of mathematical methods; and presenting more relevant examples. They also argued for greater enthusiasm from teachers: "For, if the teacher can find a challenge and an excitement in teaching mathematics to engineering students, there is a chance that the students may find a challenge and an excitement in learning." (pp. 361-2)

There was a wide range of other recommendations for appropriate teaching methods in this context. For example, small group work was supported by writers as being beneficial since it resembles the students' future working lives: "These changes train engineers to carry out mathematical tasks in an environment which is much closer to that which they will encounter in their professional practice." [26, p. 596]

The need for relevance was highlighted by many writers as being important in assisting students with learning mathematics. To make a mathematics course seem relevant to engineering students – and hence worth an investment of time – the subject has to be made to seem valuable for their own specialisation and future cases. It is also beneficial for the educator to be familiar with the other subjects the students are doing, so examples can be used from these. As Kümmerer [15] pointed out, "In most cases a translation into the other language is necessary before students can recognize that one is discussing the same subject." (p. 331) Courses need to be tailored to the career ahead, such as in terms of materials that students perceive are applicable to their profession; realistic problems will further assist with promotion of the value of numerical methods [1, 10, 18].

The use of computer and online teaching can offer benefits to the mathematics educator. Various advantages to using computer packages were presented in the literature, including:

- Students enjoy using software applications; for instance, at Wollongong University: "... during the implementation of projects and computer labs ... [it was] observed that students showed that they had an overwhelming enthusiasm in attending and using computers with algebra/numerical packages." [26, p. 597]

- Computer packages which incorporate visualisation and animation can help promote understanding of mathematics, through reducing its apparent abstract nature [4].
- Colgan [8] presented a case study on the introduction of MATLAB into mathematics teaching for engineering students and found that it offered an avenue for the introduction of applied engineering examples.
- Varsavsky and Carr [24] developed computer-based resources to provide additional materials, which also gave the opportunity for development of new learning styles. The resources were a positive addition to their course, but they asserted it is still too early to fully estimate the worth of computer-based learning in mathematics.

In terms of content, a number of writers made reference to the core curriculum in mathematics for engineering students at European tertiary institutions developed by SEFI (Société Européenne pour la Formation des Ingénieurs). SEFI established a Mathematics Working Group (MWG) in 1982, "... with the aim of furthering developments in mathematical education, assessing the impact of computer technology and monitoring the needs of industry."[2, p. 223] The MWG continues to advocate a mix of teaching strategies: "The SEFI-MWG recognizes that the design of courses with an optimum mix of teaching strategies is becoming much more difficult and expensive in terms of human resources and equipment but believes that the successful core curriculum of the 1990s should include all such elements."[2, p. 227] Some Australian universities also follow this curriculum, for instance, the Swinburne University of Technology uses the SEFI model as well as textbooks developed for the SEFI syllabus [10].

## Statistics for business students

This subject was a fruitful line of enquiry and revealed some interesting correlations with views on teaching engineering students. McAlevey and Sullivan [17], for instance, asserted that there is a need for using real-life problems since, "Students are best motivated by exposure to real applications, problems, cases and projects" (p. 426). This is affirmed by Parr and Smith [21], who presented an argument for using realistic case studies. These discussions resemble those about using real-life examples and applications for engineering students; in fact Kolb [14; cited in 6] placed business with engineering in his classification of disciplines. Parr and Smith [21] also considered the case-study approach as helpful for developing problem-solving skills and providing integration with other subjects.

Student perceptions of statistics courses were also a common topic for investigation. McAlevey and Stent [16] undertook a survey at Otago University, of business students' perceptions of the notion of good teaching in reference to statistics courses. In other studies, students of science and mathematics have given weight to the clarity of teacher explanation, and placed an emphasis on preparation by the lecturer and the organisation of course material. This survey indicated that the business students preferred similar modes of delivery. Surprisingly, the need for real applications was ranked low by the students. McAlevey and Stent's explanation for this was that an introductory statistics course was probably too early in their learning, that is, the use of applications at that point can obscure the actual statistics.

Zanakis and Valenzi [27] surveyed students at a university in Florida, USA, this time with an emphasis on anxiety, and looked at how student attitudes changed during a revised statistics course. They hypothesised that there is a certain incidence of anxiety – similar to mathematics anxiety, which is commonly recognised as a problem – as well as a certain level of fear about using a computer for analysis. The results of the survey showed that certain improvements did occur over the course, however, they also found that there was a reduced interest or belief in statistics. Their explanation for this was that, although anxiety was reduced, statistics was still perceived as being very difficult; in addition, the course had not convinced students of the worth and career relevance of statistics.

Another course restructure focused on alleviating mathematics and computer anxiety. Zeis' team [28] developed a course in business statistics, "... designed to 'put the horse before the cart', data collection and management are presented before analysis and inference" (p. 83). They also did surveyed the students, in this case to look at learning outcomes and found that there was clearly improvement in reported skill levels.

## Interdisciplinarity

Bradbeer [6] defined the different strands of the concept 'interdisciplinary':

> ...multidisciplinary study [is where] the student simultaneously studies several disciplines, each taught by specialists. No explicit attempt is made by the teachers to relate their disciplines to each other...Interdisciplinarity involves a curriculum where teaching and learning occur in the overlap areas between disciplines. Often this involves elements of one discipline being taught by a teacher of another. Students are expected not just to be able to move from one discipline to another but also to be able to synthesise insights from the various disciplines studied. (p. 382)

Thus multidisciplinarity could be seen as courses being served in an open plan area, but each dish remains firmly within the culture of the specialist. Interdisciplinarity, on the other hand, involves truly multicultural servings with blended flavours and ingredients.

There has been a great deal written on interdisciplinary innovations in teaching, most of it couched in the framework of 'our course'. It makes for interesting reading; unfortunately, it is difficult to extrapolate useful techniques. The writings are predominantly based on descriptions of either new courses or courses that have been revised in the light of new information on teaching techniques; or student dissatisfaction with previous versions of the course; or in an attempt to reduce isolation and fragmentation of subjects (see for instance [5, 9, 11, 22]). In general, articles focused on multidisciplinary approaches – that is, specialists presenting stand-alone sections – rather than taking a genuine interdisciplinary view. The articles all reported gains for students – and promotion of collaboration with other disciplines – but these assessments were usually based on student opinion rather than an objective assessment of gains in learning (exceptions are [9, 11], where they administered various tests at the end of the course).

Fitchett [12] explained the impetus for interdisciplinarity as the need to present mathematics and science in terms of how it is practiced by professionals. He theorised that in order to learn about processes and not just content, students must learn to integrate knowledge, rather than compartmentalising it. Deeds *et al.* [9] approached development of a new interdisciplinary course at Drury University (USA) in a similar spirit, and chose integration as the preferred mode to promote understanding of the subject matter, retention of material, and comprehension of the connections between science and the everyday world. Their prescription for successful ventures in this arena involved three critical elements, that is, "... building connections between departments, broadening the number of participants, and ensuring support from the administration." [9, p. 181]

Another example was an interdisciplinary science course at Alfred University (USA), as presented by Boersma *et al.* [5]. They described their rationale as wanting students to, "... see the interrelationships between different scientific disciplines and between science and mathematics because it has been shown that students often learn science better when an interdisciplinary approach is taken ..." (p. 398)

## Conclusions

Interdisplinarity is being promoted as providing significant gains in student learning. At this stage, it is difficult to ascertain how successful new courses really are. Disciplinary cultural differences between academics can pose obstacles to communication, both with each other and with students. For the students, crossing over between cultures can prove difficult and, without appropriate institutional support or other coping mechanisms of their own, they may leave university altogether. There are many ways the mathematics educator can assist student success, the first step being to recognise that each one will bring different internal and external circumstances with them (including variations in learning styles).

The major consideration must remain the student – what ingredients and styles are appropriate for their consumption – since there is no point in offering subjects that are palatable to only a few (tomato sauce does not seem to fit with wonton soup, but Worcestershire sauce might work). Current trends are towards fostering multiculturalism in many fields, but the needs of the consumer – and their palates – should remain of paramount concern.

