THE SEVENTH SOUTHERN RIGHT DELTA (ΣΡΔ '09)
CONFERENCE ON THE TEACHING AND LEARNING OF
UNDERGRADUATE MATHEMATICS AND STATISTICS

PROCEEDINGS
of Gordon's Bay Delta '09

Villa Via Hotel, Gordon's Bay, South Africa
30 November 2009 - 4 December 2009

Edited by Dirk Wessels

Mathematics in a dynamic environment

7th SOUTHERN RIGHT DELTA (ΣΡΔ’09) CONFERENCE

on the

TEACHING AND LEARNING OF UNDERGRADUATE MATHEMATICS AND STATISTICS

Mathematics in a dynamic environment

Proceedings of Gordon’s Bay Delta ‘09
30 November – 4 December 2009

VILLA VIA HOTEL, GORDON’S BAY
South Africa

Editor: Dirk Wessels

Published by

Dirk Wessels
Christelle Snyman

On behalf of the International Delta Steering Committee

November, 2009.

Proceedings of 7Th Southern Hemisphere Conference on Mathematics and Statistics Teaching and learning (Gordon's Bay DELTA '09)

Editor: Dirk Wessels

Printed in Stellenbosch, South Africa.

Foreword

Preface

The Southern Right Delta'09 conference is jointly presented by the International Delta Committee and the SAMF (South African Mathematics Foundation). The conference is the seventh in a series of conferences on the undergraduate teaching and learning of mathematics and statistics, as part of an informal collaboration between Southern Hemisphere countries.

The first conference Delta'97 (delta implying change) took place in Brisbane, Australia in November 1997. The second conference Delta'99 took place in Laguna Quays, also in Queensland, Australia. Delta'01 was in Berg-en-Dal in the Kruger National Park in South Africa followed by Delta'03 in Queenstown, New Zealand. Two years later, Delta'05 took place on Fraser Island on the Queensland coast in Australia after which we moved to South America with Delta'07 in El Calafate, Argentina.

The informal collaboration between Southern Hemisphere countries, at the moment including Australia, New Zealand, South American countries and South Africa, has grown into the formation of the International Delta Committee, with current members: Pat Cretchley and Christina Varsavsky (Australia), Derek Holton, Ivan Reilly and Greg Oates (New Zealand), Victor Martinez-Luaces (Uruguay) and Ansie Harding and Johann Engelbrecht (South Africa). The Delta conferences take place bi-annually and the next conference, Delta'11, is planned for Rotorua, New Zealand in November 2011.

The theme for Delta'09 is "Mathematics in a dynamic environment." The 150 delegates for this conference, representing 28 different countries worldwide, are responsible for more than 90 contributed presentations, workshops and round table discussions. The conference has two publications: a special issue of the International Journal of Mathematics Education in Science and Technology, consisting of peer reviewed research papers, and the Proceedings of the Conference, also with peer reviewed research reports on teaching and learning of undergraduate mathematics and statistics.

We believe that the deliberations at this meeting will influence the course of future tertiary mathematics education worldwide. It is broadly accepted that skills and training in the quantitative sciences will be crucial to success in the future. Thus our belief is that this meeting will be one of the most important of 2009.

Thank you to Dirk Wessels who was the editor responsible for the Proceedings and Tracy Craig who lend a helping hand.

Johann Engelbrecht

National Chairperson
Southern Right Delta'09

Gordon's Bay Delta '09

Local Organising Team

Jurie Conradie
Kosie Smit
John Frith
John Webb
Roland Fray
Tracy Craig (chair)
Vera Frith
Dirk Wessels
Johann Engelbrecht

Program Committee

Greg Oates
Johann Engelbrecht
Tracy Craig
Dirk Wessels
Christina Varsavsky
Victor Martinez-Luaces

International Steering Committee

Derek Holton (NZ)
Ivan Reilly (NZ)
Christina Varsavsky (Australia)
Patricia Cretchley (Australia)
Leigh Wood (Australia)
Ken Houston (UK)
Debora Hughes-Hallet (USA)
Victor Martinez-Luaces (Uruguay)
Ansie Harding (SA)
Colette Laborde (France)
Harvey Keynes (USA)
Joel Hillel (Canada)
Maria Ines Cavallaro (Argentina)
Marta Anaya (Argentina)
Johann Engelbrecht (SA)

National Committee

Jill Adler
Maritz Snyders
Steve Joubert
Michael Sears
Michael de Villiers
Sudan Hansraj
Eugen Straeuli
Jan Persens
Belinda Huntley
Cyril Julie
Johann Engelbrecht

Referees for the Proceedings and for the iJMEST Journal

Acosta	Martin
Amit	Batya
Barry	Mike
Bartholomew	Hannah
Barton	Bill
Berger	Margot
Bergsten	Christer
Blyth	Bill
Britton	Sandra
Bruyns	Jaco
Clark	Kathy
Craig	Tracy
Cretchley	Patricia
Cuthbert	Ross
D'Arcy-Warmington	Anne
Dreyfus	Tommy
Du Preez	Jeanette
Easdown	David
Edwards	Barbara
El Yacoubi	Nouzha
Engelbrecht	Johann
Fay	Temple
Freislich	Mary Ruth
Hannah	John
Harding	Ansie
Harris	Gary
Harrison	Martin
Henderson	Jenny
Holton	Derek
Hubbard	Ruth
Huntley	Belinda
Ivers	David
Joubert	Steve
Kagestan	Owe
Karakok	Gulden
Kawski	Matthias
Keyens	Harvey
Klymchuk	Sergiy
Laborde	Colette
Lock	Corrie
Martinez-Luaces	Victor
Matthews	Kelly
McCleod	Kevin
Meaney	Tamsin
Mofolo-Mbokane	Batseba
Montelle	Clemency
Muller	Eric
Oates	Greg
Oikkonen	Juha
Parnell	Sheena
Paterson	Judy
Phiri	Patrick
Pierce	Robyn
PonceCampuzano	Juan Carlos
Reilly	Ivan
Rodd	Melissa
Sangwin	Chris
Schmidt	Karsten
Smith	David
Sneddon	Jamie
Statham	Moira
Stewart	Sepideh
Steyn	Tobia
Stols	Gerrit
Tall	David
Thomas	Mike
Van Staden	Paul
Varsavsky	Cristina
Wall	David
Watson	Peter
Wessels	Dirk
Wood	Leigh
Yushau	Balarabe

Table of contents

Refereed Papers with Abstracts

Sponsers

Organised by: SAMF and the
International Delta Steering Committee

Manipulation of virtual objects for the development of connections between geometry and probability as well as between the various dimensions of space

MARIO BARRA*

Dipartimento di Matematica,
Università di Roma "La Sapienza"

Abstract

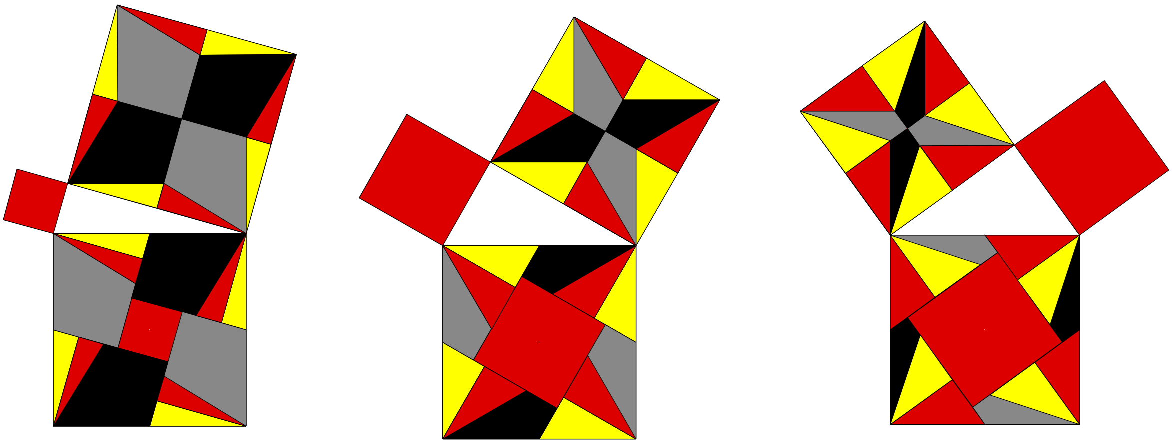
In this paper we present some examples in order to show how didactic, new technological materials, visualizations, induction, analogy, the choice of the subjects and some links among different mathematical sectors can motivate and facilitate the understanding and the development of students' ability in exposition and in argumentation. For the same aims can be very useful a presentation with many pictures, free from pain and worry, pleasant, coloured, dynamic, able to give the possibility to understand and like what is beautiful and allowing an easy reproduction in students' minds, and finally the possibility of personal discovering **[1]**. Briefly we will tray to show how this "participating and motivating environment" can improve mathematical learning and teacher's practice **[2]**. Starting from examples we hope it will be possible a better understanding of the general aspects, inductively **[3]**. Very important it's the role of the geometry and of the *fusionism* **[4; 5]**, and, in particular, the connections between probability and geometry, discrete and continuous, as well as those between the various dimensions of space **[6; 7]**.

Keywords: Infinitesimal method, transformations (area, affine, non-linear), cartographic representations.

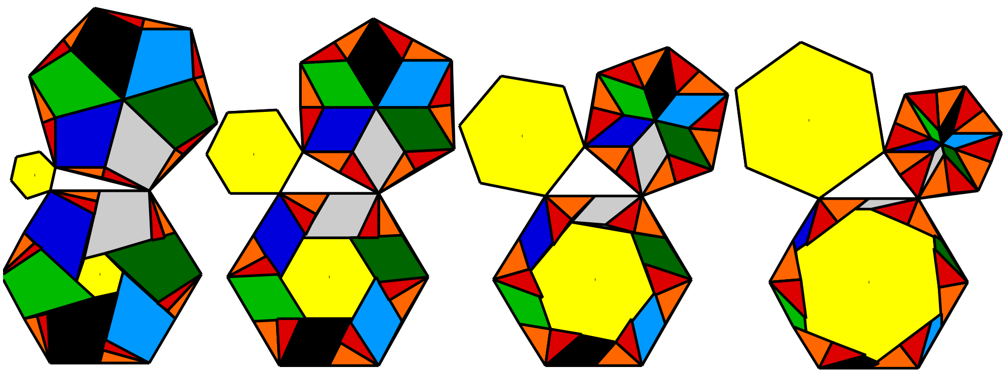
* E-mail: barra@mat.uniroma1.it

1. Pythagoras' Theorem

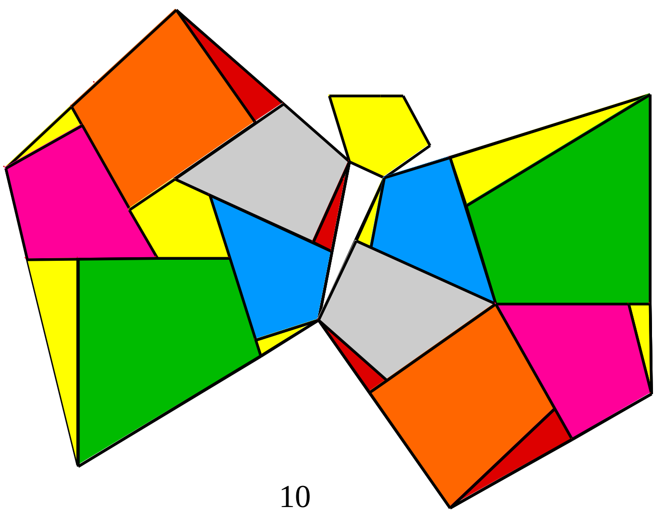
First, we present some puzzles to generalize the Pythagorean theorem, using dissection, starting from regular polygons to more general polygons inscribed or circumscribed in a circle. All the constructions are more or less the same.



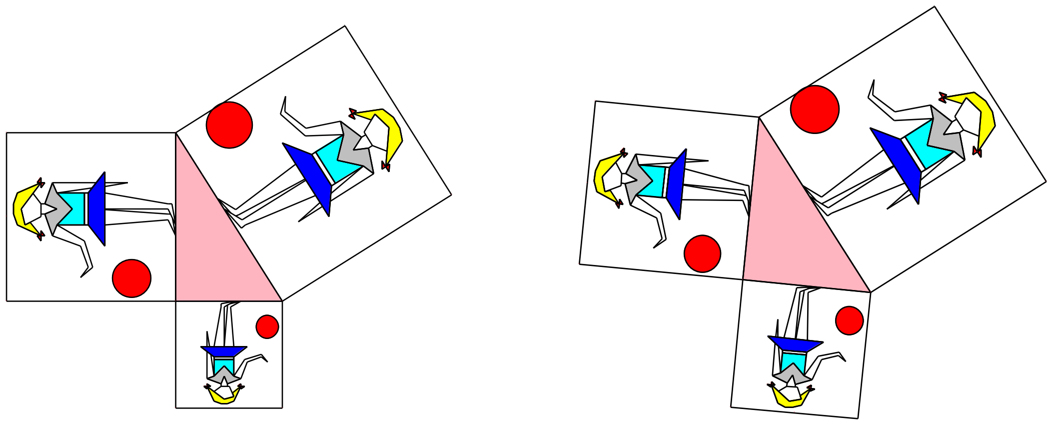
A dynamic geometry software (DGS) is a decisive factor to fascinate students and to show the generality of the construction.



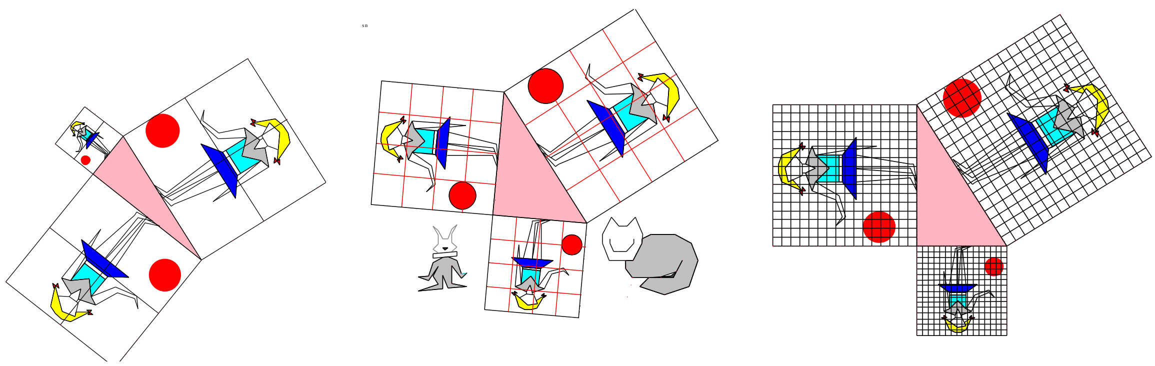
A dynamic example of the general method:



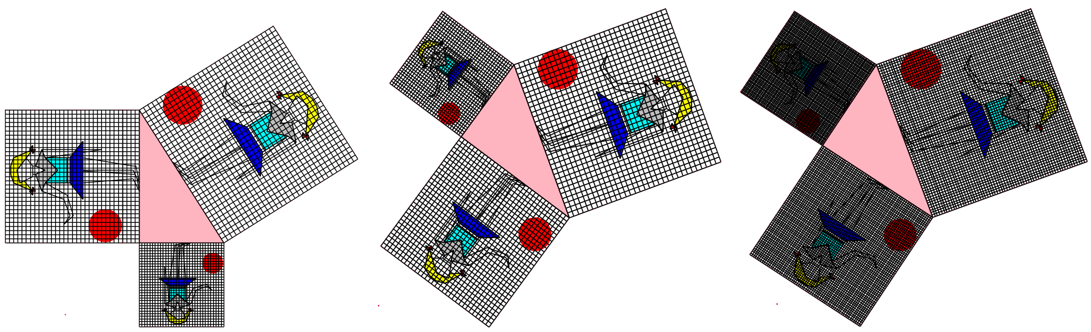
Next, we illustrate an infinitesimal method to prove Pythagoras' most general theorem on the plane. In a right-angled triangle, 3 similar general figures are built on the sides, changing relative to the side lengths



Dividing each side of the squares in two parts, and the squares into four, the areas of the littler squares together give the area of the biggest one.