## Acknowledgements

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## References


1. Bajpai, A., Mustoe, L., & Walker, D. (1975). Mathematical Education of Engineers: Part 1. A critical appraisal. *International Journal of Mathematical Education in Science & Technology, 6*(3), 361-380.
2. Barry, M., & Steele, N. (1993). A Core Curriculum in Mathematics for the European Engineer: An Overview. *International Journal of Mathematical Education in Science & Technology, 24*(4), 223-229.
3. Becher, T. (1994). The Significance of Disciplinary Differences. *Studies in Higher Education, 19*(2), 151-161.
4. Blyth, W. F. (2001). Animations using Maple in First Year. *Quaestiones Mathematicae, Suppl. 1*, 73-82.
5. Boersma, S., Hluchy, M., Godshalk, G., Crane, J., DeGraff, D., & Blauth, J. (2001). Student-Designed, Interdisciplinary Science Projects – Placing Students in the Role of Teacher. *Journal of College Science Teaching, 30*(6), 397-402.
6. Bradbeer, J. (1999). Barriers to Interdisciplinarity: Disciplinary discourses and student learning. *Journal of Geography in Higher Education, 23*(3), 381-396.
7. Clayton, B. (1995). Mathematics for Engineers: An integrated approach. In L. Mustoe & S. Hibberd (Eds.), *Mathematical Education of Engineers* (pp. 29-39). Oxford: Clarendon Press.
8. Colgan, L. (2000). MATLAB in First-Year Engineering Mathematics. *International Journal of Mathematical Education in Science & Technology, 31*(1), 15-25.
9. Deeds, D. G., Allen, C. S., Callen, B. W., & Wood, M. D. (2000). A New Paradigm in Integrated Math and Science Courses – Finding common ground across disciplines. *Journal of College Science Teaching, 30*(3), 178-183.
10. Easton, A., & Steiner, J. (1996). *A Core Curriculum in Mathematics for the Australian Engineer.* Paper presented at the Second Biennial Australian Engineering Mathematics Conference, Sydney, Australia: Institution of Engineers, Australia.
11. Elliott, B., Oty, K., McArthur, J., & Clark, B. (2001). The Effect of an Interdisciplinary Algebra/Science Course on Students' Problem Solving Skills, Critical Thinking Skills and Attitudes Towards Mathematics. *International Journal of Mathematical Education in Science & Technology, 32*(6), 811-816.
12. Fitchett, S. (2002). *Using Environmental Science to Bridge Mathematics and the Sciences.* Paper presented at the Second International Conference on the Teaching of Mathematics (ICTM2), Crete. Available on CD-ROM.
13. Kirkup, L., Wood, L., Mather, G. & Logan, P. (2003) Teaching Mathematics and Physics to Engineers: Reflections from the chalk face. In *EMAC 2003 Proceedings* (pp. 97-102). Sydney.
14. Kolb, D. (1984). *Experiential Learning: Experience as the source of learning and development.* Englewood Cliffs, NJ: Prentice Hall. Cited in J. Bradbeer (1999). Barriers to Interdisciplinarity: Disciplinary discourses and student learning. *Journal of Geography in Higher Education, 23*(3), 381-396.
15. Kümmerer, B. (2001). Trying the Impossible – Teaching mathematics to physicists and engineers. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level – An ICMI Study* (pp. 321-334). Dordrecht: Kluwer Academic Publishers.
16. McAlevey, L. G., & Stent, A. F. (1999). Undergraduate Perceptions of Teaching a First Course in Business Statistics. *International Journal of Mathematical Education in Science & Technology, 30*(2), 215-225.
17. McAlevey, L. G., & Sullivan, J. C. (2001). Making Statistics More Effective for Business? *International Journal of Mathematical Education in Science & Technology, 32*(3), 425-438.
18. McGregor, R., & Scott, B. (1995). A View on Applicable Mathematics Courses for Engineers. In L. Mustoe & S. Hibberd (Eds.), *Mathematical Education of Engineers* (pp. 115-129). Oxford: Clarendon Press.
19. Nulty, D. D., & Barrett, M. A. (1996). Transitions in Students' Learning Styles. *Studies in Higher Education, 21*(3), 333-345.
20. O'Kane, J. (1995). Engineers and Their Mathematicians – Foundations for a debate on mathematics in the engineering curriculum. In L. Mustoe & S. Hibberd (Eds.), *Mathematical Education of Engineers* (pp. 361-378). Oxford: Clarendon Press.
21. Parr, W. C., & Smith, M. A. (1998). Developing Case-Based Business Statistics Courses. *The American Statistician, 52*(4), 330-337.
22. Sjöberg, A. (1996). *Project-Oriented Courses in Scientific Computing with Applications from Natural Sciences and Engineering.* Paper presented at the Second Biennial Australian Engineering Mathematics Conference, Sydney, Australia: Institution of Engineers, Australia.

23. Varsavsky, C. (1995). The Design of the Mathematics Curriculum for Engineers: A joint venture of the mathematics department and the engineering faculty. *European Journal of Engineering Education, 20*(3), 341-345.
24. Varsavsky, C., & Carr, A. (1997). *Designing an Interactive Electronic Book for First Year University Mathematics*, available at www.atcminc.com/mPUBLICATIONS/EP/EPATCM97/ATCMPO14/paper.html
25. Wood, L.N., Mather, G., Logan, P. & Kirkup, L. (2003) Teaching Mathematics and Physics to Engineers: Reflections from the back row. In *EMAC 2003 Proceedings* (pp. 295-300). Sydney.
26. Worthy, A. (1996). *Mathematical Education of Engineers – Towards the year 2000.* Paper presented at the Second Biennial Australian Engineering Mathematics Conference, Sydney, Australia: Institution of Engineers, Australia.
27. Zanakis, S. H., & Valenzi, E. R. (1997). Student Anxiety and Attitudes in Business Statistics. *Journal of Education for Business, 73*(1), 10-17.
28. Zeis, C., Shah, A., Regassa, H., & Ahmadian, A. (2001). Statistical Components of an Undergraduate Business Degree: Putting the horse before the cart. *Journal of Education for Business, November/December*, 83-88.

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# PROBABILITY WITH CAS AS A PART OF CALCULUS – AN UNIVERSITY LECTURE FOR TEACHER STUDENTS

OTTO WURNIG

After the last reforms of the curricula in mathematics (1993, 2000), statistics and the use of the computer have been fixed in mathematical instruction for ten to fourteen year-old students in Austria. Probability & statistics have become a compulsory part in the last grades of all schools (AHS/BHS) preparing students for university level. Unfortunately probability & statistics and the use of the computer have only recently become compulsory for teacher students at university. This is the reason why a young mathematics teacher might still not have a satisfactory knowledge of it. In order to improve this situation the **ACDCA** (Austrian Centre for the Didactics of Computer Algebra) has been founded. ACDCA organises conferences and meetings and makes publications with the intention of offering a framework for university teachers, teacher trainers and school teachers, to exchange their experiences and to do research projects (homepage: http://www.acdca.ac.at*)* In addition to my work at university, I managed two research classes in the last **CAS projects (1999/2002)**. All students of these two research classes have used TI-DERIVE or PC-DERIVE in the mathematics lessons and at home. The use of a CAS calculator is a very great advantage and offers new possibilities of solving problems especially in probability & statistics.

In my lecture at university about the use of computers in probability & statistics I tried to show the teacher students these new possibilities. Out of twenty students only two had a good knowledge of at least one computer algebra system (CAS). All students learned the most important concepts in the last two years at grammar school, but now some years later after leaving school they had great difficulties in solving problems of the binomial or normal distribution. The poisson distribution, for instance, was new to most of them.