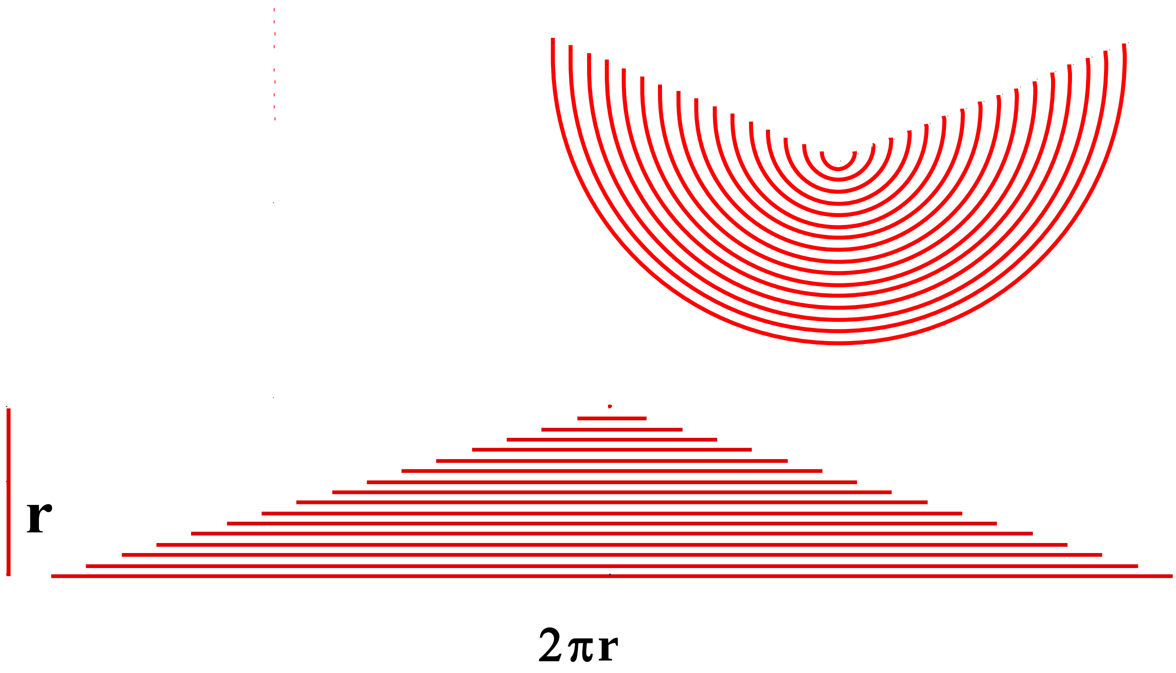
Similarly, dividing each square into multiple small squares way, we can show that the Pythagorean property is always valid for each little squares, and it is possible, for instance, to approximate. with the balls with squares becoming littler, that the sum of the areas of the two little figures gives the area of the biggest one.



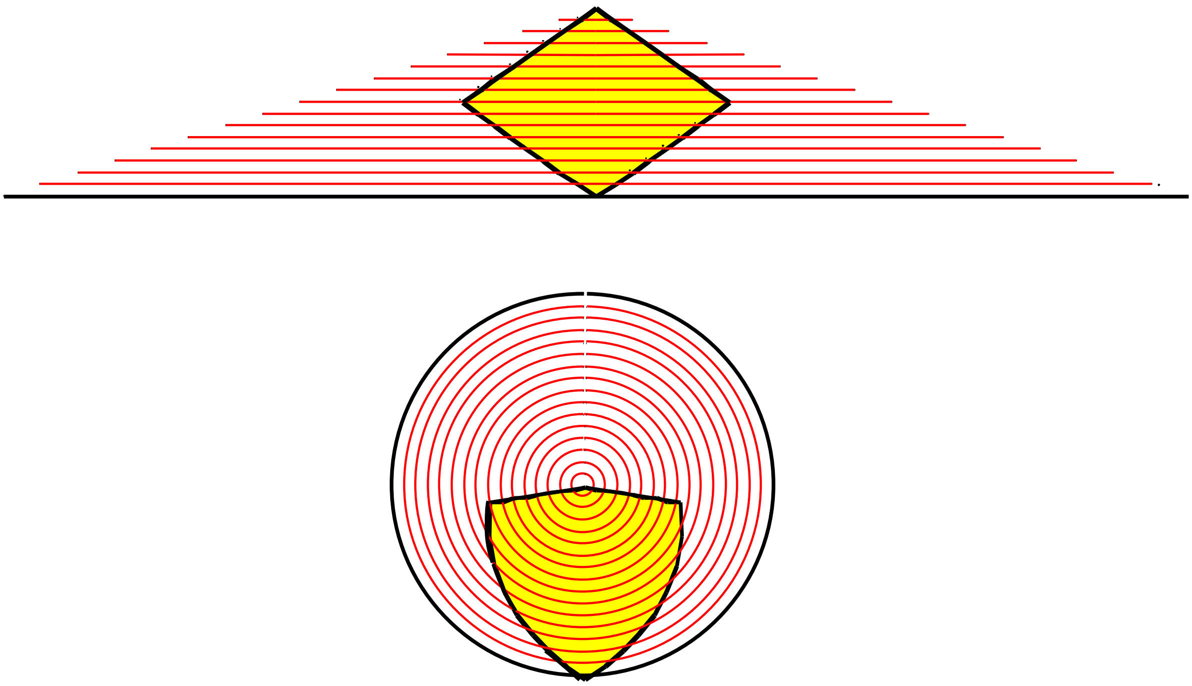
2. Transformations of area

Similar dynamic effects can illustrate transformations of the same area [8] defined by different curves – the area remains unchanged.

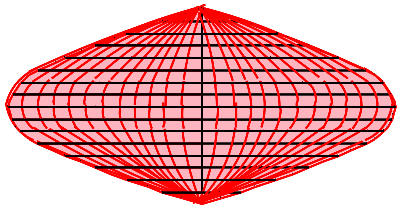
1) "opening" a circle into a triangle (discovering the formula for circle's area)

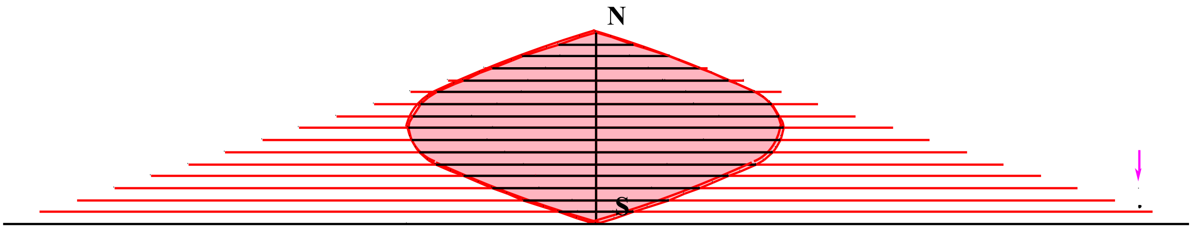


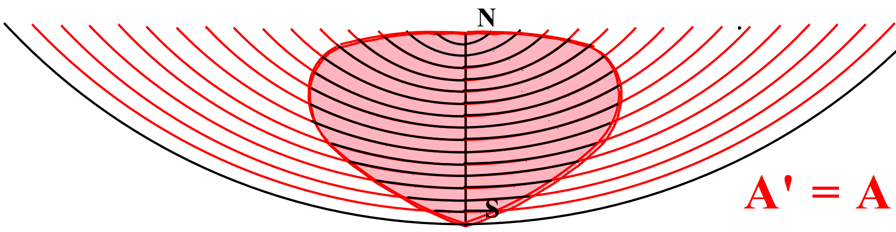
2) "closing" a triangle into a circle (discovering the area of new objects):

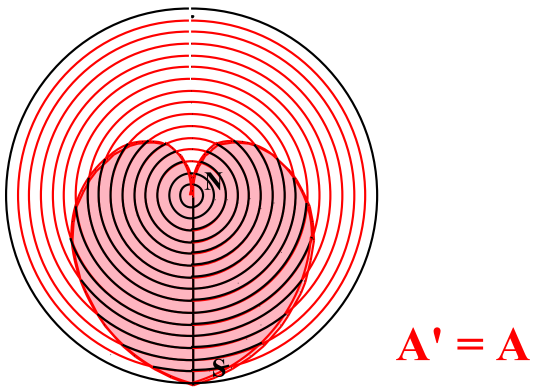


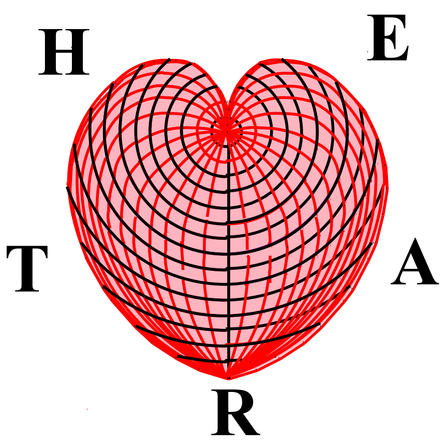
3) considering two cartographic representations (useful for illustrating multiple properties):







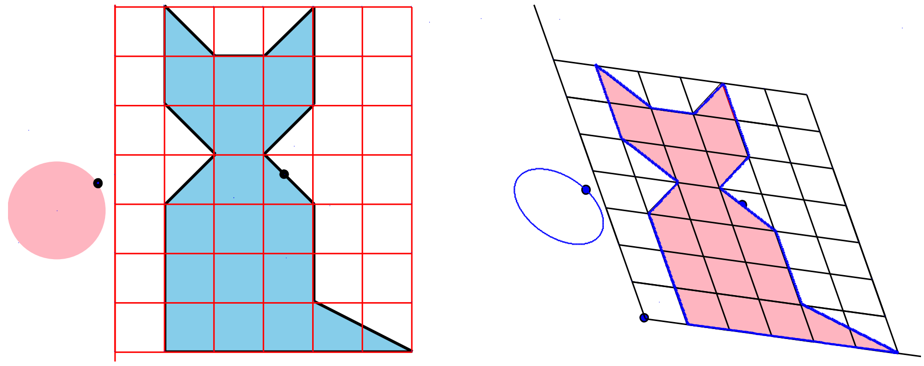


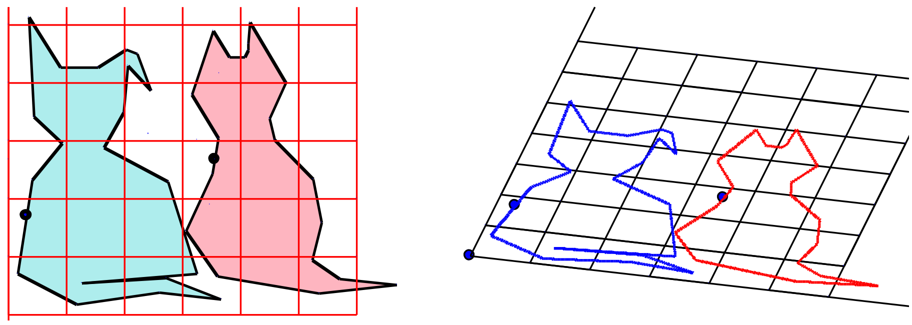


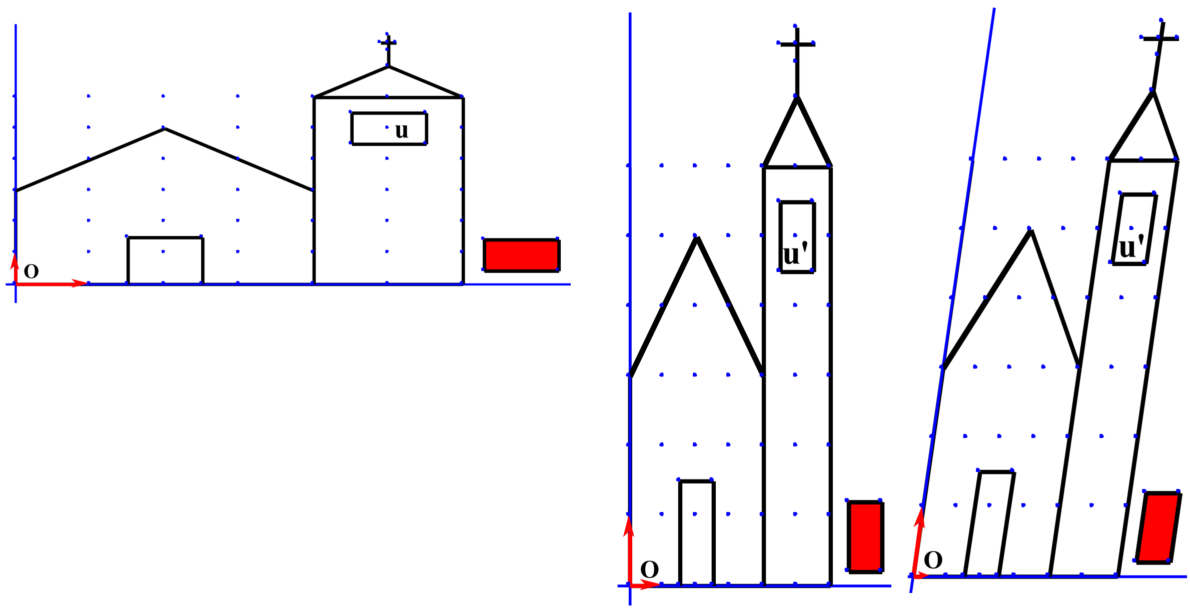
This process illustrates Werner's **cartographic representations** and the example shown above allows the reading of the words Hear and Earth.

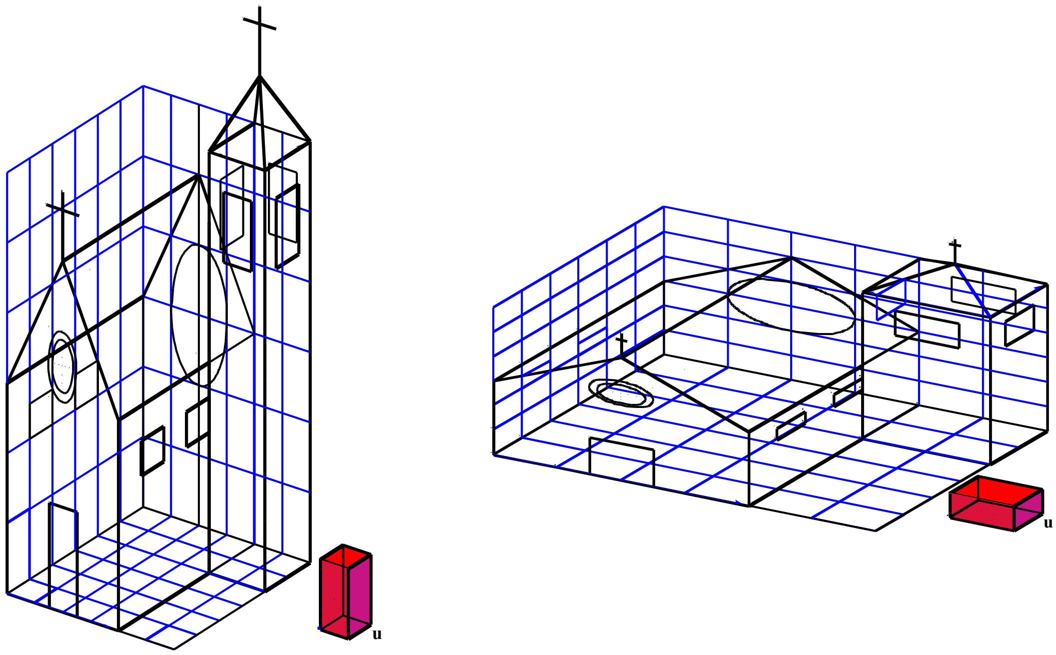
3. Affine transformations

Students can discover, with a DGS, the mathematical properties of the **affine transformation** of some objects, for example as illustrated below, animals and churches.

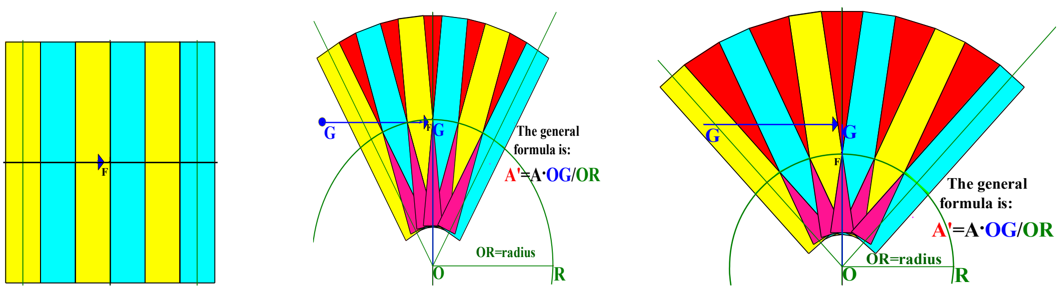




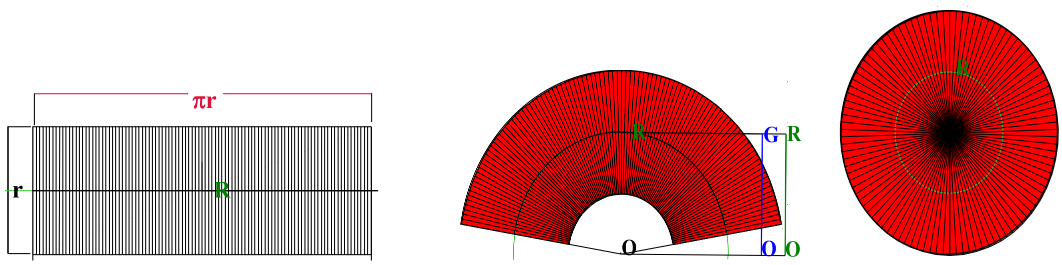




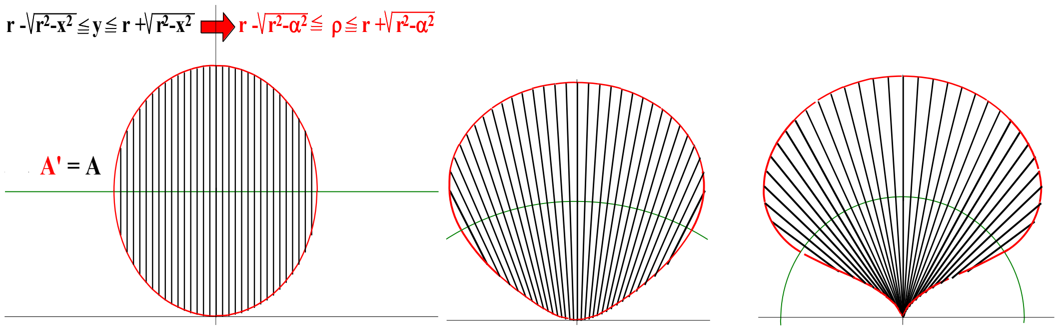
4. A new non-linear transformation and some of its properties. New curves [9].
If we divide a figure into small strips perpendicular to a straight line crossing through the centre of gravity of the figure, and if we transform this line into a circle so that the small strips remain perpendicular to this line, the **area is unchanged** because the surface which increases is equal to that which diminished through superposition.

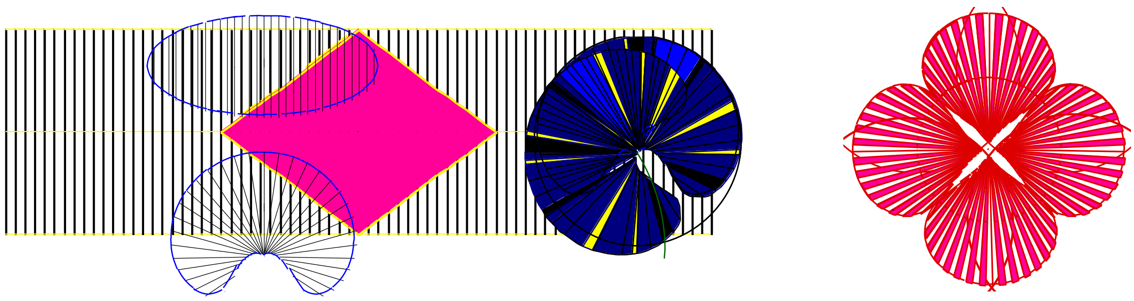


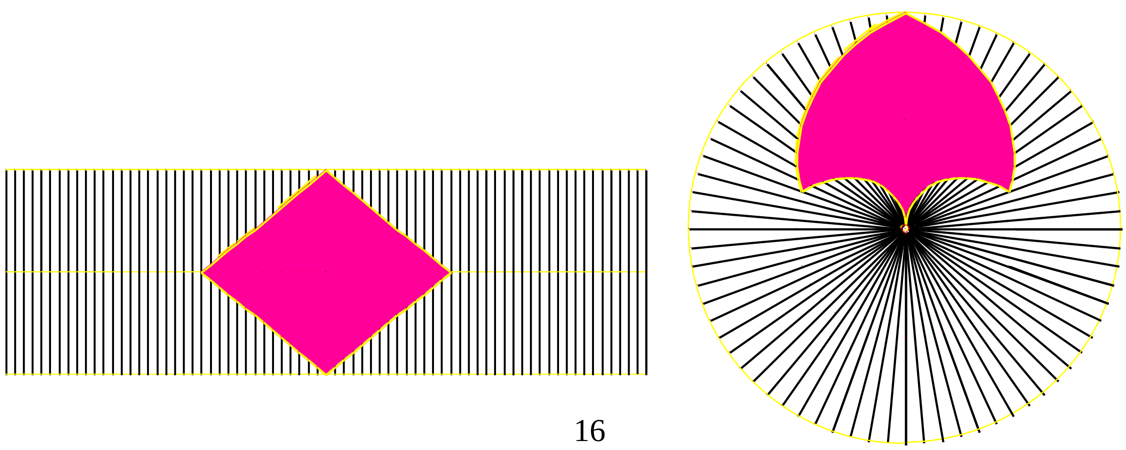
For example, the area of the rectangle in the next figure is πr^2 and the line passes through its centre of gravity. Since during the transformation there is no change in the area, we have a new way to determine the area of the circle. Moreover, we have also calculated the length of the circumference because the horizontal upper line, which initially has a length of πr, increases its length as the lower line decreases. Hence, at the end the length of this upper line doubles, thus becoming 2πr.



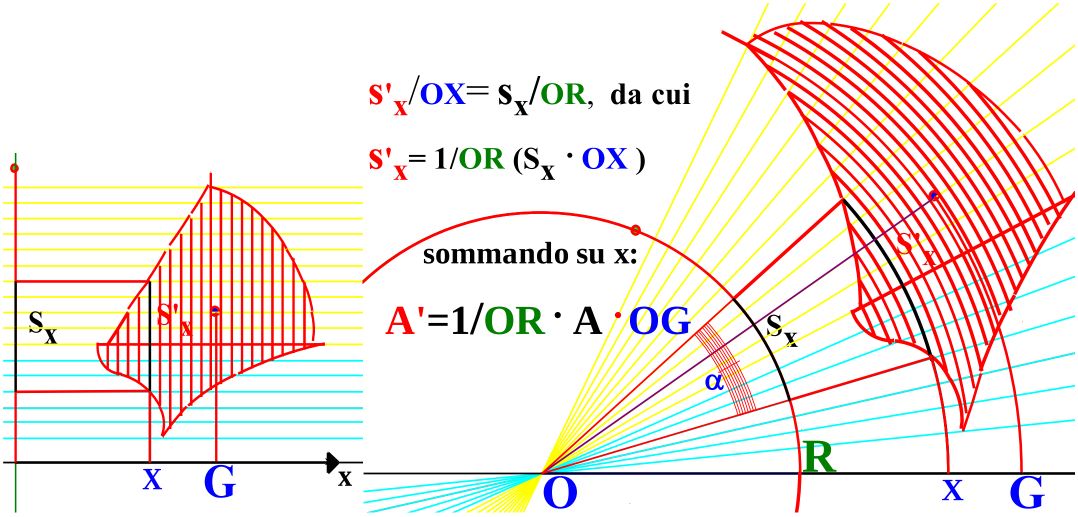
Similarly that applies to many cases to obtain the area of old or new figures.







Since the line, which will become circular, passes through the centre of gravity, the area remains unvaried throughout the transformation. Otherwise, the formula which links the area A of the initial surface with that A' of the transformed one, is as follows:
A' = A · OG/OR, as we will easily demonstrate shortly in a general theorem.

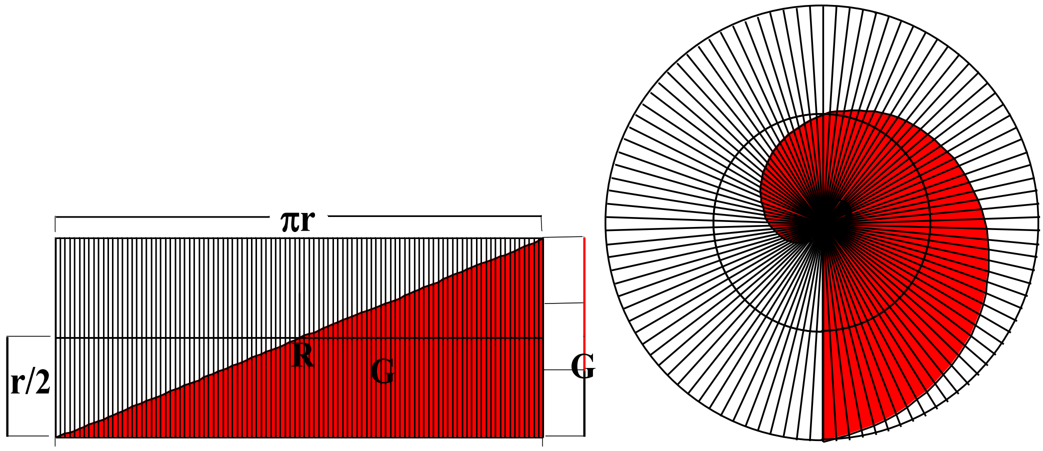


We divide a sailing boat into numerous vertical lines and take one, S_x, of these. We then project S_x on a line (in the left in the figure). When this line becomes a circle, all the vertical lines of the boat become circles, and

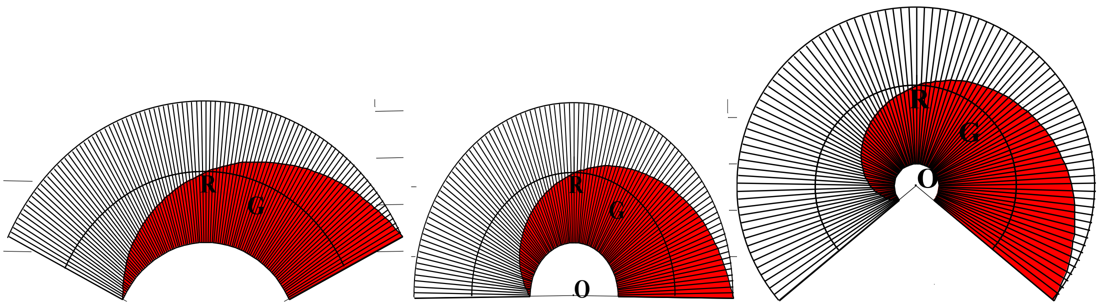
due to the similarity of the circular sections:	$S'_x/OX = s_x/OR$ that is $S'_x = 1/OR\,(S_x \cdot OX)$
Adding up all the segments, obtained by varying x (applying the Archimedes' property of a balance):	$A' = 1/OR \cdot A \cdot OG$

Let's look at two important applications of this new formula: **A' = A·OG/OR**

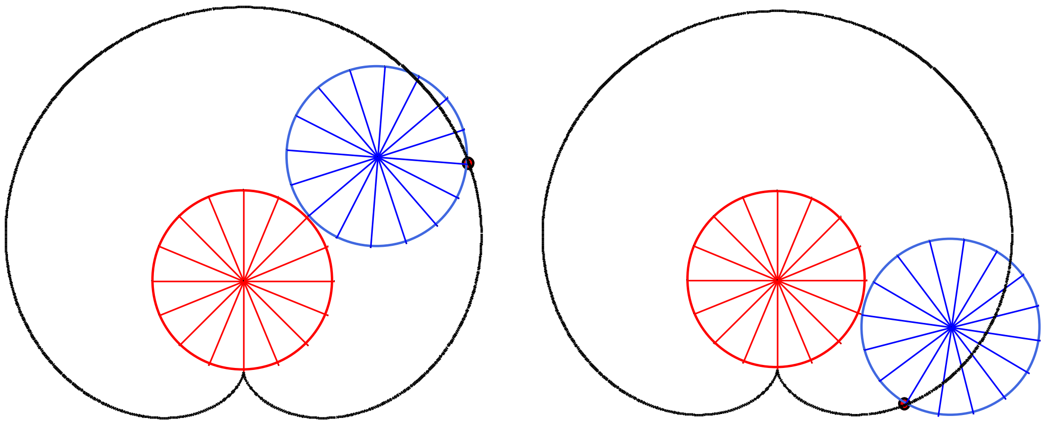
1) Archimedes' Spiral



As we saw before, the area of the rectangle is equal to that of a circle. The triangle has its centre of gravity at 1/3 of its height. Therefore, after our transformation which changes the triangle into the “Archimedes' Spiral”, this latter will have an area equal to 1/3 of the circle's. Naturally, knowing OG/OR, we can calculate the area of many other curves: hook-bills and fingernails...

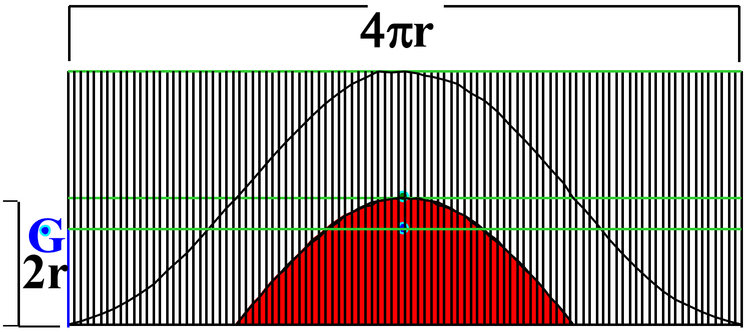


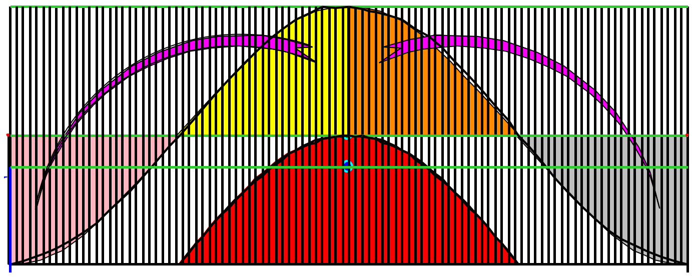
2) The **Cardioid** is a beautiful curve which we can see inside a cup of milk.



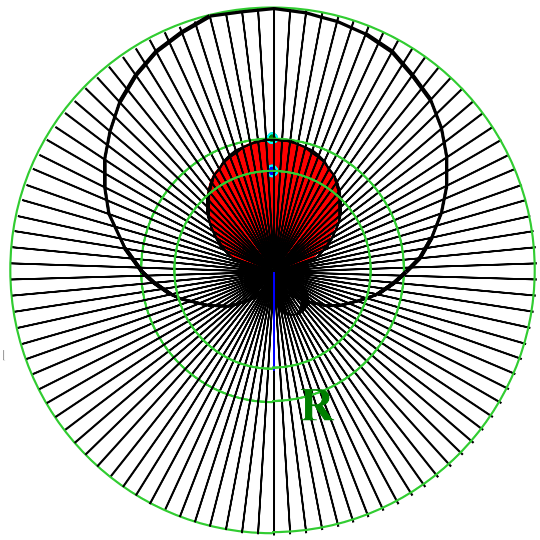
The previous one is one way to draw a Cardioid. Let's now try to determine its area. The next are the polar equation of a Cardioid and the corresponding cartesian equation, which represents a sinusoid shifted upwards.

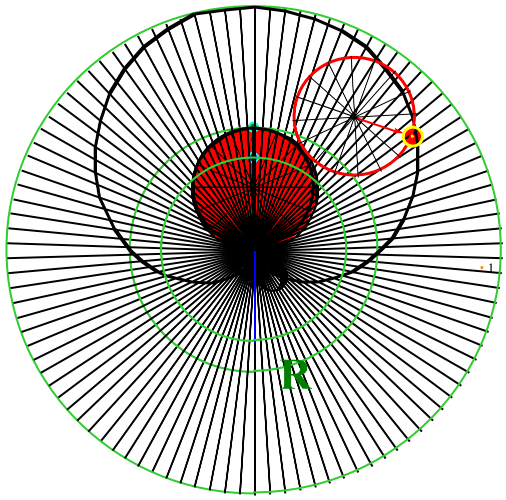
The area A of the second function is $8\pi r^2$ because the area of the sinusoid above the straight line is equivalent to that of a rectangle, and the height of its centre of gravity G is three-quarters of the height of the rectangle.





The cardioid is obtained after applying our transformation. We can see that it can be found through the construction we saw previously.





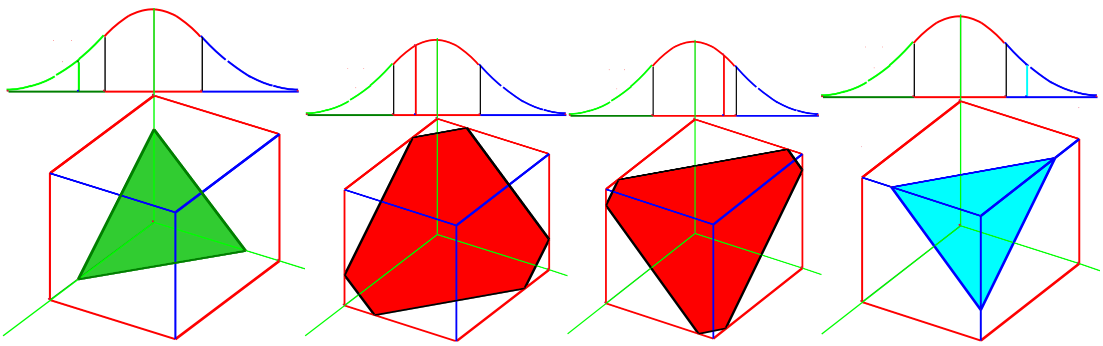
Finally, by applying the formula determined, we see that the area A' of the Cardioid is $8\pi r^2 3/4 = 6\pi r^2$, that is **six times** greater than the area of one of the two circles which generated it.

The densities of the sum of equal, uniform, independent, distributions, or the box-spline, and some of their properties [10], [11].