Our computer lab at the Institute of Mathematics in Graz is equipped with eleven PCs one of which is connected with a beamer. In the summer term 2003 two students shared a PC in my lecture. As the coordinator of the action "Teachers Teaching with Technology" for my county, I could also organize twenty TI-89/92+ for the students. After an introduction in TI-DERIVE and PC-DERIVE ten students worked with CAS on TIs and ten on PCs in the following lessons. This was the reason why I sometimes used the PC with the beamer and the TI with the view screen simultaneously.

The fact that the teacher students at university have very different experience in working with a computer and have rarely experienced how to teach statistics with the help of a computer at school, turned out to be other an advantage as they were gradually willing to learn concepts of probability & statistics with CAS. The new possibilities offered by CAS, especially in understanding the concept of the bell shaped curve in plotting it so quickly on the screen, made some of the students really enthusiastic. Some of them said,

*"This really helps us to understand the concept of the normal distribution at last."*

1. **With CAS calculators the students can quickly manage summations** and understand more easily and more independently **algebraical proofs**, e.g. the derivation of the formula of the expectation and the variance of a binomial distribution, in the algebra window.

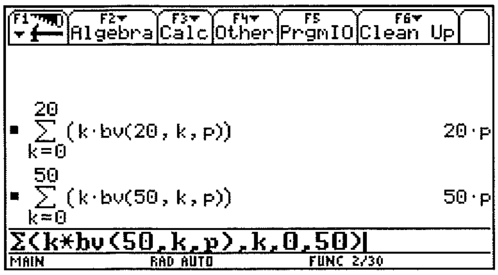


*FIGURE 1.* Expectation $\mu = n \cdot p$

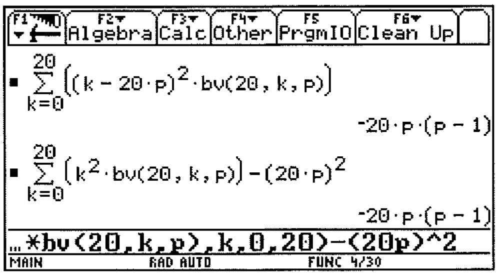


*FIGURE 2.* Variance $\sigma^2 = 20.p.(1-p)$

2. **With CAS calculators the students can quickly manage the limit of sequences.**

Therefore it is possible to show the students a way to find the formula of the poisson distribution by themselves. The binomial distribution can be approximated with the poisson distribution, if the number n of the trials is very large and the probability of the experiments is very little and if it is granted that **n . p → μ** with μ as a real and finite number. This approximation is very good for $0 < p < 0.1$. In 1975 A. ENGEL suggested a new way to derive the formula, which is easy to calculate with the TI-92+.

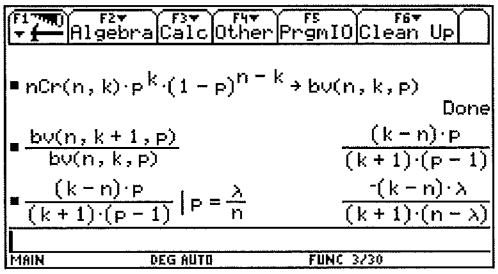


*FIGURE 3.* bv(k+1)/bv(k) with p = μ/n

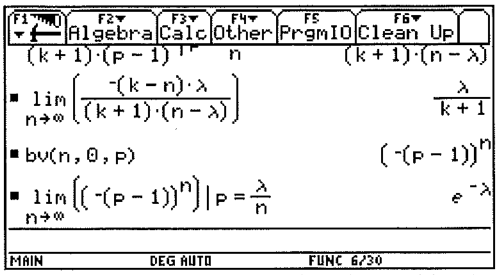


*FIGURE 4.* p(μ,k+1)/p(μ,k) = μ/(k+1)

**formula of recursion:** $\frac{p(\mu,k+1)}{p(\mu,k)} = \frac{\mu}{k+1}$ **with** $p(\mu,0) = e^{-\mu}$

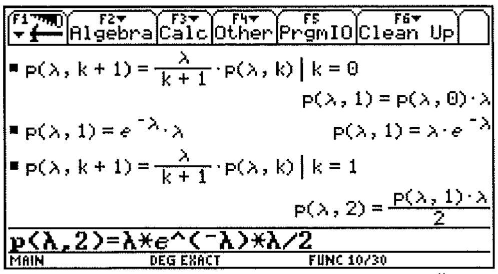


*FIGURE 5.* k=1 → p(μ,1)=μ.e$^{-\mu}$

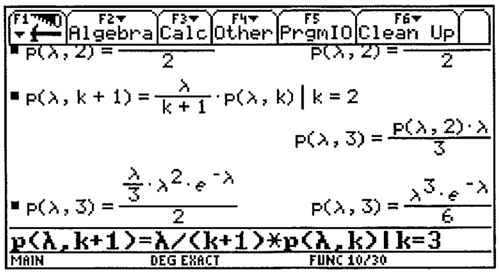


*FIGURE 6.* k=3 → p(μ,3) = e$^{-\mu}$ . μ³/6

Given k = 1, 2, 3, ...you can easily find the formula of the poisson distribution.

$$pv(\mu, k) = \frac{\mu^k}{k!} \cdot e^{-\mu}$$ with **μ = n . p as expectation and as variance σ²**

## 3. The solution of a problem with the TI-92+.

4 % of all the air passengers who have reserved their seats usually do not turn up.

The airlines know this and so they sell 75 tickets for 73 available seats.

What is the probability that all passengers find a seat ? (KIRSCHENHOFER/ARNOLD)

The probability of a passenger collecting the ticket is 0.96, of his not turning up is 0.04. Two variants of calculation are possible: All passengers get a seat, if 73 passengers turn up, or if at least two of the 75 passengers do not turn up.

**CALCULATION OF $\mu$:**

<u>N = 75</u> | **FOR** P = 0.96 **IS** $\mu$ = 75 * **0.96** = 72 | **FOR** P = 0.04 **IS** $\mu$ = 75 * **0.04** = 3

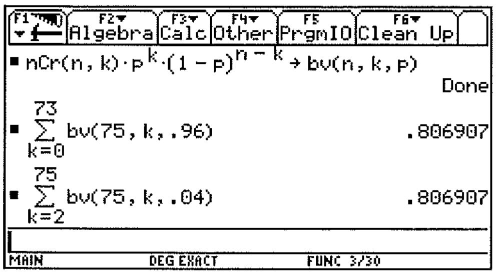


*FIGURE 7.* binomial distribution bv(n,k,p)

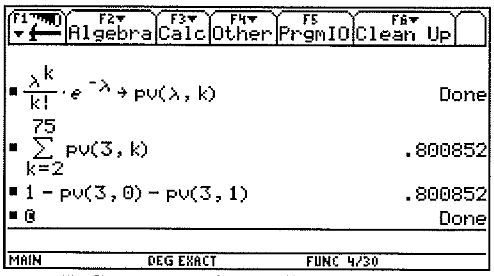


*FIGURE 8.* poisson distribution pv($\mu$,k)

The calculation with the TI-92 is now possible with both distributions in different variants.
The passengers turning up get a seat with a probability of about 80%.

<u>Result of binomial distribution 80.7%</u> <u>Result of poisson distribution 80.1%</u>

## 4. With CAS-calculators you can quickly manage a sequence of figures.

If a table for the binomial distribution e.g. n=20 and p=0.5 (**k = 0,1,2, ..., 20**) is created, a point diagram, a histogram and at last a probability density polygon can quickly be plotted.