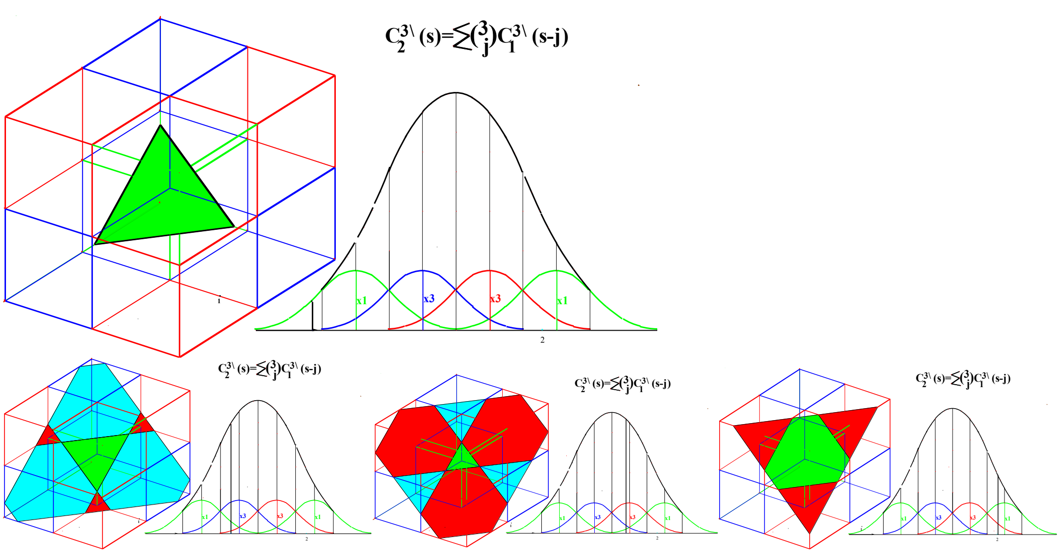
In a cube C^3_e (**e** is the length of an edge) a box-spline $C^{3\backslash}_e(s)$ expresses the area of the section with a plane that is perpendicular to one of the principal diagonals of C^3.

$C^{3\backslash}_e(s)$ is proportional to the **density $f_{3;e}(s)$ of the sum of three equal uniform distributions** in $0 \le s \le e$, and it is a first approximation of **Gauss**' "**Curve of errors**".

$$C^{3\backslash}_e(s) = (C^3_e = \{x = (x_1, x_2, x_3) : (x_i - e) \geq 0,\ i = 1, 2, 3\})(\cap \sum_{1}^{3} x_i = s) \cong f_{3;e}(s$$



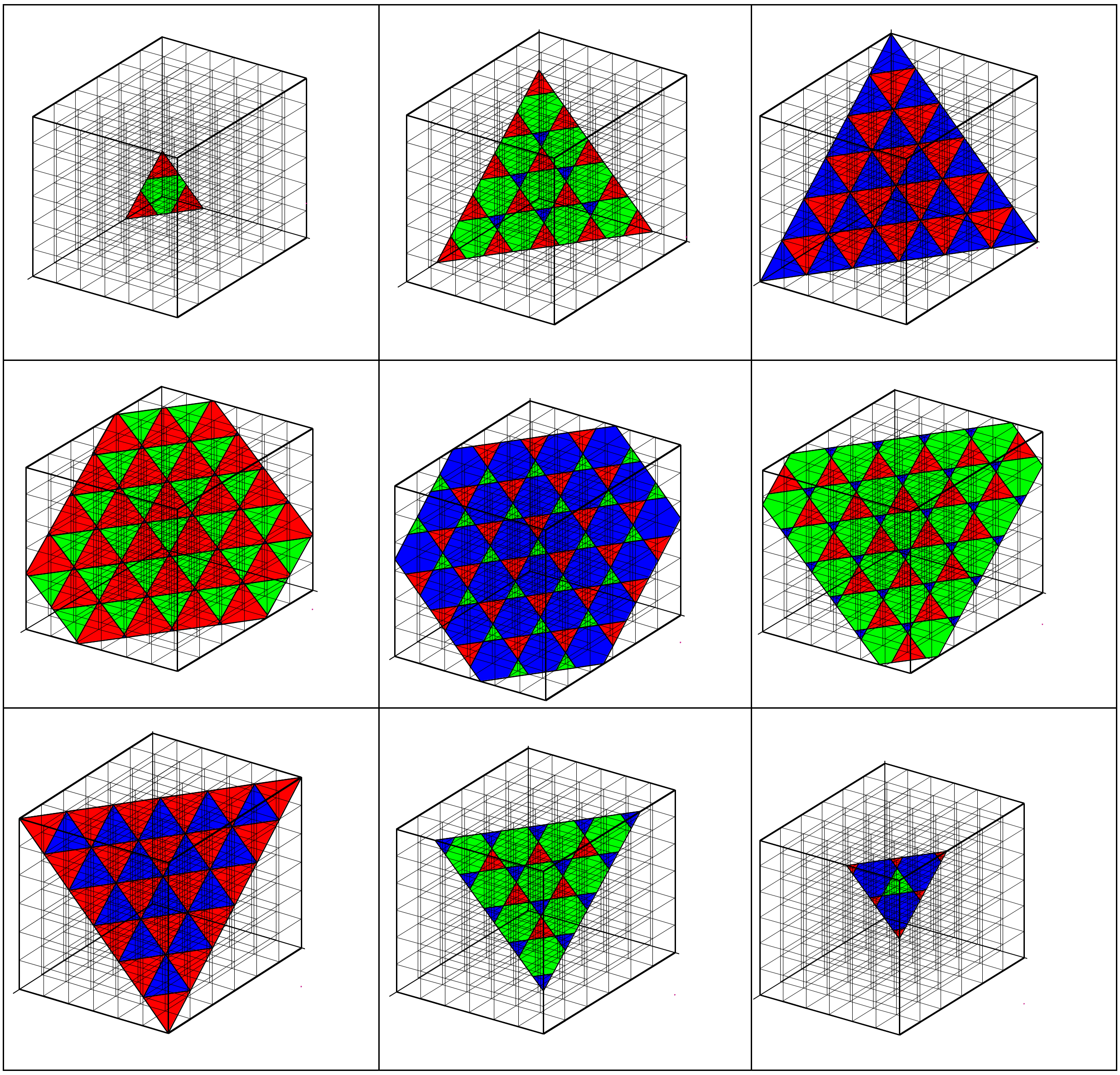
The connection with geometry makes it possible to identify many of the spline's properties. Some of these couldn't be known otherwise.



For instance, we can obtain our 'bell-shaped' curve, as the sum of many similar curves. It suffices to divide up a cube into smaller cubes. Here has been divided into 8 smaller cubes. Consider that the area of the sections of C^3 is equal to the sum of the areas of the sections of the 8 small cubes: first we have only one cube, near the origin of axes, then 3, then 3 again and finally yet another one. It can be seen that the "big" b-spline can be found by adding eight smaller ones. In similar ways we find that, adding b-splines, it is possible to obtain a constant, a straight line and a parabola.

If we divide up a cube into 216 smaller equal cubes we find: $\mathbf{C^{3\backslash}{}_6(s)} = \sum_0^{15}\binom{3}{j}_6 \mathbf{C^{3\backslash}{}_1(1;s-j)}$,

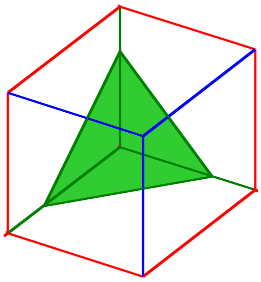
where $\binom{3}{s}_6$ is the number of ways to obtain **s** as the sum of 3 dice (each with the values 0, 1, 2,…5). So we can see that the density of an error can be the distribution of a "sum of errors".

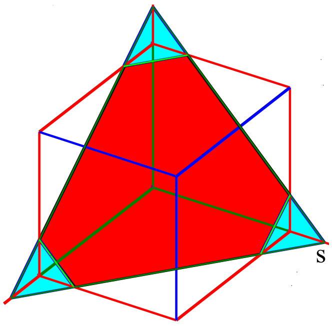


It's wonderful lto see the sections dynamically.
Mathematics is beauty. G. H. Hardy [12]

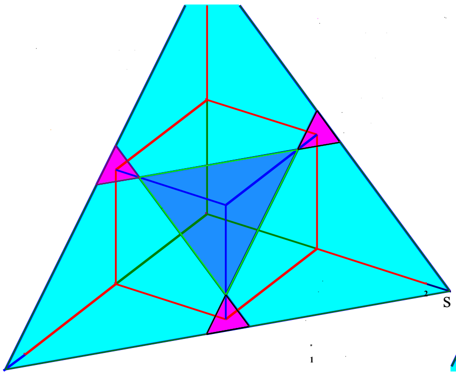
All these properties are valid in any given dimension and are also true in discrete space as shown in the distribution of the sum with dice.

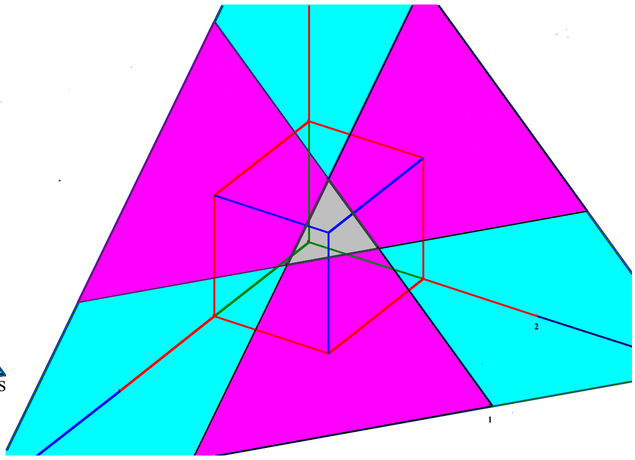
To find the area of the sections $C^{3\backslash}{}_e(s)=C^3{}_e \cap \sum_{1}^{3} x_i = s$, we start with a parabola, that is the area of the regular triangles $T^2(s)$, growing with s: $C^{3\backslash}{}_e(s)=T^2(s)$, $0 \leq s \leq e$). When $e \leq s \leq 2e$, $C^{3\backslash}{}_e(s)=T^2(s)-3T^2(s-e)$, where $T^2(s-e)$ is a littler triangle growing with s-e.





When: $2e \le s \le 3e$, $C^{3\backslash}{}_e(s)=T^2(s)-3T^2(s-e)+3T^2(s-2e)$,
$3e \le s$, $C^{3\backslash}{}_e(s)=T^2(s)-3T^2(s-e)+3T^2(s-2e)-T^2(s-3e)$.
Putting the area of the equilateral triangles: $T^2(s)=0$, $s \le 0$, we can write:





$0 \le s$, $C^{3\backslash}{}_e(s)=T^2(s)-3T^2(s-e)+3T^2(s-2e)-T^2(s-3e)=\nabla_e^3 T^2(s) \cong \mathbf{f_{3;e}(s)}$,

where: $\nabla_e T^2(s)= T^2(s)- T^2(s-e)$, and $\nabla_e^2 T^2(s)= T^2(s)- 2T^2(s-e)+ T^2(s-2e)$

The same in discrete space to have the numbers of ways to obtain **s** as the sum of 3 dice:

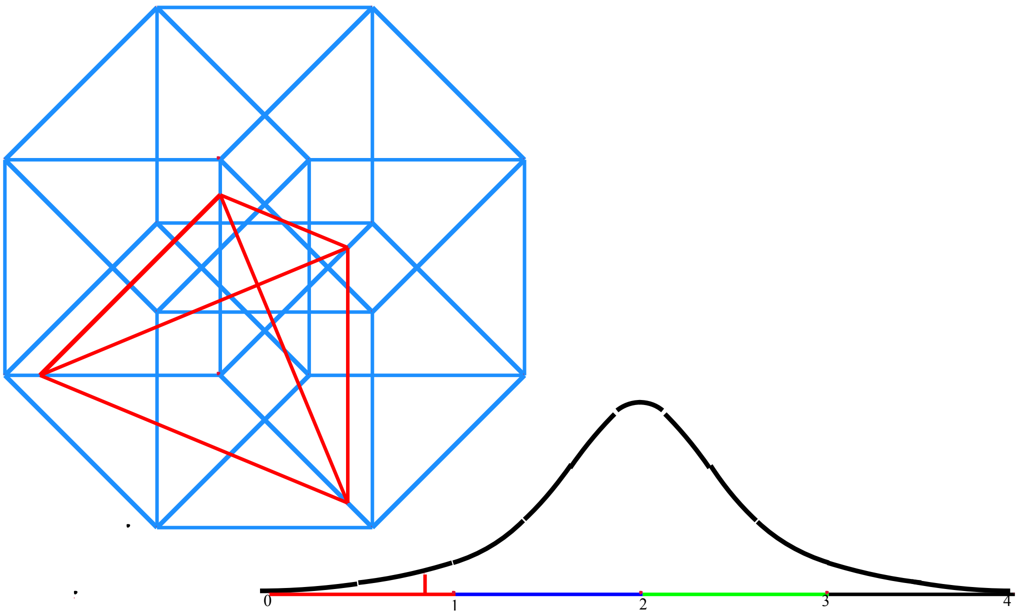
s =	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**…**
	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190	…
							- 3(1	3	6	10	15	21	28	36	45	55	66	78	91	…
													+3(1	3	6	10	15	21	28	…
																			-1 (1	…
$\binom{3}{s}_6$ =	**1**	**3**	**6**	**10**	**15**	**21**	**25**	**27**	**27**	**25**	**21**	**15**	**10**	**6**	**3**	**1**	**0**	**0**	**0**	**…**

The same in the continuous space in four dimensions, in a hypercube $C^4{}_e$:

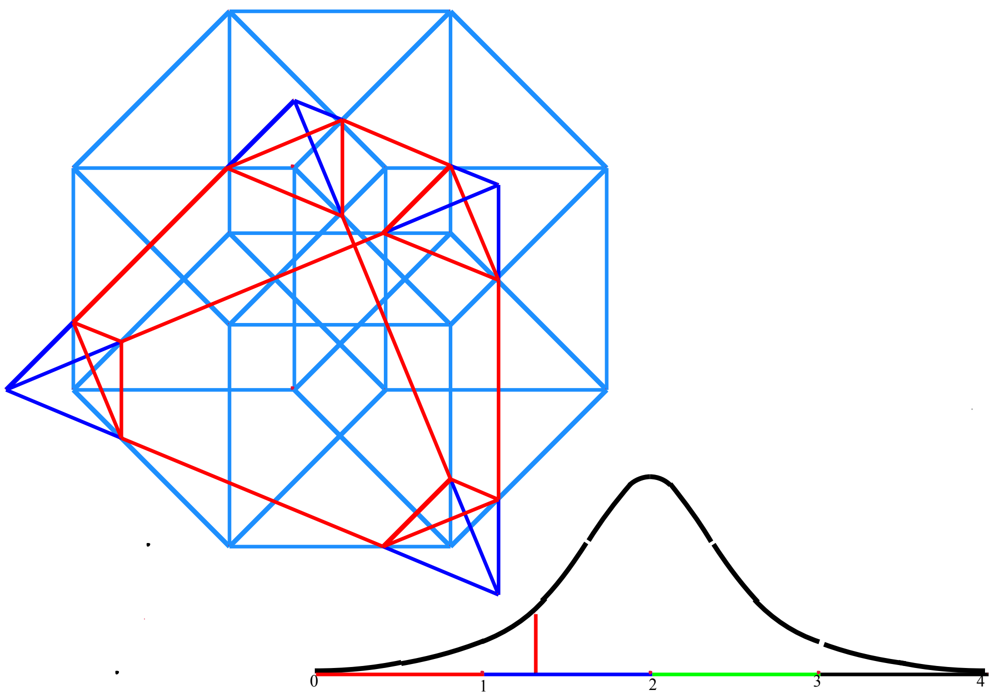
$$0 \le s,\ f_{4;e}(s) \cong \nabla_e^{4} T^3(s) = T^3(s) - 4T^3(s-e) + 6T^2(s-2e) - 4T^2(s-3e) + T^2(s-4e) = C^{4\backslash}_e(s)$$

where $T^3(s)$ is the volume of a regular tetrahedron, and $T^3(s)=0$, per $s\le 0$.