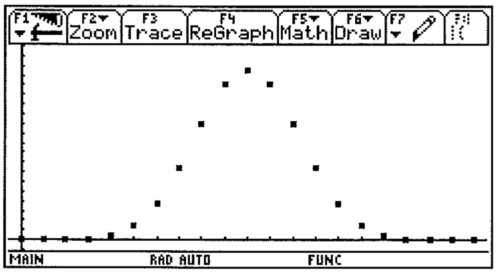


*FIGURE 9.* Point diagram (square)

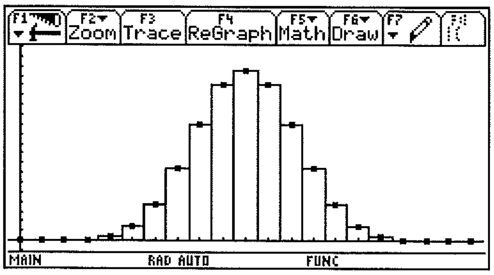


*FIGURE 11.* Points as midpoints of the bars

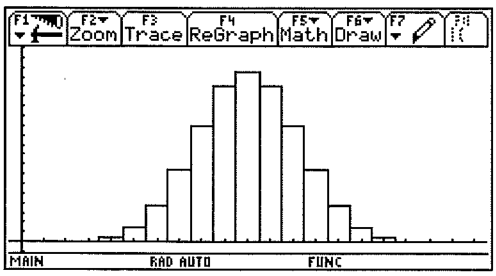


*FIGURE 10.* histogram, width of the bars = 1

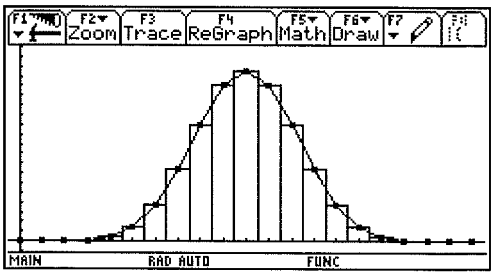


*FIGURE 12.* Points connected (xy-line)

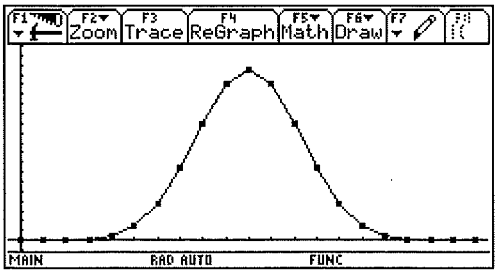


*FIGURE 13.* probability polygon

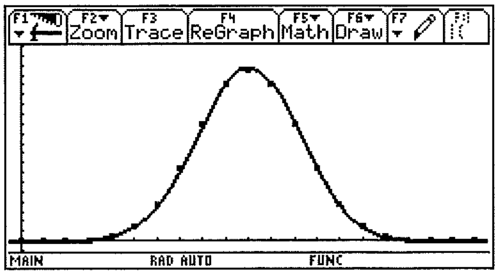


*FIGURE 14.* Does a function exist?→ **nv(x,μ, σ)**

With this sequence of figures it is possible **to get an imaginable transition from the discrete distribution to the continuous distribution** including the important correction. In order to replace the polygon to the fitted curve of the probability density function **nv(x,μ,σ) = 1/(√(2π)σ) . e** $^{-0.5((x-\mu)/\sigma)^2}$ in figure 14, it is necessary to look for the standardized function with $\mu=0$ und $\sigma=1$.

**5. With CAS-calculators you can quickly give an impression of standardization.**

If there is no time to derive the function nv(x,μ,σ), **nv(x,0,1) = 1/(√(2π)) . e** $^{-0.5x^2}$ , the standardized function, can be entered by the students on their own screens and they can see that the curve is running through almost all the points.

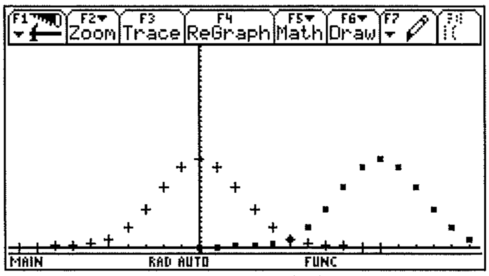


*FIGURE 15.* Translation with (x-μ)

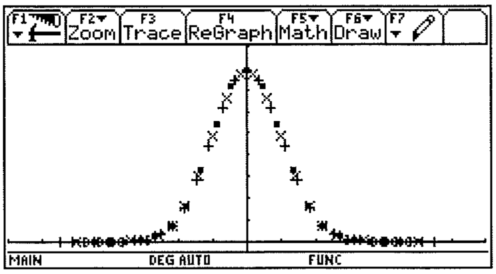


*FIGURE 17.* standardized points: n=20, 25, 30

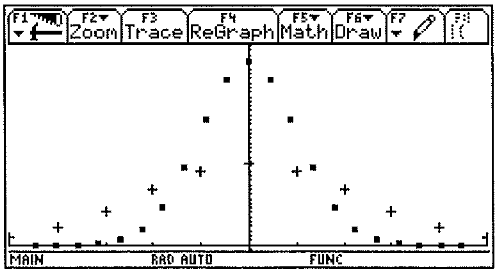


*FIGURE 16.* x → (x-μ)/σ, y →σ.y

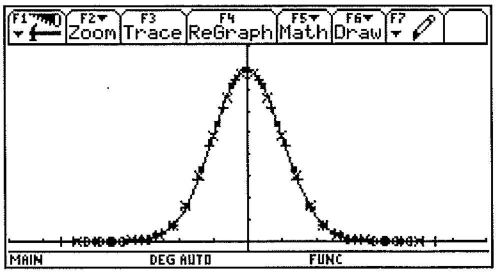


*FIGURE 18.* The graph of nv(x,0,1) fits !

**6. With CAS-calculators you can derive the function nv(x,0,1).**

If there is time to derive the function nv(x,0,1), it is possible to go the way of standardization with all students stepwise (*WURNIG, 2001, 2002*). Regarding the figure with the standardized points, many students want to know how you can derive a function so that the graph will closely approximate the points.

In order to achieve this, you have to find a function f with the following qualities:

- $f(x) \geq 0$ for all x and $f(0) = d > 0$ (y-intercept);
- the graph of f has to be „bell shaped" and symmetrical to the y-axe.
- the area between the bell-shaped curve and the x-axe has to be 1.

Calculation of **f(0) = σ(n).b(μ)**
p.e.: n=40, p=0.5 → μ=40*0.5=20,
σ(40)=√(40*0.25) → σ(40).b(20) ≈ **0.396**
*Does a limit of σ(n).b(μ) exist for n → ∞?*

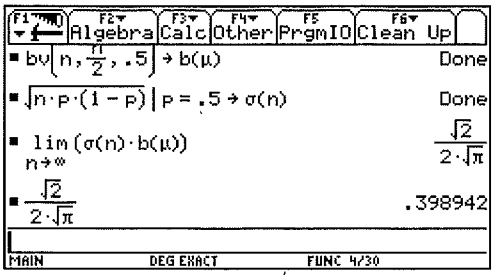


*FIGURE 19.* d = 1/√(2π) = ≈ 0.398942

which is symmetrical to the y-axe. But the area between the curve and the x-axe from -∞ to +∞ is 0.7 and not 1.

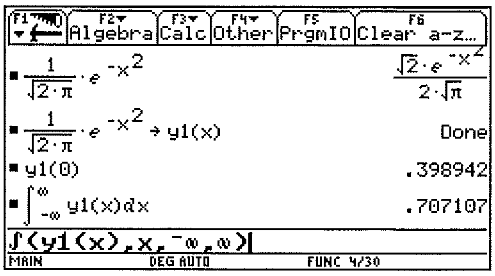


*FIGURE 20.* the area is calculated with 0.7 (≠ 1)

y1(x) = **(1/√(2π)) . e$^{(-x^2)}$ is a bell-shaped curve,**
*Therefore the exponent of the Function y1(x) has to be corrected by the factor a. The newly defined function*

*nv(x,a) = (1/√(2π)) . e$^{(-a.x^2)}$ has to be integrated by the bounds -∞ and +∞ . If the CAS-calculator cannot*

*solve the entered equation, the bounds have to be changed to -10 and +10. This offers the result:* **a = 0.5**

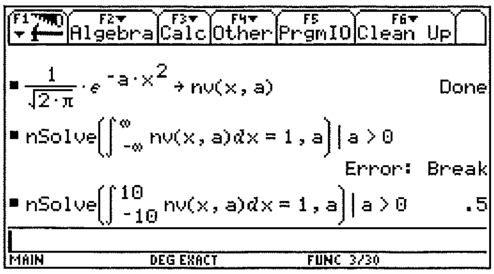


*FIGURE 21.* Calculation of a: a = 0.5

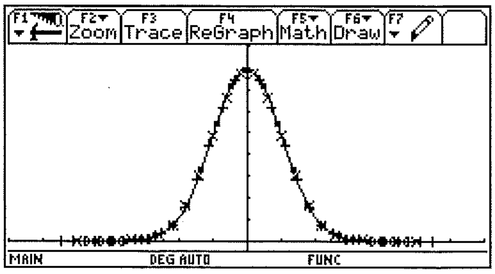


*FIGURE 22.* The graph of nv(x,0,1) fits.

**nv(x,0,1) = (1/√(2π)) . e$^{(-0.5x^2)}$**

**7.** **With** CAS calculators **the students can quickly** integrate functions.

Problems of probability can thus be solved **without tables**. This increases the number of different ways of solving problems and stochastics loses its isolated position. Probability becomes a part of calculus.

The death rate of newborns, this means the probability of babies to die during their first year of life, is 1,8% in a country. What is the probability that of 1000 newborns chosen at random more than 950 and less than 980 will live to see their first birthday ? (*REICHEL*)

Normally this problem is solved with the binomial distribution (p = 0.982). But you can also solve it with the poisson distribution, if you calculate the probability that of 1000 newborn babies less than 21 will not live to see the first birthday. After doing this you have to find the contrary probability.

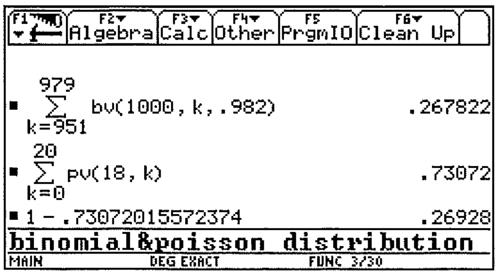


*FIGURE 23.* PV: μ = n . p = 1000 . 0,018 = 18

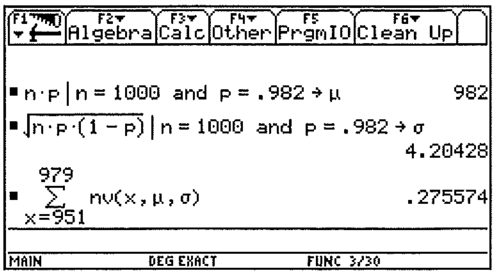


***FIGURE 24.* NV: summation with nv(x,μ,σ)**

The binomial distribution can also be approximated by the normal distribution. But with DERIVE you can choose between two ways: you can interpret nv(x,μ,σ) as a closely approximating function and make a summation or you compute the area by integration with corrected bounds.

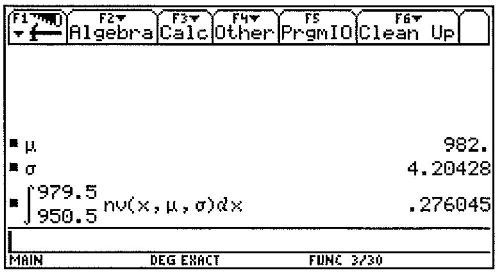


*FIGURE 25.* Integration of nv(x,μ,σ)

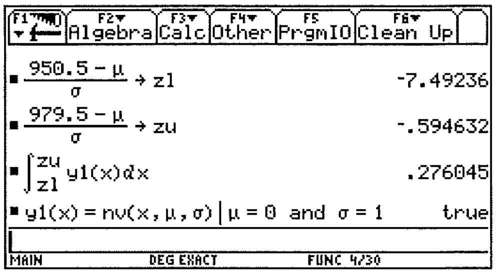


*FIGURE 26.* Integration of nv(x,0,1)

## 8. The TI-92 can calculate in the graphic window.

The TI-92 can plot the Gaussian bell-shaped curve of the normal distribution in the graphic window for every μ and σ. As soon as the curve is plotted, the TI-92 is able, with the command **∫f(x).dx**, to calculate the area under the curve after the input of the lower and upper bounds.

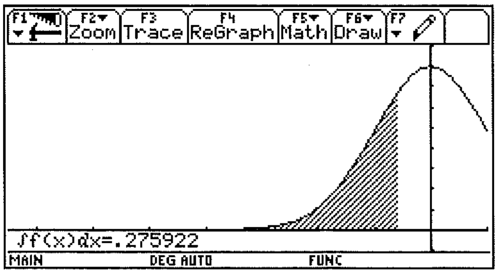


*FIGURE 27.* standardized corrected bounds

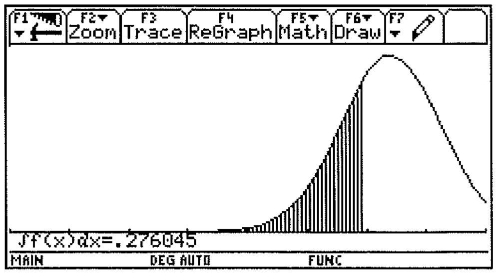


*FIGURE 28.* corrected bounds 950.5;979.5

Many students like this way of solution because they can immediately see at once the area which is the measure of the probability on the screen.

## 9. **Solving a problem of the** normal distribution **if μ is to be calculated.**

Engines have an average life μ of 100000 km and a standard deviation σ of 20000 km.
How is the average life to be improved so that by a constant σ more than 80% of the engines have a life longer than 90000 km ? (FIALA/MOSER)

Such problems, if μ or σ is asked, you can only solve with CAS-calculators if you have an idea of the result. Then you can go two ways:

- first integration and then solution of the equation with the command **nsolve**,
- first **calculated testing**, then interpolation and at the end control with integration.

The better students like these two ways. An average student prefers the Φ(z)-tables, because it is a receptive way, but very often with a wrong result if there is no control at the end.

```
F1 F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ ∫[90000,∞] y1(x)dx | σ = 20000 → f(μ)          Done
■ nSolve(f(μ) = .8, μ) | μ > 90000           106832.
■ ∫[90000,∞] y1(x)dx | σ = 20000 and μ = 106840
                                             .800106
MAIN        DEG EXACT        FUNC 3/30
```

*FIGURE 29.* integration, nsolve with μ>90000

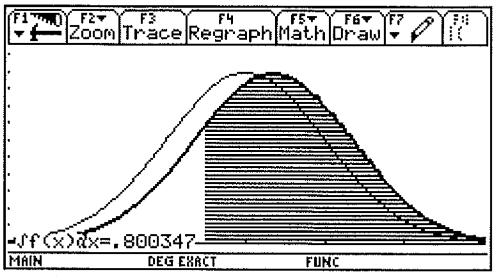


*FIGURE 31.* control: μ=100000 and μ=106840

```
F1 F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up
■ nInt(nv(x, 100000, σ), x, 90000, 200000)
                                       .691462
■ nInt(nv(x, 105000, σ), x, 90000, 200000)
                                       .773372
■ nInt(nv(x, 106000, σ), x, 90000, 200000)
                                       .788143
■ nInt(nv(x, 107000, σ), x, 90000, 200000)
                                       .802336
MAIN        RAD AUTO        FUNC 13/30
```

*FIGURE 30.* testing from μ=100000 to 107000

```
F1 F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

■ 802336 - 788143                        14193
■ 14193/1000                            14.193
■ 2336/14.193                          164.588
■ 107000 - 160                          106840
MAIN        RAD AUTO        FUNC 4/30
```

*FIGURE 32.* interpolation [106000, 107000]

The average life μ of the engines is to be improved to 106840 km.

## 10. Final Remarks

On the basis of the accompanying research of the CAS II project with the centre of school development (ZSE), it results that students who are mainly good at maths make positive observations and see advantages in the use of the TI-92 more frequently than those students who have good marks in their certificate in other subjects as well. Furthermore the report points out that contrary to the girls, boys experience a stronger increase of fun and a better stimulation of their interests and suffer less additional stress with the use of the TI-92. Besides boys hardly notice a loss of mathematical basis knowledge. *(GROGGER, 1999)*

All these reasons require a careful moving on when using utilities or functions and programs defined by the teacher. There is a great danger that such functions and programs are quickly used as a blind tool by the pupils at school.