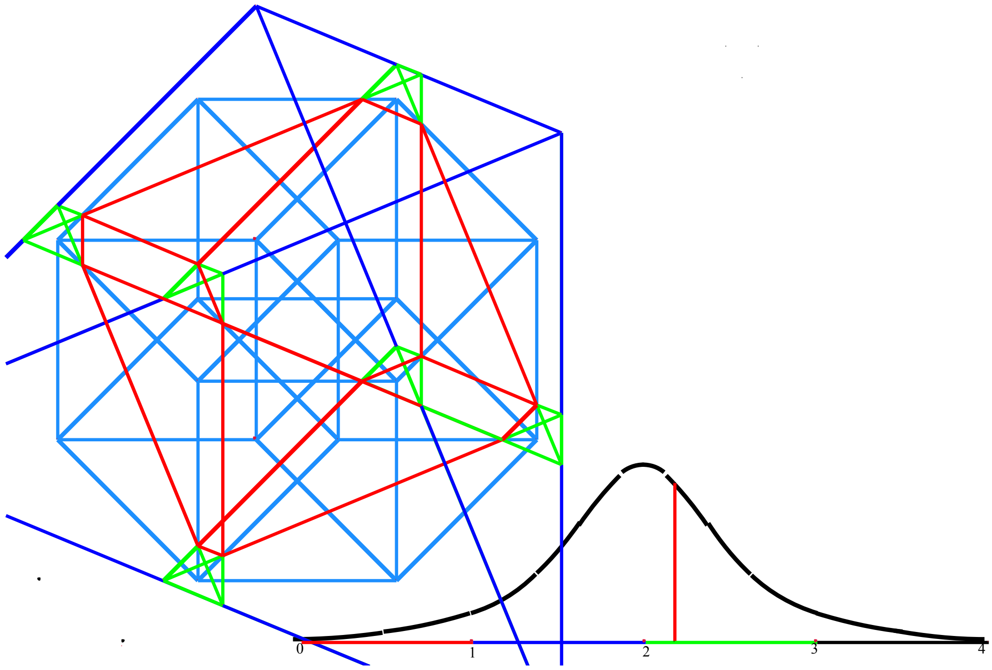
$0\le s\le e,\ C^{4\backslash}_e(s) = T^3(s)$



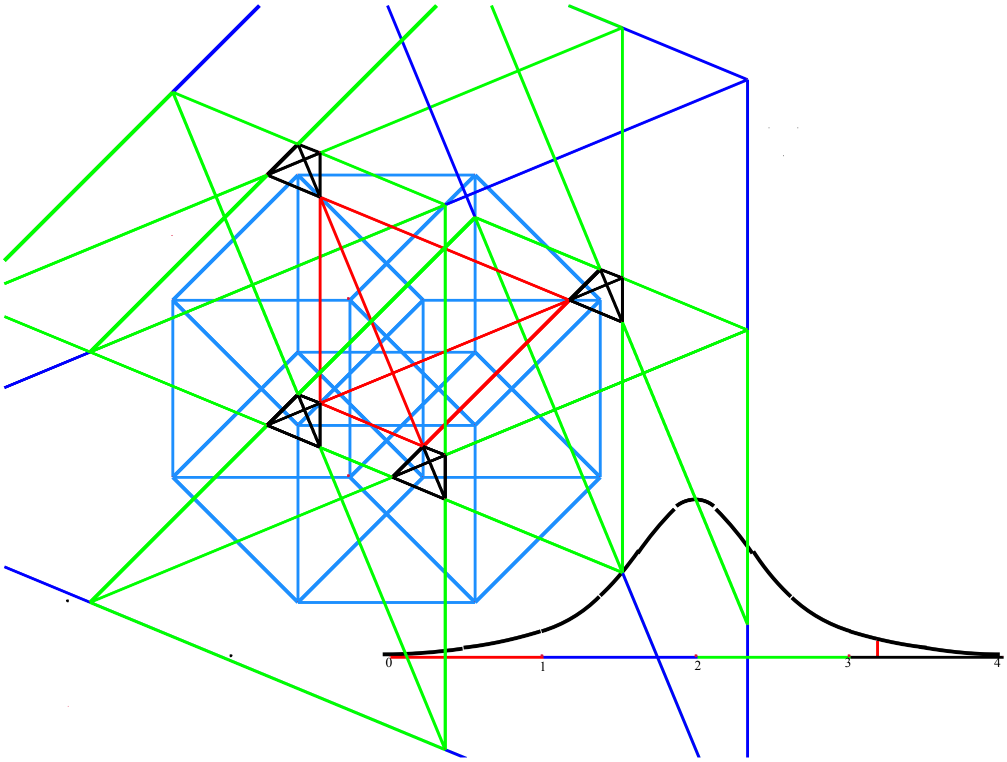
$e\le s\le 2e,\ C^{4\backslash}_e(s) = T^3(s) - 4T^3(s-e)$



$2e\le s\le 3e,\ C^{4\backslash}_e(s) = T^3(s) - 4T^3(s-e) + 6T^2(s-2e)$



$3e \le s \le 4e$, $C^{4\backslash}{}_{e}(s) = T^3(s) - 4T^3(s-e) + 6T^2(s-2e) - 4T^2(s-3e)$



It's very easy to prove in general [13] and in a geometric way, that:
when $0 \le s$, $\mathbf{f_{d;e}(s)} \cong \mathbf{C^{d\backslash}{}_{e}(s)} = \nabla_{e}^{d} \mathbf{T^{d-1}(s)}$

References

[1] Barra M., 1975, The cycloid - A didactic Experience - A new proof, *Educational Studies in Mathematics*, 6, pp. 93-98.

[2] Laborde C., 2000, Dynamic Geometry environments as a source of rich learning contexts for the complex activity of proving, *Educational Studies in Mathematics*, 44: (1-3), pp. 151-161.

[3] Polya G., 1954, *Induction and Analogy in Mathematics*, Vol. I of *Mathematics and plausibile reasoning*, Princeton University Press

[4] De Finetti B., 1973, Le fusionnisme aujourd'hui. In: *Atti del III Convegno CIEM*, Echternach, 4-9 giugno, pp. 91-109.

[5] Kleine F., 2004 (1939), *Elementary mathematics from an advanced standpoint*, Geometry, Dover Publications.

[6] Barra M., 1994, Being able to see in d-Dimentional Spaces, Proceedings of the *First Italian-Spanisch Research Symposium in Mathematics Education*, Modena 15-19 febbraio, pp. 21-32.

[7] Barra M., 1995, Random images on mental images, nel libro "*Exploiting Mental Imagery with Computers in Mathematics Education*", Edited by R. Sutherland, J. Mason, Series F: Computer and Systems Sciences, Vol. 138, Springer in cooperation with NATO Scientific Affairs Division, pp. 26-277.

[8] Castelnuovo E., Barra M., 2000 (1976), *Matematica nella realtà*, Boringhieri, Torino, Trad. *La mathématique dans la réalité*, CEDIC Ed., 1980, and Nathan Ed., 1986.

[9] Barra M., 2004, Una nuova trasformazione non lineare resa possibile dalle proprietà dinamiche di Cabri, gli avvolgimenti radiali. Baricentro di una sinsoide, area del cerchio, della spirale di Archimede, della Cardioide e di altre curve, determinati in modo nuovo. Fiori matematici, Atti delle Plenary Conference di "CabriWord 2004", Roma 9 - 12 settembre, *Progetto Alice*, N.14, Vol. 5, Ed. Pagine, pp. 305-330.

[10] Barra M., 2005, Importance of the use of didactic, new technological materials in mathematics, *in Atti del Convegno CIEAEM 57: Changes in Society: a Challenge for Mathematics Education*, http://math.unipa.it/~grim/cieaem/cieaem57_barra.pdf

[11] Barra M., 2003, The importance of the use of didactic, new technological materials in mathematics classrooms and how this "participating environment" can improve mathematical learning and teacher's practice, *Acts of the 20th Panhellenic Conference on Mathematics Education, organized from the Hellenic Mathematical Society*, November 7 - 9, 2003, pp. 649-659.

[12] Hardy, G. H., 1992, *A Mathematician's Apology*, Cambridge University Press.

[13] Barra M., 2001, Ipersolidi e altri argomenti, *Progetto Alice*, N. 5, Vol. 2, Ed. Pagine, pp. 191-246.

Activities with CAS: Construction, transformation and interpretation

MARGOT BERGER†

School of Education, University of Witwatersrand, Johannesburg, South Africa.

Abstract: Mathematics tasks with a Computer Algebra System (CAS) usually involve several key activities: the construction of an initial sign, transformation of this sign or of the CAS output into another sign, and interpretation of the CAS output. Using these categories of activity, I examine the written answers to a CAS−based mathematical task of a heterogenous group of first−year university mathematics students. This analysis shows that although more than three−quarters of the class were able to construct an adequate initial sign, more than half the class did not appropriately interpret the CAS output in terms of the demands of the task.

Keywords: Computer algebra systems; tasks; semiotics; first-year university mathematics students

1. Background

In the last decade or so, mathematics educators have become increasingly aware that transforming CAS into a tool for learning mathematics is a complex process [1]; that the successful use of a CAS requires new and different skills to that of paper and pencil mathematics [1], [2]; that its use may enable new forms of learning [3] and that its use may stimulate whole−class discussion [4].

In this paper, I examine the types of activities in which a large heterogeneous group of first−year university mathematics students engage whilst using the CAS, *Mathematica*. My concern is with the extent to which the different students are able to successfully partake in certain CAS−based activities.

2. Theoretical framework

I regard mathematics as a semiotic system, that is, as a system of signs. According to Ernest [5] mathematics consists of three components: a set of signs which may be written or uttered or encoded electronically, a set of rules for sign production and a "set of relationships between signs and their meanings embodied in an underlying meaning structure" [5, p. 70].

Within this semiotic framework, I use Peirce's notion of mathematical reasoning to deconstruct a CAS−based task into its major components. C.S. Peirce (1839−1914), an American mathematician and founding father of semiotics, proposed that all thinking is performed upon signs of some kind or other, imagined or perceived. Peirce regarded signs not only as a means of signifying or referring to an object; rather they were "means of thought, of understanding, of reasoning and of learning" [6, p. 45]. Peirce argued that all deduction and mathematical reasoning involves the **construction** of an appropriate sign or diagram, **experimentation** on this set of signs through manipulations and transformations of signs (written, spoken or imagined), and observation (which of necessity includes **interpretation**) of the transformed set of signs.

† E-mail: Margot.Berger@wits.ac.za

Peirce's categorization of mathematical reasoning is particularly useful for isolating key components of mathematical tasks and examining their implications for teaching and learning. In the context of a CAS–based task, these activities assume a very particular form and function.

Construction – Transformation – Interpretation

Broadly speaking most uses of CAS for problem–solving involve the **construction** of one or more CAS–based signs for the mathematical object or operation of interest and **interpretation** of the CAS output. This interpretation may result in the **transformation** of the CAS output (the CAS sign) into further signs.

I briefly explain here what these activities entail. Note: All CAS examples that I offer are generated using *Mathematica*.

Construction of sign

A student engaging in CAS–based activities will need to construct a suitable mathematical sign on CAS. This may be a representation of the mathematical object (eg a graph or the definition of a function) and/or it may be an operation (for example, Solve [*f*[*x*]==*g*[*x*], *x*]).

Construction of an appropriate sign may be problematic in that different commands and constructions require a specific syntax; this may be an impediment to the effective use of the CAS [7].

Also, certain types of mathematical knowledge are necessary for certain constructions. For example, if the student wishes to plot arcsine, she needs to know that the domain of arcsine is [–1, 1].

The process whereby a student chooses an appropriate domain in the CAS environment differs from the process in the paper and pencil environment [1]. In the paper and pencil environment, drawing an unfamiliar function by hand requires a prior analysis of the function. That is, the student first needs to use algebra to find key features of the graph such as turning points, points of inflection, asymptotes, intercepts, if they exist. From this the student deduces an appropriate domain. With graphics software, prior algebraic work is not essential: students can use greater or lesser knowledge about the functions to presume a roughly useful domain and then use trial and error to get an appropriate domain. The problem is that key features (e.g. turning points) of the graph are often missed in this experiential environment.

Transformation of sign

Mathematical tasks largely involve the transformation of mathematical signs. With CAS the user may need to use the CAS output to generate another sign. For example, consider a beginner–calculus learner who uses CAS to find (symbolically) the derivative of $f(x) = 3x^2$. In order for that student to appreciate the relationship between the derivative of the function and the original function, the student may be required to transform the derivative, $f'(x)$ and the original function $f(x)$ into their graphical representations (on the same system of axes). This transformational activity itself entails the construction of a representation.

A different aspect of the activity of transforming signs involves a change in epistemic status of certain transformational activities when using CAS. That is, as a result of CAS' role as a tool for outsourcing processing power [8], various mathematical activities which have a high epistemic value in the paper and pencil environment have a different epistemic value in the CAS environment. Such mathematical activities may require knowledge of appropriate CAS commands (construction of representation) rather than knowledge of the rules of mathematics. For example to solve the equation: $\boldsymbol{x}^5 - 10\boldsymbol{x}^3 + 9\boldsymbol{x} = -4\boldsymbol{x}^4 + 40\boldsymbol{x}^2 - 36$ in *Mathematica*, one needs to input: Solve [$\boldsymbol{x}^5 - 10\boldsymbol{x}^3 + 9\boldsymbol{x} == -4\boldsymbol{x}^4 + 40\boldsymbol{x}^2 - 36$, *x*]. The computer will output: {{*x*→-4},{*x*→-3},{*x*→-1},{*x*→1},{*x*→3}}. The user then needs to interpret this as $x = -4$ or $x = -3$ or $x = -1$ or $x = 1$ or $x = 3$. Clearly the epistemic value of hand–solving an equation (in this case, an application of Factor Theorem to $\boldsymbol{x}^5 - 10\boldsymbol{x}^3 + 9\boldsymbol{x} + 4\boldsymbol{x}^4 - 40\boldsymbol{x}^2 + 36 = 0$) is lost. But the learner is able to use the results of her CAS' transformations in yet other mathematical activities, which may have their own epistemic value.

Interpretation

In order to use CAS effectively, the user needs to interpret the CAS output appropriately. This may involve simply being aware of certain conventions in the output format of the CAS. For example, the user needs to know that the '→' symbol in the output above is equivalent to the paper and pencil symbol '='. Or it may involve a mathematically aware interpretation of the output. CAS may produce the same factored and simplified expression for two different algebraic expressions despite the fact that the domain of one of the expressions excludes a particular *x*–value [4]. For example, asking *Mathematica* to Simplify or Factor $\left(x^2 - 1\right) / \left(x - 1\right)$ yields the output $x + 1$. There is no mention of the restriction, $x \neq 1$, in the output and so the user needs to interpret the output with a critical eye.

Interpretation of graphical representations in a computer environment also entails its own challenges [9]. In the task below we see that many students did not interpret the points of intersection of two graphs as the endpoints of a particular interval.

3. Research context

In 2005 I introduced *Mathematica* into the first–year Mathematics Major Course at Wits University where I was lecturing mathematics. Every two weeks students used this CAS during a tutorial to solve mathematical problems and to consolidate or anticipate new mathematical material. The students in the course came from diverse backgrounds both in terms of the quality of mathematics education that they received prior to entry to university and in terms of their exposure to technology. To this extent, a Maths I Major course survey in 2007 revealed that 69% of students regarded themselves as computer literate on entry to university; 19% did not regard themselves as computer literate and 12% did not respond to this question. All students accepted into the Mathematics Major course have achieved a predetermined grade for Mathematics in their final (state–endorsed) school examinations.

In 2006 I set up a research project to examine aspects of the evolving relationship between students and CAS. In this paper my concern is with whether the heterogeneous class was able to use CAS as a tool for construction of signs and for transformation of signs and whether they were able to appropriately interpret the mathematical relationships or results generated by the CAS. I also audio– and screen–recorded several pairs of students as they engaged in the task. For analysis of these transcriptions, see [10, 11].

The task

Near the end of the 2007 academic year, all students in the class were given an assignment involving the use of CAS and paper and pencil. The assignment was designed to introduce students to the concept of the Maclaurin polynomial before the students had been introduced to the concept in regular mathematics lectures. It was adapted from a laboratory project in the course textbook [12, p. 212]. Students worked in self–selected pairs on this assignment. Students could work on the assignment during two tutorial sessions and in their own time. Ultimately they were expected to hand in their assignments for grading.

In this paper, I examine all students' handed–in answers (in the form of printed–out computer output) to Task 4 of the assignment.

In Task 4 students were required to determine the interval in which the second–order Maclaurin polynomial, $p(x)$ of the Cosine function, is accurate to within 0.1 of $f(x) = \text{Cos } x$. (Note: Previously, in Task 3, students were required to generate the second order Maclaurin polynomial $p(x) = 1 - ½\, x^2$ and to plot a graph of $p(x)$ and $f(x) = \text{Cos } x$ on the same set of axes.)

> Determine the values of x for which the quadratic approximation $p(x)$ found above is accurate to $f(x)$ within 0.1.
> [Hint: Graph the functions, $f(x) = \text{Cos } x$, $p(x)$, $y1 = \text{Cos } x + 0.1$ and $y2 = \text{Cos } x - 0.1$ on a common screen.]

Figure 1: Task 4

To ease the task of the reader, I outline a possible method of solution (used by many students in part) before analyzing students' responses.

Possible Solution

The student uses CAS to sketch the four functions (the construction) as per the hint – see Figure 2.

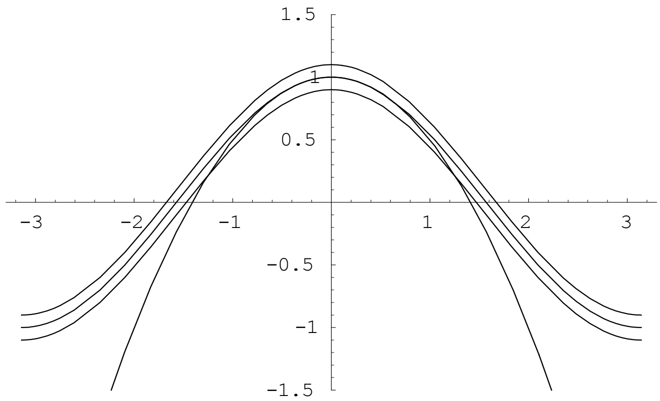


Figure 2. Graphical representation of four functions

The learner uses this graph, or otherwise, to see that she has to find the points of intersection of $y2 = \text{Cos } x - 0.1$ and $p(x)$. She finds the points of intersection either by visually estimating the points of intersection (visual transformation of CAS output) or by using symbolic algebra to equate the two graphs. This symbolic algebra takes the form of the use of the FindRoot command (transformation through construction of operation). Either way, the student interprets the points of intersection as the endpoints of the interval in which $p(x)$ is within a distance of 0.1 from Cos x. This leads to an answer such as as $x \varepsilon (-1.26, 1.26)$ or $-1.26 < x < 1.26$ (interpretation and conclusion). Open or closed intervals are acceptable as are approximations to ± 1.26.