Some of my students at university, who worked with CAS in my lecture, also did School Mathematics and Didactics offered by the chairman of the department of Mathematics. In their final examination they had to solve five problems. Two of the problems were taken from probability & statistics. Having studied the concepts of probability & statistics in the above described new way, my students solved those two problems significantly better than the others.

## References


1. A. Engel, *Wahrscheinlichkeitsrechnung und Statistik*, Band 1, Verlag Ernst Klett, Stuttgart, 1975, p. 118.
2. F. Fiala and W. Moser, *Mathematik Maturaaufgaben*, Verlag Hölder-Pichler-Tempsky, Wien, 1982³, p. 314.
3. G. Grogger, *Evaluation zur Erprobung des TI-92 im Mathematikunterricht an AHS,* ZSE-Report 40, Zentrum für Schul-Entwicklung (Center of School-Development), Graz, 1999
4. W. Kirschenhofer & H. Arnold, *Aufgabensammlung zur Wahrscheinlichkeitsrechnung mit didaktischen Beiträgen*, Band II, Institut für Mathematik, Universität Linz, 1978/79, p. 22.
5. H-Ch. Reichel, R. Müller & G. Hanisch, *Lehrbuch der Mathematik 8*, Verlag Hölder-Pichler-Tempsky, Wien, 1992, p. 124.
6. O. Wurnig, Wahrscheinlichkeitsrechnung und Statistik. ACDCA, 2001. (CAS III project, draft plus report of the results, 1999/2000, in German, http://www.acdca.ac.at)
7. O. Wurnig, *Advantages and dangers in the teaching of stochastics by using CAS,* in: M. Borovcnik and H. Kautschitsch (eds.), Technology in Mathematics Teaching, Schriftenreihe Didaktik der Mathematik, vol. 26, öbv & hpt, Vienna, 2002, 79-82.



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# SHOULD STRUCTURAL EQUATION MODELING BE TAUGHT TO UNDERGRADUATE STATISTICS STUDENTS?

STUART YOUNG

Structural Equation Modeling (SEM) is a general approach to data analysis that attempts to incorporate a range of standard multivariate methods, including regression, analysis of variance and factor analysis. Increasing use is being made of easy-to-use software such as AMOS to visually specify, view and modify a model using simple drawing tools. AMOS accepts a path diagram as a model specification and displays parameter estimates graphically. The software makes use of bootstrapping and can fit multiple models in a single analysis. In SEM interest often focuses on unobserved variables rather than on observed variables, which allows for modeling the linkage between a wide variety of response and explanatory variables. Thus models can be tested in fields where measurement is difficult and error-prone, such as econometrics, biometrics, health sciences and social sciences. SEM is a relatively young field with a methodology that is still developing. This paper attempts to discuss the advantages and disadvantages of teaching SEM at the undergraduate level.

## Introduction

The emphasis of this paper will be placed on the possible benefits of increasing the exposure of undergraduate students to Structural Equation Modeling. No attempt is made to describe in any detail the methodology, extent and sophistication of the technique. In addition, computer software such as AMOS is not explained in any depth. Interested readers are referred to [1] and [3] as initial texts. Instead, after a short introduction/review of SEM, several points are presented which argue in favour of increasing attention to the SEM procedure. This is followed by a summary that lists some of the concerns raised by critics of the SEM approach.

## Structural Equation Modeling

*Overview*

SEM is an extension of several multivariate techniques that largely takes a confirmatory (i.e., hypothesis-testing) approach. It has the ability to assess multiple and interdependent relationships simultaneously and efficiently, and to provide a transition from exploratory to confirmatory analysis [6]. The process typically begins with the researcher specifying theoretical conceptions of complex patterns of relationships amongst relevant variables [8]. Variables may be observed or unobserved, and the *a priori* model specifies which independent variables predict each dependent variable (which may themselves predict other dependent variables). A path diagram is constructed, and a series of structural equations are then formulated from the proposed relationships. The hypothesized model of relations is carefully tested to evaluate the degree to which the specified model appears to fit the observed data [4, 9]. For this purpose, several indices are calculated. SEM does not try to establish causality, and may rather be viewed as a complementary and useful approach in the absence of experimental data. If the model is theoretically strong and is well supported by the data, the conclusion is that it is a plausible model (not disproved). If modifications to the model are indicated (by the software), these can be tested providing theoretical justification is evident.

*Software*

Several computer programs are available for the estimation process, including the most widely known LISREL, EQS and AMOS. The latter has gained popularity in recent years [8] and has been used for this paper. AMOS is short for Analysis of Moment Structures.

*Simple Regression Example*

A path diagram from Arbuckle [1] (pp.109, 287) shown in Figure 1 below, shows that *performance* scores depend, in part, on *knowledge*. Since *performance* is not an exact linear combination of the three predictors, an error variable is included (enclosed in a circle since it is not directly observed), representing measurement error and any other factors on which performance may depend. Double-headed arrows indicate correlations/covariances. Notice that the error term is assumed uncorrelated with all predictors – a fundamental assumption in regression.

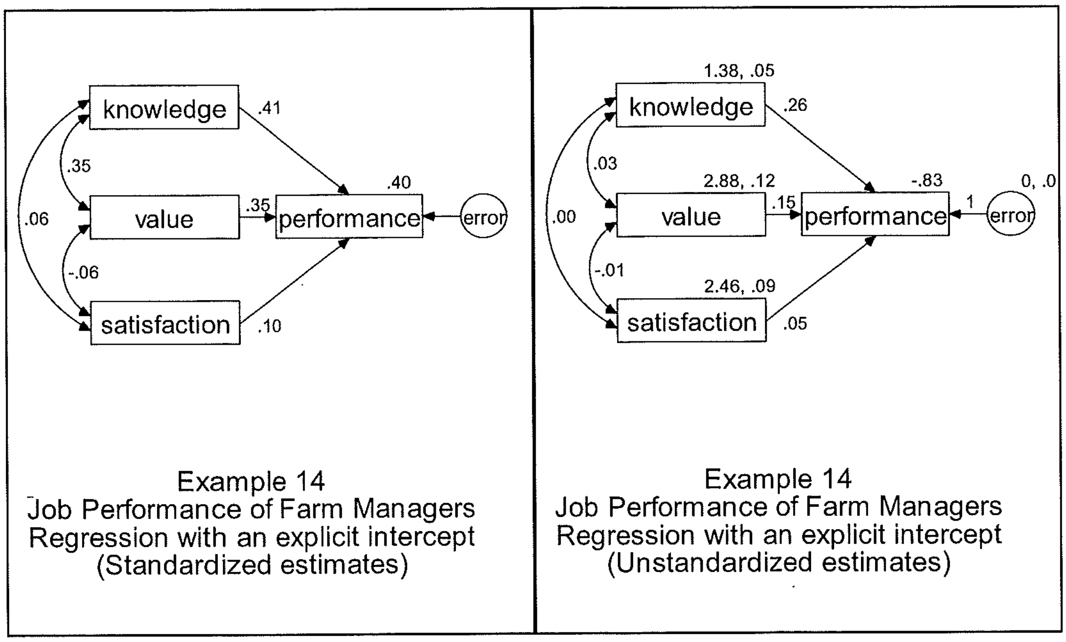


*FIGURE 1. Standardised and unstandardised estimates of data from Warren, White and Fuller (1974)*

The standardised estimates diagram on the left shows an R-square of 0.4, estimated beta weights as well as correlations (double-headed arrows). The right diagram shows estimates of the three regression weights (single-headed arrows), an intercept of -0.83, and *satisfaction* has estimated mean score of 2.46 and variance of .09. Output is available to test the significance of each estimate (not shown).

## Discussion

*Popularity*

SEM comprises a family of models, which have emerged as an integral tool for managerial, academic and non-experimental research [3, 5, 6, 9]. Hershberger concludes that it has become the pre-eminent multivariate method of data analysis [7]. His study of the growth of SEM examined articles published during 1994-2001 in over 1200 journals for multivariate techniques:

By 2001, the number of studies employing SEM had risen precipitously to 381, whereas in usage the other four multivariate techniques remained relatively stagnant: cluster analysis (121); MANOVA (100); discriminant analysis (54); and multidimensional scaling (52). [7] (p.41)

SEM has demonstrated growth also in number and variety of journals during this period, and is undergoing more development than other techniques including exploratory factor analysis. Hershberger described the clear majority of articles as substantive i.e., those which "apply SEM to real data" (p.36), Dilalla confirms "SEM has become a popular research tool in the social sciences, including psychology, management, economics, sociology, political science, marketing and education, over the past two or three decades" [4]. (p.439). Other areas of application include linguistics, environmental studies, biometrics, health, hospitality and tourism [5, 8, 9].

At AUT, almost all students who study statistics at any level are majoring in one of the behavioural sciences described above. Increasing numbers of postgraduate students and staff are undertaking research indicating structural modelling, as evidenced by the University's Statistical Advisory Service, as well as a site license for AMOS. The applied nature of SEM and the increasing availability of easy-to-use visual SEM software have raised awareness of the approach. These factors have led to the question of whether an introduction to SEM might be appropriate for undergraduate statistics students.

*Visualisation*

Schematic representations of models are termed path diagrams. They provide a visual portrayal of relations assumed to hold for the variables under study. Computer programs such as AMOS are easy-to-use since the model can be specified graphically with simple drawing tools. The path diagram is accepted as a model specification. Publication-quality graphics of both the input final model and the output are available, as are well-presented additional estimates and summaries. Byrne [3] comments that relations modelled pictorially enable a clearer conceptualisation of the problem. In summary, the path diagrams facilitate clearer communication and understanding of the model to students and fellow researchers and are enjoyable to work with [1, 8].

*Enriched understanding*

Using SEM has the potential to enhance and broaden the understanding of concepts encountered with many multivariate techniques. In Figure 2 below, $r_{12}$ is given more prominence than is usual with conventional multiple regression. If the predictors were independent then the partial regression coefficient $p_{31} = r_{31}$. Typically, $r_{12}$ is not zero (especially in non-experimental research or in experimental studies with unequal cell sizes) and the fact that $X_1$ and $X_2$ are related is what makes multiple regression interesting – trying to disentangle the various influences [8]. Thus $p_{31} = r_{31} - p_{32}\, r_{21}$. Problems of multicollinearity may be understood more clearly and remedied (discussed later). The concept of partial regression as an attempt to spread or partition common variance across the predictors is reinforced.

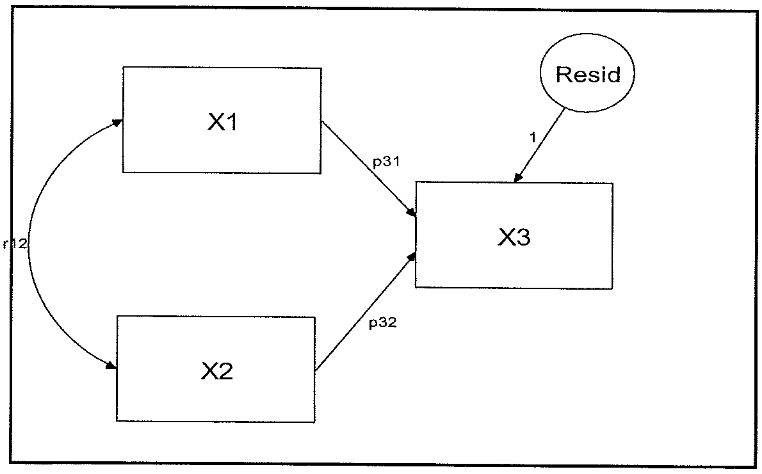


*FIGURE 2. Partial regression*

As another example of how ideas may be reinforced visually, Figure 3 below clarifies how the partial correlation $r_{23.1}$ is estimated – i.e., correlations of residuals after regression

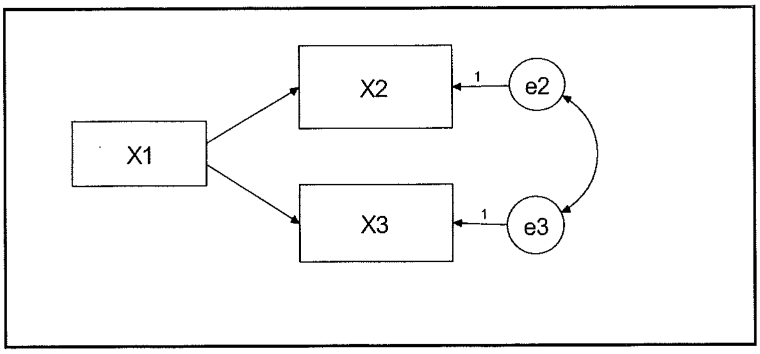


*FIGURE 3. Partial Correlation*

*Effects*

Path analysis affords the opportunity to examine indirect as well as direct effects. Using Figure 4 we can consider the causal effects: the direct effect of $X_1$ on $X_4$ is $p_{41}$ while an indirect (mediated) effect is $p_{43}$ $p_{31}$. There are also non-causal effects: $X_3$ has a relationship with $X_4$ by virtue of the fact that both are predicted by $X_1$; also there are effects which Maruyama [8] calls 'unanalysed prior association' – for example, $X_3$ is related to $X_4$ through the correlation of $X_1$ and $X_2$. Decomposing effects, i.e., tracing paths, allows the researcher to examine total effects within the model, and is software output.

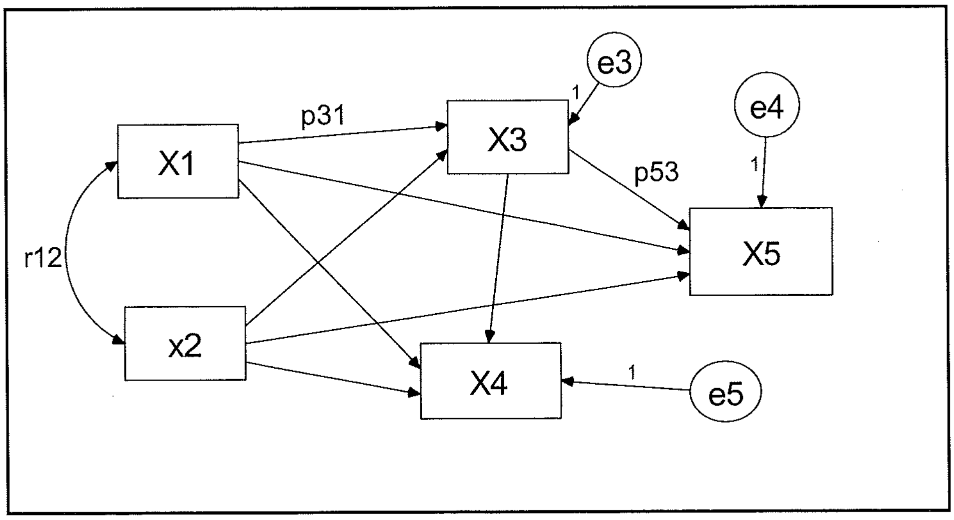


*FIGURE 4. Path Diagram of five observed variables*

*Model Specification*

There are efforts in many fields of study toward developing a more holistic view of relevant problems, corresponding with a more systematic methodology of testing a series of relationships [6]. Increased emphasis on operationalisation of concepts and careful forethought in developing a model highlights the importance of theoretical justification. Hair, Anderson, Tatham and Black [6] note that theory is "rooted in experience and practice obtained by observation of real-world behaviour" (p. 589).