Indicators

I describe the possible types of (written) outputs and how I used these outputs as indicators of activities.

Construction of sign

I classified the initial representation of the task (usually in the form of a graph of four functions as per the Hint) into one of six categories, mostly based on the size of the window.

A representation was Illuminating if the size of window allowed a user to see the points of intersection of $p(x)$ and $y2$, such as in Figure 2. A representation was Opaque if the window extended over such a large domain that one graph could not be easily distinguished from another. A representation was Partial if the domain of the functions was so small that the points of intersection were outside the window. Some students elected to only zoom in on the points of intersection. I classified this as a Zoom representation. Presumably the latter two groups used the graphic representation from Task 3 (in which they had sketched $p(x)$ and $f(x)$=Cos x on the same system of axes) to deduce that they needed to find the points of intersection of $p(x)$ and $y2$, which they then did using the FindRoot command. Some students did not present any graphic interpretation of the task. That is, there was No Representation. Other students used incorrect functions in plotting the graphs. This included students who had not deduced that $p(x)$ was $1 - 0.5x^2$ in Task 3.

Of course, many students performed several iterations (not visible in the handed in assignment) before they found a suitable window. But my concern was with whether the students could *ultimately* generate a useful representation, or not.

Interpretation 1

An appropriate interpretation of the graphs required students to notice that the required values of x were those x–values lying between the two points of intersection of $p(x)$ and $y2$. From such an interpretation, or otherwise, they may have deduced that they needed to equate $p(x)$ and $y2$. One cannot see 'Interpretation' directly in the written record, only its consequences. These consequences are described in Transformation of CAS output below.

Transformation of CAS output

Two avenues of transformation were likely: the use of a Symbolic command like FindRoot, to solve the equation, $p(x) = y2$ (the construction of a new sign), or the use of visual estimation to find the points of intersection of $p(x)$ and $y2$.

Visual estimation could take one of two forms: either the students could move the cursor to the points of intersection and read these off, or the students could visually estimate the points of intersection (yielding a less accurate reading) by just looking at the graphs. This visual estimation would be visible to the researcher in that the students would record something about the values of x at the points of intersection (for example "x lies between $-$ 1.26 and 1.26" or "$x = -1.26$, $x = 1.26$").

Interpretation 2

In the case of the output from the FindRoot command, ie {$x \rightarrow 1.26124$}, the student needs to observe that the points of intersection are symmetrical about the *y*–axis and so $p(x)$ is within 0.1 to Cos *x* graph for x ε (–1.26, 1.26). I regarded such an answer as an indication of successful interpretation. On the contrary if the students found the values of one or both roots ($x = 1.26$ or $x = -1.26$) but did not interpret these roots as the endpoints of an interval, I regarded them as not having successfully interpreted the signs generated by CAS.

With regard to those students who found the points of intersection of $p(x)$ using visual estimation, they too needed to interpret the *x*–values as the endpoints of the interval for which $p(x)$ is within 0.1 to Cos *x* graph. If the student merely wrote down the values of the endpoints, eg $x = -1.26$, $x = 1.26$, they were regarded as not successfully interpreting the transformation.

Epistemic Transformation

As a maths education researcher, it is necessary to make certain inferences about the epistemic value of the mathematical activities. If the student was able to correctly interpret the mathematical signs resulting from their transformations, I regarded the task as having positive epistemic value for that learner. That is, if the student gave a final answer which indicated that $p(x)$ was approximate to within 0.1 of cos *x* over the interval (–1.26, 1.26), I regarded that learner as having undergone the intended epistemic transformation.

Of course, other students who only partially completed the task may have derived epistemic benefit from their activities. But it was not the intended benefit and so I do not classify this as epistemic transformation.

4. Results and Discussion

There were 203 pairs of students in the class. Of these 27 (13%) pairs consisted of two previously computer non–literate learners. I do not disaggregate the data according to level of computer literacy – the numbers involved are too small to be useful in a broad analysis as in this paper. Suffice to say that these formerly computer non–literate learners appear in almost every category given below.

Construction of sign – briefly

Most students (171, ie 84%) drew an adequate representation of the four graphs. That is, they drew an Illuminating graph or they were able to use the graphs from Task 3 (Cos *x* and $p(x)$), with an additional plot of the relevant graphs (Opaque, Partial or Zoom) to deduce that they needed to find the points of intersection of $y2 = \text{Cos } x - 0.1$ and $p(x)$. Eighteen (18) students used an incorrect $p(x)$ and seven (7) students graphed the wrong functions, but with correct $p(x)$. In most of these cases, this made a proper interpretation of the task impossible. Nevertheless, I analysed the transformations and interpretations of these latter two groups in terms of the functions they used. Seven (7) students did not attempt the task at all.

Table 1: Summary of (brief) results concerning construction of sign $N = 203$

Category	Number of pairs of students
Adequate construction of task using CAS (mostly through graphical construction of four functions).	171 (84%)

Incorrect $p(x)$	18 (9%)
Wrong functions drawn but correct $p(x)$	7 (3%)
No attempt at construction of graphical representation	7 (3%)

Note: Due to rounding off errors, the percentages do not total 100%.

Epistemic transformations

According to the indicators presented above, 80 pairs of students (39%) were able to use the CAS–based activities for epistemic transformation. That is, they constructed an appropriate representation for the mathematical task (mostly in the form of a CAS–based plot of the four graphs) and then used suitable transformations to determine the endpoints of the required interval. Of the 80 pairs, 33 pairs (16%) used the CAS operation, FindRoot, to equate the relevant graphs; 47 pairs (23%) visually estimated the points of intersection. All these students interpreted the results from their transformations appropriately, presenting their final answer as an interval, i.e.x ε (–1.26, 1.26) or $-1.26 < x < 1.26$.

In terms of my indicators, the CAS–based task thus provided epistemological access to the notion of Maclaurin polynomial, to the notion of its approximating nature and to a strategy for finding intervals of approximation to about 39% of the whole class – a rather disappointing percentage.

Adequate Transformation followed by poor Interpretation 2

Some students successfully transformed the CAS graphs that they had generated but did not make any explicit observations on their transformations. Specifically, 37 pairs (18%) used FindRoot command correctly to equate the correct functions. However they either failed to interpret the FindRoot output (33 pairs) at all, or they (4 pairs) used the value given by FindRoot to present the endpoints of the interval ie –1.26 and 1.26 as their answer (eg $x = 1.26$ or $x = -1.26$). Six pairs of students (3%) used visual estimation to give the values of the endpoints, again without explicit interpretation, merely writing down the values of the endpoints.

This evidences the difficulties that students may have in interpreting CAS information; in this case, the values derived, symbolically or visually, from equating two functions in the light of the particular task. Perhaps for those students who found the points of intersection of the two graphs but failed to interpret these as endpoints of an interval, the process of using the CAS became the aim of the task rather than a means to solving the given problem in the task.

Poor Interpretation 1 followed by poor or no Transformation

Thirty-six pairs of students (18%) drew the graphs but did not attempt any form of transformation. As such I regard them as not having interpreted the task and/or the graphs adequately.

To determine the endpoints of the interval, students had to first find the points of intersection of $p(x)$ and cos x – 0.1. The students discussed above did not find these endpoints, stopping their activities after they constructed their initial representation. Perhaps these students (as in the previous category) put so much effort into drawing the graphs that they regarded the task as complete once they had achieved their aim of constructing a representation.

An additional 37 pairs (18%) used inappropriate transformations. Of these, 8 pairs used FindRoot command with the incorrect functions (for example, they equated $p(x)$ with 0.1 or with Cos x +0.1). These students probably did not recognise which graph belonged to which function in their graphical representation or they did not know how to go about determining "the values of x for which … $p(x)$ …is accurate to $f(x)$ within 0.1". A further 18 pairs used visual estimation but looked at an incorrect interval (eg some used the intercepts of $p(x)$ with the x–axis and gave their answer as $x \varepsilon \left(-\sqrt{2}, \sqrt{2}\right)$); 11 merely gave the answer as one or two (incorrect) points.

The graphical representation of the functions, as per the Hint, did not give these students insight into the task, as one may have hoped.

No representation, no experimentation, no interpretation:

Seven students (3%) made no attempt to do the task at all.

Table 2: Summary of results concerning transformation and interpretation of signs
$N = 203$

Category	Total number of pairs of students	Details	Number of pairs of students
Adequate Epistemic transformation	80 (39%)	Used symbolic Algebra (FindRoot command) and interpreted output correctly	33
		Used Visual Estimation to present appropriate answer to task	47
Adequate Transformation followed by poor Interpretation 2	43 (21%)	Used Symbolic Algebra (FindRoot command) but did not interpret output.	33
		Used Symbolic Algebra (FindRoot command) but interpreted output as points (rather than as endpoints of an interval)	4
		Used Visual Estimation to find values of intersection points. Did not interpret these intersection points as endpoints of an interval.	6
Poor Interpretation I followed by poor or no Transformation	73 (36%)	Drew graphs adequately and stopped, ie no subsequent transformation of information in graphs.	36
		Drew graphs adequately but equated incorrect functions (using FindRoot command) eg $p(x)$ and $y = 0$ to get incorrect interval.	8
		Drew graphs adequately but equated incorrect functions (using visual means) eg $p(x)$ and $y = 0$, to get incorrect interval.	18
		Drew graphs adequately but gave answer as one or two incorrect points.	11
No attempt at task	7 (3%)		

Note: Due to rounding off errors, the percentages do not total 100%.

5. Concluding comment

The above analysis and Tables 1 and 2 show that the deconstruction of a CAS–based task into its key components, that is, construction of sign, transformation of sign and interpretation of sign, illuminates how various students are able to engage or not with a CAS–based task.

In particular, we see that although 84% of the class were able to construct an adequate initial representation of the task with CAS, only 39% of the students were able to complete the entire task successfully (epistemic transformation). Of the remainder, 21% of the students were able to successfully use symbolic algebra or visual estimation to find the intersection points of the relevant functions but were unable to interpret these intersection points in terms of the task requirements; 36% of the students either did not proceed beyond the stage of representing the relevant functions graphically or found the points of intersection of incorrect functions. The remaining 3% did not even embark on the task.

In this exemplar, we see that we cannot assume that students will be able to adequately interpret CAS output, even if they are able to generate the appropriate CAS commands and output. One implication for CAS use at first–year university level is that more emphasis is placed on the interpretation of CAS– based outputs with reference to the requirements of the relevant task.

6. References

[1] M. Artigue, *Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work*, International Journal of Computers for Mathematical Learning, 7(2002), pp. 245-274.

[2] C. Kieran and P. Drijvers, *The co–emergence of machine techniques, paper and pencil techniques, and theoretical reflection: a study of CAS use in secondary school algebra*, International Journal of Computers for Mathematical Learning, 11(2006), pp. 205–263.

[3] C. Winsløw, *Semiotic and discursive variables in CAS–based didactical engineering*, Educational Studies in Mathematics, 52(3) (2003), pp. 271–288.

[4] C. Kieran, J. Guzman, A. Boileau, D. Tanguay, P. Drijvers, *Orchestrating whole–class discussion in Algebra with the aid of CAS technology*, Proceedings of the Joint Meeting of PME 32 and PME–NA XXX, Morelia, Mexico, Vol. 3(2008), pp. 249–256.

[5] P. Ernest, *A semiotic perspective of mathematics: The case of number*, Educational Studies in Mathematics, 61(1-2) (2006), pp. 67–101.

[6] M. Hoffmann, *Signs as means for discoveries* in *Activity and Sign– Grounding Mathematics Education*, M. Hoffmann, J. Lenhard and F. Seeger eds., Springer, New York, 2005, pp. 45–56.

[7] R. Pierce & K. Stacey, *A framework for monitoring progress and planning teaching towards the effective use of computer algebra systems*, International Journal of Computers for Mathematical Learning, 9 (2004), pp. 59-93.

[8] J. Kaput, *Technology and mathematics education* in *Handbook of Research on Mathematics Teaching and Learning*, D. Grouws, ed., Macmillan, New York, 1992, pp. 515–556.

[9] E. P. Goldenberg, *Mathematics, metaphors and human factors: Mathematical, Technical and Pedagogical Challenges in the educational use of graphical*

representations of functions, Journal of Mathematical Behavior, 7 (1988), pp. 135–173.

[10] M. Berger, *Mathematical Activity with a Computer Algebra System: A Semiotic Perspective*, African Journal of Research in Mathematics, Science and Technology Education (in press)

[11] M. Berger, *Computer algebra systems, semiotic activity and the cognitive paradox*, Proceedings of the Joint Meeting of PME 32 and PME−NA XXX, Morelia, Mexico, Vol. 2(2008), pp. 153–160.

[12] J. Stewart, *Single Variable Calculus*, Fifth edition, Brooks/Cole, Belmont, 2003.

An APOS analysis of students' constructions of the concept of continuity of a single-valued function

DEONARAIN BRIJLALL[‡] AND ANESHKUMAR MAHARAJ[§]

**School of Science, Maths and Technology Education, University of KwaZulu-Natal, South Africa*

***School of Mathematical Sciences, University of KwaZulu-Natal, South Africa*

Abstract: This article reports on the use of APOS theory to analyse data captured in a study which investigated second-year students' understanding of the concept of continuity. The study is qualitative in that it reports on students' mental constructions of the concept of continuity of single-valued functions, obtained from analysis of their responses to structured activity sheets. The twelve students in this study specialise in the teaching of mathematics for the FET high school curriculum at a South African university. A two-tiered concurrent approach was attempted; one through student-collaborations and the other through an instructional design worksheet, to develop mathematical understandings of the concept of continuity.

Keywords: APOS theory, single-valued function, continuity, collaborative learning

1. Introduction

Some studies, for example [14], analysed student mathematical learning on an individual basis. This study however analysed teacher-trainees' understanding, after they carried out investigations first individually and then in a collaborative manner. This is to address the learner-centred approach which underpins Curriculum 2005 [11]. We report on an investigation based on the use of activity sheets and group-work to construct the concept of continuity. To collaborate is to work with another or others. In practice, collaborative learning has come to mean students working in pairs or small groups to achieve shared learning goals [5].

Vidakovic [24, 25] used APOS theory in the context of collaborative learning. Those investigations focused on the differences between group and individual mental constructions of the inverse function concept. Vidakovic described the construction processes for developing schema (genetic decomposition) of the inverse function. In particular, genetic decompositions which predict the mental constructions are a part of every good APOS based study.

Wu [27] argued that pre-service development of teachers for grades 6 to 12 require courses which consolidate, *mathematically*, those topics which do not stray far from the high school mathematics curriculum. In particular, they should revisit all the standard topics in high school from an advanced standpoint, and enliven them with motivation, historical background, inter-connections and above all, proofs [27]. Bezuidenhout [6] points out that misconceptions relating to students' understanding of

‡ E-mail: brijlalld@ukzn.ac.za

the concepts of *limit* and *continuity* are impediments to the development of deeper understandings in differential calculus. It seems that many students perform poorly in mathematics because they: (a) are unable to adequately handle information given in symbolic form which represent objects [abstract entities], for example mathematical expressions, equations and functions, and (b) lack adequate schema or frameworks, which help to organize and link different objects [15]

2. Background

Knowledge Base for Educators

One of the expectations of the Norms and Standards for Educators [10] is that the educator be well grounded in the knowledge relevant to the occupational practice. She/he has to have a well-developed understanding of the knowledge *appropriate* to the specialism. Many mathematics educators find themselves in a position requiring them to implement the syllabus, which includes certain topics they are unfamiliar with. According to Adler [1], educators with a very limited knowledge of mathematics need to develop a base of mathematical knowledge. They need to relearn mathematics so as to develop conceptual understanding. Taking this into account we attempted to make certain that trainee-teachers leave with a base of knowledge relevant to their occupational needs. Mwakapenda [18] concurs when stating that a significant concern in school mathematics is learning with understanding of mathematical concepts.

The National Curriculum Statement (NCS) emphasises a learner-centred, outcomes-based approach to the teaching of mathematics to achieve the critical and developmental outcomes [11]. The following question guided our inquiry into preservice students' understandings in their construction of the concept of continuity.

How does the graphical representation learning approach facilitate students' learning process with regard to the construction of the concept of continuity of single-valued functions in differential calculus?

The main intention of the study was to observe how learning of mathematics content, whether effective or not, took place under these circumstances. In order to answer the above research questions an APOS analysis of the data was conducted.

3. Theoretical Basis

This study was carried out in accordance with a specific framework for research and curriculum development in undergraduate mathematics education advocated by Asiala, et al [4] which guided our systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contributed to student learning. The framework consists of the following three components: instructional treatment, theoretical analysis, and observations and assessment of student learning.

Instructional treatment

Visualization plays an important role in learning [26] and in particular, the learning of mathematics. This idea is espoused in the old adage, "*a picture is worth a thousand words*". The role of graphs in the teaching of mathematics is complex and has multi-fold dimensions [8]. In this regard, we adopted the Dual-Coding Theory (DCT) as discussed by [19], to motivate our design of the structured worksheet. Dual-Coding Theory, a theory of cognition, postulates that visual and verbal information are processed differently and along distinct channels with the human mind creating separate representations for information processed in each channel. Both visual and verbal codes for representing information are used to organize incoming information into knowledge that can be acted upon, stored, and retrieved for subsequent use. We designed the worksheet to create graphical representations of continuous and non-continuous functions as visual information. Then both students (apprentices) and tutor (the experienced teacher) engaged collaboratively [26] to provide verbal information, allowing mathematical connections for a deeper conception of contuinity to emerge.

According to Paivio [19], mental images are analogue codes, while the verbal representations are symbolic codes. *Analogue codes* represent the physical stimuli we observe in our environment and in this study viewed as graphical representations of *continuous and non-continuous* mathematical functions. These codes are a form of knowledge representation that retains the main perceptual features of what is being observed. *Symbolic codes*, on the other hand, are a form of knowledge representation chosen to represent something arbitrarily as in the *concept definition* of continuity in calculus [22]. Continuity has a pre-formal visual meaning in that it has elements of dynamic movement, being all in one piece, not changing suddenly (either in direction or in its formula) and having no holes [22, 23]. Supporting evidence comes from research [26] that shows that memory for some verbal information is enhanced if a relevant visual is also presented or if the student can imagine a visual image to go with the verbal information. Verbal information can often be enhanced when paired with a visual image, real or imagined [2, 3]. This paper also uses collaborative learning from Vygotsky's theory as a framework for classroom interactions to occur fruitfully [26]. Unlike traditional teaching approaches, collaborative learning rewards all individual students participating in the group by explicitly ensuring that all have achieved the intended lesson outcomes [5].

Theoretical analysis

Piaget, cited in Bowie [7], expanded and deliberated on the notion of *reflective abstraction*. Reflective abstraction refers to the construction of logico-mathematical structures by a learner during the process of cognitive development [12]. Two features of this concept are: (a) It has no absolute beginning but appears at the very earliest ages in the coordination of sensori-motor structures, and (b) It continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction [13].

We define the following four concepts that are used in APOS theory of conceptual understanding [7]:

- Action: an action is a repeatable physical or mental manipulation that transforms objects
- Process: a process is an action that could take place entirely in the mind.
- Object: the distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made, and such actions are made in the mind of the learner.
- Schema: a schema is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema could aid students to "... understand, deal with, organise, or make sense out of a perceived problem situation" [12, p.102].

Observations and assessment of student learning
This followed the instructional treatment and allowed us to gather and analysis data. The data was used in two ways. Firstly the results of the data analysis were used to test our initial genetic decomposition. Secondly the data gathered was used to report on the performance of students on mathematical tasks related to the concept of continuity.

4. Methodology

The structured design of worksheet used an examples and non-examples approach. In particular, we focus on sorting, reflecting and explaining, generalizing, verifying and refining. The methodology adopted five stages: (a) Design of worksheet, (b) Facilitation of group-work, (c) Capture of written responses and (d) Interviews. The data collection relied to a large extent on what students could say or write about their learning experiences. The worksheet task was completed over two double periods, each of one and a half hour duration.. This included the individual work by students, the discussions in the groups, the group class presentations and the final discussion involving the tutor. The interviews were done with individuals a week later during the free periods involving both the student and tutor. All the interviews were video recorded.

Design of worksheet
A worksheet was designed in accordance with ideas postulated by a guided problem-solving model suggested by the work of Cangelosi [9]. This work modelled how meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and affective domains when students solve problems. An interpretation and modification of the guided problem-solving model [16], illustrated in Figure 1, has the following three interlinking levels or phases: (a) inductive reasoning; conceptual level processing occurs, (b) inductive and deductive reasoning; where simple knowledge and knowledge of a process level occurs, and (c) deductive reasoning; occurring at an application level.

In our case, to provide a structured approach in an inductive manner, we implemented the graphical representations as tools to guide the discussion in arriving at the concept definition of continuity. However, we note that there is always interplay between inductive and deductive reasoning for the different levels. They are continuously present and constantly following each other in mathematical thinking. For example, in an inductive process, very often a preliminary 'generalising' step is reached; the finalisation of an inductive part is the beginning of the deductive part [16]. Therefore, generalising at each of the different levels implies that the deductive mode of reasoning comes into play.

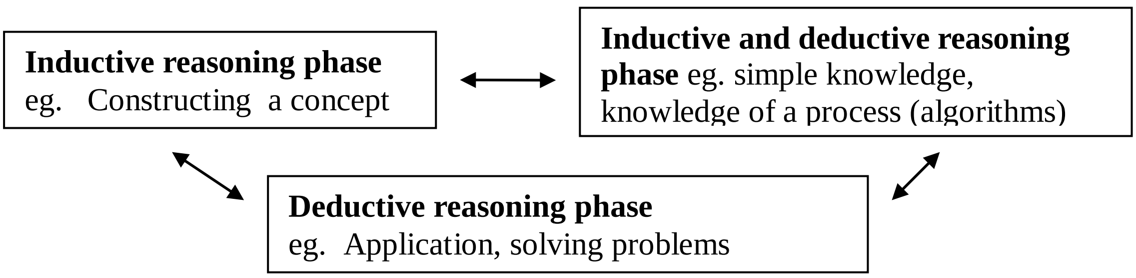


Figure 1: A Guided Problem Solving Teaching Model

In creating constraints for the examples and non-examples in the guided worksheets, we implemented the concept of boundaries [17] that included the characteristics of the existence of function values and limits (see Figures 3 and 4 which are extracts from the worksheet handed to students). For the design of the worksheet, inductive learning activities were used to construct the concept of continuity of functions. This design promoted visualisation and verbalisation. These activities had the following stages within the inductive level (a) comparison with examples and non-examples and categorising, (b) reflecting and explaining the rationale for categorising, (c) generalising by describing the concept in terms of attributes, that is, what sets examples of the concept apart from non-examples, and (d) verifying and refining the description and definition by testing and refining it. Those stages were chosen since they could be exploited to facilitate the following which form the framework for APOS theory and contribute to conceptual understanding: action, process, object, schema.

Facilitation of group-work

The second year class of twelve preservice students engaged with the activities individually for approximately fifteen to twenty minutes. This was to allow students to make contributions when working in a group setting. The groups were formed by the lecturer using the marks attained in a mathematics education module from the previous semester (Mathematics for Educators 210). The purpose was to ensure that the groups had members of different ability levels, mixed race, mixed gender and different home language. Preston [20] notes in this regard that the teacher should carefully group students that “can potentially develop in collaboration” with more capable persons. When constructing the concept of continuity they worked in four groups, comprising of three members, each. Each group, after discussing and reaching a collective decision, presented their mathematical ideas to the class. The student facilitator was to report on the collective ideas or thoughts of their groups. The students were given time limits set by the facilitator to encourage them to focus on the task on hand. The groups were similar in that they had members with a spread of ability levels. At the end of the group presentations, an intensive classroom discussion including responses from the lecturer led to students in a group establishing the *concept definition* of continuity.

Capture of written responses

Each preservice student was given a guided activity sheet. When they were in groups, they were
provided a separate worksheet that required the collective group response to the activities. The following five instructions appeared on the worksheets: (a) Complete each worksheet on an individual basis. (b) Now form groups of three. (c) Discuss your findings within the group to reach consensus. (d) Write down a collective response and elect a leader to discuss with class. (e) Finally conclude findings as a class with lecturers. This involved the tutor, who is a PhD student and a mathematics lecturer, who clarified, using the worksheets, the mathematically acceptable definition of continuity. The group response worksheets were then collected by the lecturer for analysis of preservice students’ construction of the continuity concept, within a group context.

Interviews

The interviews took place after the written responses were analysed. After categorising them in Table 1 and Table 2, it was then that we employed verification interviews.

5. Genetic decomposition

At this stage in the course, most of the second year preservice students already had adequate knowledge of existence of limits, which was verified orally by the lecturer. They also sketched graphs of piecewise functions comprising of linear, quadratic, hyperbolic, semi-circular and absolute-valued functions. This involved an inter-play between graphical illustrations and algebraic notations. The graphical approach provided a visual representation of the algebraic expression of the function. As an example when finding $\lim_{x \to 2} f(x)$ where $f(x) = 2x^2$, the students normally proceed using a substitution algorithm without a possible graphical representation of the function. In this regard, when algebraic notations of functions were alone presented, the two-sided approach in the limit concept was not immediately perceived. The students had no formal prior knowledge of the concept of continuity. During the guided problem solving activity, students were expected to develop the following definition in full sentences, for example, "there is a y-value for x = a", and in notation form as follows: A real single-valued function is continuous at $x = a$ if: (1) $f(a)$ exists, (2) $\lim_{x \to a} f(x)$ exists, and (3) $f(a) = \lim_{x \to a} f(x)$. So, a function f is continuous if it is continuous at every point in its domain. The thorny question of whether a function can be considered discontinuous outside its domain arises. Yes as a global gestalt because there is a hole, but no from the formal definition of a continuous *function* since continuity only refers to points in the domain.

Based on the above the following genetic decomposition of the concept of continuity was used to guide our instructional treatment.

As part of his/her function schema the student:

1. has developed a process or object conception of a function, and
2. has developed at least an action conception of graphs of piece-wise functions.

As part of his/her limit schema the student:

3. has developed a process conception of the limit of a function,
4. has developed at least an action conception of the existence of a limit of function,
5. recognises and uses suitable notation and their respective applications to specific situations, and
6. then coordinates previously constructed schemas of a function, limits of functions and appropriate notation to define continuity of a function. See figure 2.

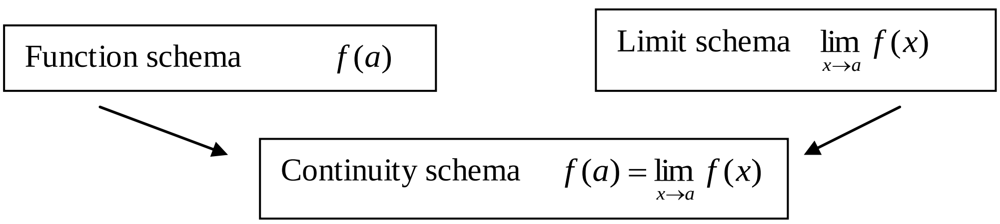


Figure 2: Schema for continuity

6. Analysis and discussion of results

The following is an extract from the students' worksheet, labelled *Stages A to D*. In particular, the mathematical stages are sorting, reflecting and explaining, generalizing, verifying and refining. In *Stage A* (see Figures 3 & 4), the researcher demarcated the examples and non-examples and the students then compared these distinguishing features, namely, the existence of a function value, the existence of a limit and the equality of the function value with its limit that characterize continuous functions from non-continuous ones.

Extracts taken from students' worksheet covering the four stages

Stage A: Sorting

The following are examples of graphs of *continuous* functions:

Example 1 Example 2 Example 3 Example 4

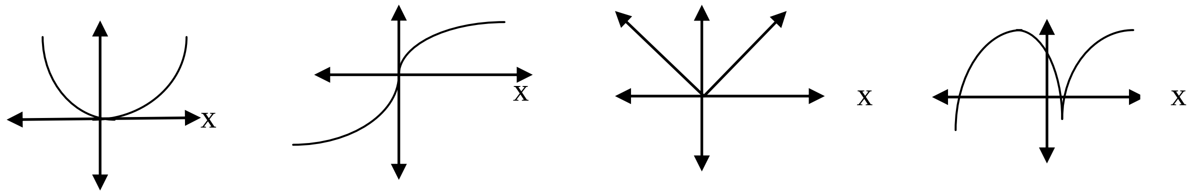


Figure 3: Examples of continuous functions

The following are examples of graphs of functions, which are *not continuous*:

Example 5 Example 6 Example 7 Example 8

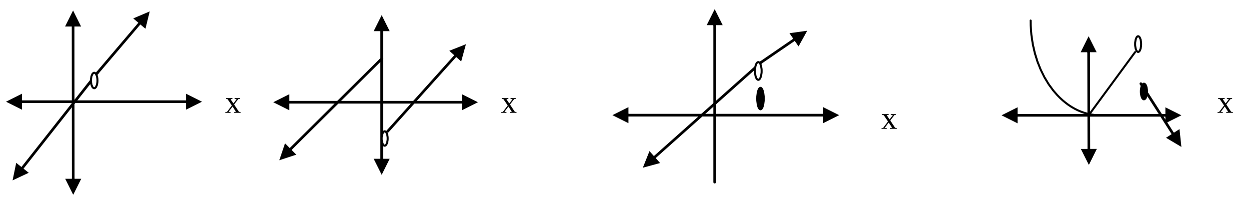


Figure 4: Examples of non-continuous functions

Stage B: Reflecting and explaining

After interrogating the above examples and non-examples of graphs of continuous functions, explain why one would categorize them as such.

Stage C: Generalizing the description of continuous functions

Now provide mathematical conditions which a function need satisfy in order for it to be called continuous at x = *a*.

Stage D: Verifying and refining

Check whether the following functions are *continuous or not* by using the conditions you have derived in the generalization above:

Example 9
$$f(x) = \begin{cases} x+1 & , if\ x \geq 2 \\ x+4 & , if\ x < 2 \end{cases}$$

Example 10
$$f(x) = \begin{cases} x^2 & , if\ x \neq 2 \\ 1 & , if\ x = 2 \end{cases}$$

Example 11
$$f(x) = \begin{cases} x^3+1 & , if\ x \geq 2 \\ 2x+5 & , if\ x < 2 \end{cases}$$

The summary of responses covering the above stages appear in Table 1 and Table 2. Table 1 summarizes the four group responses, with the data captured from the video as well as from the group activity sheet. The reflecting and explaining stage, and generalizing the concept continuity of functions is tabulated. Characterization of coded categories is as follows: (a) *none* was used for no response, (b) *inadequate* codes implied an incorrect or unclear response with features which are not in accordance with our genetic decomposition, (c) *partial* codes indicated gaps in description where responses had features that resembled our genetic decomposition, and (d) *complete* codes implied a mathematically correct response in accordance with the concept of continuity.

Table 1: Results on group constructions for continuity of functions (n = 4)

Stages	None	Inadequate	Partial	Complete
Reflecting & Explaining	0	2 (groups A and B)	2 (groups C and D)	0
Generalizing	0	1 (group A)	2 (groups B and D)	1 (group C)

From the table above, two groups provided inadequate explanations of continuity after they studied the examples and non-examples provided on the guided worksheet. Group B did not consider limits or function values when reflecting and explaining the rationale for categorizing. Their response given was:

> *The first four graphs are continuous where the x and y intercepts are included and the graph passes completely through the x and y intercepts. The next four graphs are not since some parts of the graphs are excluded and included. In some cases there are more than one sketched graph on the same set of axes, indicating the graph is not continuous.*

This has three separate sentences. The first says the first four graphs are continuous (with some extra observations about intercepts) but does not say *why* they are continuous. The second sentence, which seems separate from the first, says that the next four are not continuous since some parts are excluded. In other words, there are gaps in the graph contrary to the pre-conception of continuous operations going on smoothly without gaps [22]. The third sentence says there are more than one sketched graph in each picture where a picture has a single set of axes. This relates to both the idea of a graph that ‘continues’ and the long experience that the student will have had of a function given by the same formula that ‘continues’ through its domain. It is a consequence of how the students have been previously taught functions as being given by a single formula. Thus all the comments of group B relate to preconceptions of functions as a formula, drawn smoothly and having no gaps. In Dubinsky’s work a formula is only an *action*, so the above response from Group B is not even at the process level, since the students possibly think there are two functions because each piece is defined by a different rule; which suggests an action level response in APOS.

Group B used symbolic language to generalize the definition of continuous functions as confirmed below:

> *straight line graph* $y = mx + c$; *parabola* $ax^2 + bx + c$
> $f(x) = f(a)$
> $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, *if the limit always exists*
> $\lim_{x \to a} f(x) = f(a)$
> *point ‘ a’ on a domain of f(x) , f(a) = exist*

Firstly, this group’s reference to the existence of limits occurs from their existing schema. This suggests that these students had an object conception of the existence of a limit of function. They correctly chose suitable notation to illustrate their conception of continuity as an object, since they wrote $\lim_{x \to a} f(x) = f(a)$.

However, two instances of misuse are evident in the second and last lines of their response. The last line implies that they meant *f(a)* exists, so this line should replace the second line. Secondly, there is a clear difference between the group's concept image in the reflecting and explaining activity with that portrayed during generalizing the definition. In the former activity, the group responses indicate that emphasis was placed on a visual analysis rather than on the algebraic meaning, as was the case during generalizing. This would imply that at this point the group used only one representation. Thus the students' conception of continuity is limited, as it does not include a number of different representations. They did not integrate the symbols that are associated with an algebraic representation of continuity. However, they later resorted to graphical representations as evidenced in the first two functions in Table 2 when provided with examples of piecewise functions in the verifying stage. They provided correct responses and applied sketches of graphs. We suggest that the algebraic statements made in their arguments were not clear in their mental constructions but when using graphs they had a better sense of what they were arguing. This seems to justify the need for visual representations when trying to understand and make sense of the concept of continuity.