There may be less need to learn an array of different techniques for different types of models [8]. Conventionally, students must choose carefully, whether to conduct a multiple regression analysis, perhaps exploratory factor analysis, perhaps some other multivariate technique. For each analysis, assumptions must be considered. For example, in regression it is critical to identify multicollinearity and remedy it if it exists. Also the assumption is made that error in the independent variables vanishes [3]. Both of these issues are dealt with naturally as part of a SEM procedure.

Undergraduate students traditionally have been minimally involved in the modelling process. They play a passive, analytical role where technical expertise threatens to overshadow the nature of the problem under study. Often statistical programs use default values in the analysis which further distance the researcher from some estimation issues. Because SEM flexibly encompasses an entire family of models it is well suited to a 'theory-driven' approach.

*Latent Variables*

Researchers from both academia and industry are confronted with specifying a clearly defined structural model as well as a *measurement model* – that is representing latent concepts with minimal measurement error. To illustrate how SEM is well-suited for this, part of the Hatco dataset in Hair, et al. [6] is used. There are seven variables measuring customer perception of the Hatco business (delivery speed, price level etc.), while two further variables measure customer usage and satisfaction. The question is asked 'What influences purchasing level of customers?' Time, thought and expertise is expended on this structural model. Initially, it may be hypothesized that customer's purchase relationships are predicted by *store service* and *store image*. These latent or unobserved variables are not measured directly and must be operationalised by observed, measured variables. Measures must then be proposed/chosen that share considerable variance with the unobserved variable. This is the underlying factor that affects the measured (manifest) variables. Having to label conceptual variables should warn against inclusion of overlapping measures; a latent variable would be formed that combines the highly correlated measures.

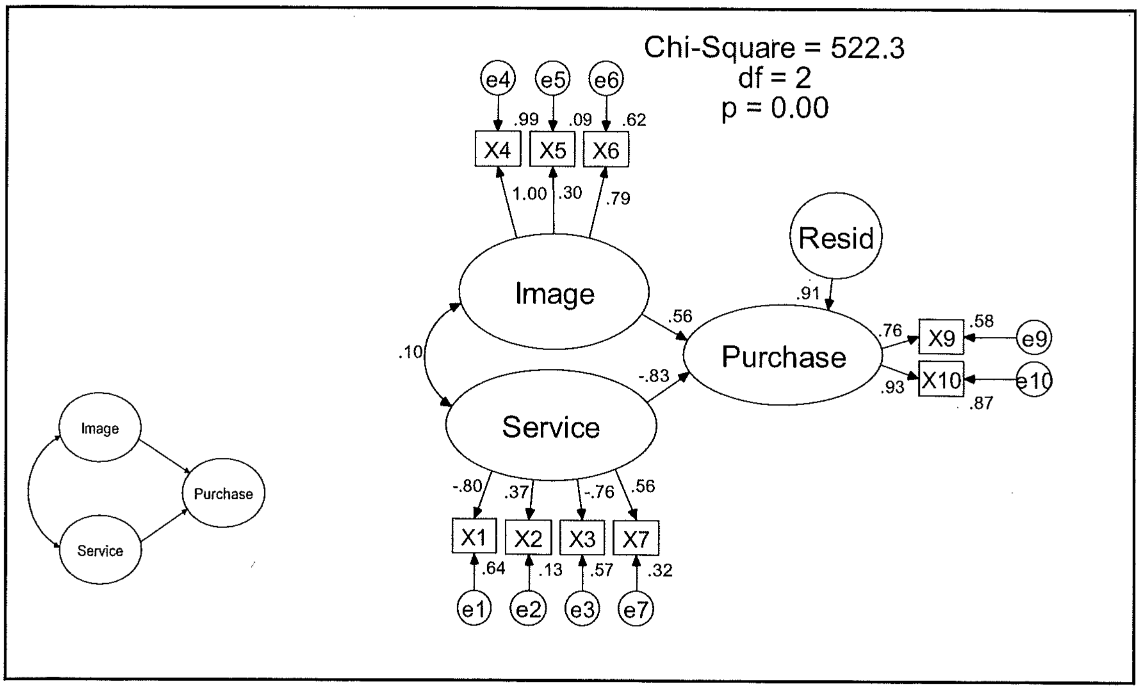


*FIGURE 5. Structural model and its elaboration into a measurement model with standardised estimates*

The plausibility of the model may now be tested, and if it has acceptable fit by suitable criteria the researcher has only confirmed that it is one of several possible models [8]. SEM allows development of several modeling strategies. The estimated model may be compared with alternative models in a competing models strategy. There may be alternative interpretations of the underlying theory, and models may be tested across groups. A nested model approach may be adopted whereby, for example, in Figure 5 a restriction is added that factor intercorrelation is zero. A model development strategy may be followed where from a weakly theoretical starting point, modifications to structural and measurement relations are made resulting in "development of a theoretically justified model that can be empirically supported" [6] (p.592)

*Cautions*

Much has been written [4, 5, 6, 8, 9] regarding the misuse and limitations of SEM, and students will need to be warned of these. However, discussing shortcomings can have positive effects for later research efforts, and provide a good learning experience.

General data requirements are often not met. In their enthusiasm, researchers often infer causality or state that a model 'has been confirmed'. Some fail to recognise that there is always a gap between theoretical named variables and the measures that operationalise them. Although using multiple indicators in a latent variable model helps to improve reliability and validity, often these indicators are collected using a single method i.e., method and trait are intertwined [8]. Critics are uneasy that a confirmatory technique is now increasingly used for model development with a single set of data. Dilalla mentions it is too easy to misinterpret results when examining the plausibility of a model. [4]. There is often an emphasis on overall model fit (at the expense of particular path coefficients), which encourages overfitting of data to attain the best possible fit (at the expense of generalisation). As Box cautions: "All models are wrong, but some are useful" [2] (p201).

## Conclusion

SEM is a flexible, analytic tool which, when coupled with graphical software that is user-friendly, encourages a holistic, theory-driven, modelling approach to analysis of real world problems. It is especially useful with non-experimental data and provides a unifying overview of many multivariate techniques. The emphasis on a theory-driven process and interpretation of a theoretically justified/empirically supported model involves students more fully in the entire research process. It is acknowledged that expertise in the technique involves many more complexities that have not been discussed in this paper. However, being exposed to some of the ideas and methodology at undergraduate level will help to give students an action-oriented view of how statistics is integral to many applied fields of study. The application of SEM to complex research questions will increase [4]. Although it has historically been considered an advanced technique, it may be beneficial for undergraduates to be shown its potential.

## References


1. J.L.Arbuckle, & W.Wothke, (1995). *Amos 4.0 User's Guide*, Chicago: Smallwaters Corporation.
2. G.E.P. Box (1979). *Robustness in the strategy of scientific model building*, In R. L. Launer & G. N. Wilkinson, (eds.), *Robustness in Statistics*. New York: Academic Press.
3. B.M.Byrne, (2001). Structural Equation Modeling with AMOS : Basic Concepts, Applications and Programming, NJ: Lawrence Erlbaum Associates.
4. L.F.Dilalla, (2000). *Structural Equation Modeling: Uses and Issues*. In H.E.A.Tinsley & S.D.Brown (Eds.), Handbook of Applied Multivariate Statistics and Mathematical Modeling (pp. 440 – 462), California: Academic Press.
5. B.S.Everitt, & G.Dunn, (2001). *Applied Multivariate Analysis (2nd Ed.)*, London: Arnold.
6. J.F.Hair, R.E.Anderson, R.L.Tatham, & W.C.Black, (1998). *Multivariate Data Analysis (5th Ed.)*, NJ: Prentice-Hall.
7. S.l.hershberger, *the growth of structural equation modeling 1994 – 2001*, structural equation modeling, 10(1), 35 – 46.
8. G.M.Maruyama, (1998). *Basics of Structural Equation Modeling*, California: Sage Publications.
9. L.M.Wolfle, (2003). *The Introduction of Path Analysis to the Social Sciences, and some Emergent Themes: An Annotated Bibliography*, Structural Equation Modeling, 10(1), 1-34.



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