From Table 1, we observe that two out of four groups when reflecting and explaining constructed partial understandings of the concept of continuity. These were:

Group C: *The continuous functions have no disturbance and their limits exist at all points. Not continuous functions have disturbance (hole) on the graph or either whose limits do not exist at certain points or f(x) doesn't exist at some points. There are points which are not on the domain of the function.*

Group D: *One can recognize the first four graphs as continuous only because at point of, for example,* $x \to a$*, we have the fixed y-value which means at every point of x there is also a fixed y-value. For the last four graphs one can say they are discontinuous because at some of the y- values, x does not exist. This means that at the point of x, there is no unique y- value.*

Group C relate continuity to 'no disturbance', at the same time referring to the existence of limits and points not in the domain, which could relate to an intuitive notion of limit as given in the course earlier. Group D use the intuitive language of limits but assert that for the last four graphs, for some *y* values, there are no *x* values (*x* does not exist) again referring to difficulties relating to what they may perceive as holes in the graph. We note that the response of Group C was incomplete, since this group seems to assume that the existence of $\lim_{x \to a} f(x)$ is a sufficient condition for continuity of the function *f* at $x = a$. It is likely that this option arose from previously learnt concepts that they intended to link now. On the other hand, this misconception could be sourced by an understanding that if $\lim_{x \to a} f(x) = c$, then $f(a) = c$ so f is then continuous at $x = a$. It seems that this group did adequately reflect on example 7 in Figure 4. As a result they were unable to satisfy part 6 of our genetic decomposition for continuity, namely 'then coordinates previously constructed schemas of a function, limits of functions and appropriate notation to define continuity of a function'. They were unsuccessful in linking the function schema with the limit schema as illustrated in Figure 2. With reference to the Dual Coding Theory, note that this group could represent visual information by verbal codes since they described points of discontinuity as *disturbance*.

The response given by Group D may translate to a view that at $x = a$, $f(a) = c$ so the limit exists there because it has a value. This means that Group D incorrectly linked $f(a)$ or the *y*-value at $x = a$ with the $\lim_{x \to a} f(x)$. They did not consider the behaviour of a function about points, but they only focused on the value of the function at $x = a$. This might also explain other shortcomings we come across when teaching calculus. An example of this is when dealing with procedures, like using substitution to find limits algebraically, students seem to believe that the value of the function at a point (in this case $x = a$) is of greater importance, rather than how function values behave around the point. Students in this group were unable to proceed beyond the function schema in our genetic decomposition, as illustrated in Figure 2. They did not realize that the existence of the limit of a function $f(x)$ as $x \to a$, does not depend on whether $f(a)$ is defined. The response clearly shows that Group D employed a correspondence between an interval about an *x*-value with an interval about a *y*-value. However, they do not use the independent and dependent variables satisfactorily and conclude that $\lim_{x \to a} f(x)$ and $f(a)$ must be equal. It is important to help the student(s) move on from colloquial to mathematical insight in a meaningful way as argued by Tall [21].

Group A's response when explaining and reflecting was as follows:

A continuous function is a function without a break in co-ordinates or a function that goes on without end to either $-\infty$ *or* ∞.

The response above seems to be derived from the colloquial use of the word 'continuous' in phrases like 'goes on' (meaning that there were no stops). It is observed that Group A views the visual representation of a continuous function as a graph in one piece, with domain the set of real numbers. This contradicts examples 1, 2 and 4 in Figure 3. With reference to part 5 of our genetic decomposition, this group's use of symbols was limited to expressing the domain of the function in example 3 of Figure 3. This led to their inadequate generalization.

Even though the other three groups offered inadequate or partial explanations when responding to the reflecting and explaining processes in the construction of the concept of continuity, their generalizations displayed evidence of features included in our genetic decomposition of continuity. Group C, in particular, generalized the description of continuity concisely and in accordance with our genetic decomposition. These findings display an interplay existing between graphical and symbolic representations. This was assisted by the guided design using inductive reasoning within the framework of the guided problem-solving teaching model (see Figure 1) and the collaborative learning approach to facilitate the development of the concept of continuity.

Table 2: Results on *verifying* and *refining* the concept Continuity of single-valued functions

Function	No group response	Incorrect group response	Correct group response
1. $f(x) = \begin{cases} x+1 & , if\ x \geq 2 \\ x+4 & , if\ x < 2 \end{cases}$	0	0	4
2. $f(x) = \begin{cases} x^2 & , if\ x \neq 2 \\ 1 & , if\ x = 2 \end{cases}$	0	1 (group A)	3 (groups B, C and D)
3. $f(x) = \begin{cases} x^3 + 1 & ,if\ x \geq 2 \\ 2x + 5 & ,if\ x < 2 \end{cases}$	0	2 (groups A and D)	2 (groups B and C)

In the verifying and refining aspect, three functions were defined in questions 1, 2 and 3 (Table 2). The majority of students identified the first two functions correctly while the third function was identified correctly by half of the groups. What was pedagogically interesting was that groups A and D could get the first one correct but not the third. This might be a sign that the students confuse continuity (connectedness of the graph) with differentiability (smoothness of graph). Example 1 has an obvious jump in the middle so it is discontinuous (in a visually coded sense not necessarily symbolically coded). Example 3 is problematic because there is no jump in value but it has what may be conceived as a 'discontinuity' in the change in the formula. Thus the distinction between examples 1 and 3 is self-evident. The first is clearly discontinuous for any reason one cares to name (visual or formal), the second is mathematically continuous but may 'feel' discontinuous visually and dynamically.

Group A tried to reason graphically or geometrically for the first two functions. However, this group of preservice students portrayed an inadequate conception of piecewise functions as they considered the second function to be two separate graphs due to the definition provided. Group D supported this notion when their response was *"f(2) does not exist from the second graph in question one"*. These groups did not comply with part 2 of our genetic decomposition of continuity. Group D used visual-pictorial processing to check whether functions 1 and 2 were continuous. Group B considered 'limit existing' to be an adequate condition for continuity without investigating whether $f(x)$ exists or whether it was defined at $x = 2$. Therefore with regard to Figure 2, it seems that this group was unable to link their function schema and limit schema to verify the continuity of the function.

Group A further displayed their inadequate conception of piecewise functions when looking at the second function, by stating that "*f(x) is 2 graphs"*. Their answer was that the one definition of the graph is continuous while the second definition is not continuous. In a separate interview, the representative for group A said "*we did not know what to do with x = 2*". This implies that the students either did not know how to read/represent the point (1;2) or that that part of the function was not considered vital in deciding on the continuity of the function. They see what they believe to be two graphs (actually two formulae), one is x^2 for $x \neq 2$. This clearly 'continues' in the sense that it continues off to infinity in both directions. The other is $f(x) = 1$ (for $x \neq 2$) which is a single point and so stays in place and does not 'continue' at all. The reason for the distinction in terms of global visual coding is self-evident, and involves not the mathematical definition of continuity but the colloquial preconception of discontinuity. Group B used both symbolic or numeric and geometric reasoning to give a concise answer to the second function under verifying and refining. Group C used numerical reasoning to arrive at the same concise answer for function 2. Group D, on the other hand, considered the fact that f (2) did not exist on their graphical representation of function 2 (i.e., 'open dot on graph), to be a sufficient condition for it not being continuous.

Groups B and C did not use visual or pictorial modelling to answer the third question, but the algebraic manipulations done to find limits and function values were correct. Using this they correctly concluded that the function was continuous. This implies that their schema for continuity satisfies the illustration in Figure 2. Groups A and D, on the other hand, did not use the generalizations they made about continuity earlier to determine whether the third function was continuous or not. Both of these groups reflect on the inequalities in the definition of $f(x)$ as shown below:

Group D: *The use of inequality disturbs movement of our graphs to infinity, therefore, the graphs do not flow (move) freely. There are restrictions therefore the function is not continuous. There is an open gap (dot) in the second function in exercise 3.*

Group A: *not continuous because* $\geq$ *and* < *means open dot on the graph.*

A beautiful expression of dynamic movement is suggested by the response of Group C. 'The use of inequality disturbs movement', suggesting the changing of the formula disturbs dynamic continuity. The graphs *do not flow,* so the function is *not continuous* according to their own personal concept definition. Notice that there are separate statements here. The first two sentences refer generally to disturbing changes of formula. The last sentence refers to the open gap in the third function and also the dot which gives a discontinuity.

7. Conclusion

The findings of this study showed that some students demonstrated the ability to make use of symbols, verbal and written language, visual models and mental images to construct internal processes as a way of making sense of the concept of continuity of single-valued functions. On perceiving functions as mathematical entities, students could manipulate these entities, which were understood as a system of operations. The study provided some valuable insights into the mental constructions of preservice students with regard to 'limit of a function' and 'continuity at a point' in calculus. These insights should be analysed and understood keeping in mind the specific methodology that was used. The verifying and refining stages in the construction of the 'continuity' concept required a conceptualisation of the concept of continuity as a meaningful mathematical entity. This conceptualisation enabled the formulation of a new mathematical idea that can be applied to a wider range of contexts.

The responses received in the four stages A to D of the worksheet indicate that most of the preservice students were able to construct the concept of continuity and hence were in fact capable of definition-making with some degree of success. This was evident by the overlap in ideas arising from the mental constructions formulated and those that are encapsulated in the definition. The worksheet possibly nurtured their creativity by encouraging and providing opportunities for them to share, to value and to discuss the new concept freely without fear of being judged or embarrassed by anyone. They were able to assist each other in addressing the common misconceptions at certain stages of the worksheet. This approach offered opportunities for them to collectively recognise previous knowledge, as well as for them to engage in alternative conceptions with group members.

Acknowledgements: The authors thank David Tall for his input and suggestions to the earlier draft of this paper, and Nalini Maharajh for assisting in the collection of data.

8. References

[1] Adler, J. (2002). Inset and mathematics teachers conceptual knowledge in practice. In P*roceedings of the 10 th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology*. (Vol. II, pp.1-8).

[2] Anderson, J. R. (2005). *Cognitive Psychology and its implications*. New York: Worth Publishers.

[3] Anderson, J. R., & Bower, G. H. (1973). *Human associative memory*. Washington, DC: Winston.
[4] Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D. & Thomas, K. (1996). A framework for research and development in undergraduate mathematics education. *Research in Collegiate Mathematics Education*, 2, 1-32.
[5] Barkley, E.F., Cross, K.P., & Major, C.H. (2005). *Collaborative Learning Techniques. A Handbook for College Faculty*. USA: John Wiley & Sons.Inc.
[6] Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first year students. *International Journal of Mathematics Education, Science and Technology, 32* (4), 487-500.
[7] Bowie, L. (2000). *A Learning theory approach to students' errors in a calculus course,* Pythagoras no. 52. Centrahil:Pretoria.
[8] Brijlall, D. (1997). Graphical solutions to algebraic inequalities. *Pythagoras, 44,* 23-27.
[9] Cangelosi, J. S. (1996). *Teaching Mathematics in Secondary and Middle School: An Interactive Approach*. (2nd ed). New Jersey: Prentice Hill.
[10] Department of Education., (1999). Norms and Standards for Educators, Pretoria, National Department of Education.
[11] Department of Education. (2003). *Revised National curriculum statement for Grades 10 –12 (general) Mathematics*. Pretoria: Government Printer.
[12] Dubinsky, E. (1991a). Reflective Abstraction in Advanced Mathematical Thinking. In: Tall, D. (ed), *Advanced Mathematical Thinking*. The Netherlands: Kluwer.
[13] Dubinsky, E. (1991b). Constructive Aspects of Reflective Abstraction in Advanced Mathematical Thinking. In: Steff, L. (ed), *Epistemological Foundations of Mathematical Experience*. Yew York: Springer Verlag.
[14] Dubinsky, E., Weller, K., McDonald, M.A., & Brown, N. (2005). Some historical issues and paradoxes regarding the concept of infinity: an APOS analysis. *Educational Studies in Mathematics, 60*, 253-266.
[15] Maharaj, A. (2005). Investigating the Senior Certificate Mathematics Examination in South Africa: Implications for Teaching. Phd thesis In Mathematics Education: Unisa.
[16] Maharaj A. (2007). Using a task analysis approach within a guided problem-solving model to design mathematical learning activities. *Pythagoras*. Number 66(Dec), 34-42.
[17] Mason, J., & Watson, A. (2004). *Getting students to create boundary examples*. Retrieved 15 April, 2007, from http:/www.bham.ac.uk/ctimath/talum/newsletter/talum11.htm.
[18] Mwakapenda, W. (2004), Understanding student understanding in mathematics, *Pythagoras*, No 60, 28-35.
[19] Paivio, A. (1986). *Mental representations: a dual coding approach*. Oxford. England: Oxford University Press.
[20] Preston, D. F., & Robert, M. V. (2003). *Methods of Teaching: Applying Cognitive Science to Promote Student Learning.* New York: McGraw-Hill.
[21] Tall, D. (2003). Using technology to support and embodied approach to learning concepts in mathematics. In L.M. Carvalho and L.C. Guimarães *História e Tecnologia no Ensino da Matemática*, vol. 1, pp. 1-28, Rio de Janeiro, Brasil.

[22] Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12* (2), 151-169.

[23] Tall, D., & Bakar, M. (1992). Students' Mental Prototypes for Functions and Graphs. *International Journal of Mathematics Education in Science & Technology*, 23 **(**1), 39–50

[24] Vidakovic, D. (1996). Learning the concept of Inverse Function. *The Journal of Computers in Mathematics and Science Teaching, 15*(3), 295-318**.**

[25] Vidakovic, D. (1997). Learning the concept of inverse function in a group versus individual environment. In Dubinsky, E., Mathews, D. & Reynolds, B., (Eds.), *Readings in Cooperative Learning* , MAA Notes No 44, 173-195.

[26] Vygotsky, L.S. (1978). *Mind in Society: The Development of Higher Psychological Process*. Cambridge: Harvard University Press.

[27] Wu, H. (2003). *Preservice professional development of mathematics teachers*. Retrieved on 26 June 2007, from http://www.math.berkeley.edu/~wu/.

"Expert" vs. "Advanced": Investigating Differences in Problem-Solving Practices

KATHLEEN M CLARK,[††] ALEX JAMES and CLEMENCY MONTELLE

Department of Mathematics and Statistics; University of Canterbury;

Christchurch, New Zealand

Abstract: This study investigated ways in which students enrolled in an advanced calculus course participated in weekly problem-solving sessions associated with the course. In particular, we sought to identify positive participatory actions as well as the conditions that promote them. Identification and description were key goals of the research. As such, the study was primarily qualitative in design and nature. Two types of data were collected:
(1) Observational data from 10 weeks of problem-solving sessions (50 minutes each) for each of four small groups (3-5 students each); and
(2) Focus-group interview data from six students representing three of the four problem-solving groups observed during the study.

For the purpose of reporting initial findings we analyzed the data from one of the small groups using a framework created by Carlson and Bloom [1]. The first version of their framework, the "Initial Problem-Solving Taxonomy", was constructed from the salient research literature describing the behaviors that experts (i.e., research mathematicians) exhibit when engaged in problem solving. Using Carlson and Bloom's taxonomy, we determined the degree to which the students exhibited expert behaviors. Initial results indicate that the advanced calculus students also exhibited a large number of the expert problem-solving attributes; the results also signaled attributes not included in the taxonomy. By conducting further analyses, we will not only contribute to amending Carlson and Bloom's taxonomy but will suggest improvements for future problem-solving session iterations, with the goal of providing students with the opportunity for more expert performance.

1. Introduction

Almost every tertiary level calculus and linear algebra course includes an additional classroom contact component that supplements and takes place outside of the core lectures. Typically, this component may be in the form of tutorials, problem-solving sessions, help sessions, or the like, in which students meet, often in groups significantly smaller than their lecture streams, to work on assigned problems and examples associated with the material covered in class. What often occurs with the tutorial model of this supplemental instruction is that the tutor is likely to become the center of the sessions. For example, tutorial sessions can morph into "the tutor conducting a question and answer session, with the tutor delivering an off-the-cuff and inexpert lecture" [2, p. 88]. Thus, the learner takes on a passive role even though the intent of the supplemental course components has always been clear: to provide a forum for students to apply the more theoretical aspects of the mathematics syllabus that they are led through in lectures to actual problems with minimal guidance. The aim is to encourage and promote active problem-solving behaviors and to facilitate

[††] E-mail: k.clark@math.canterbury.ac.nz

the transition from applying rules and mimicking procedures and routines to applying the more general precepts in minimally familiar contexts. However, identifying the best way to effectively nurture students and encourage their independent problem-solving skills has been a more challenging issue to address.

In order to explore this issue, we investigated ways in which students enrolled in an advanced[‡‡] calculus course participated in weekly problem-solving sessions associated with the course. In particular, we sought to identify positive participatory actions and behaviors as well as the conditions that promote them.

Drawing upon recent research, we catalogued the various behaviors and strategies exhibited by groups of students and to collate these we searched for inspiration in frameworks already established. Perhaps one of the most apt was the taxonomy established by researchers Carlson and Bloom [1]. Their framework, an "Initial Problem-Solving Taxonomy" (see Appendix 1), was constructed from the salient research literature describing the behaviors that *experts* exhibit when engaged in problem solving. Carlson, Bloom and Glick [3, p. 278] focused on research mathematicians as the experts in their research, observing that these were "individuals with a broad, deep knowledge base and extensive problem solving experience". Furthermore, their study raises the legitimate question of how, when, and through what means these experts have learnt and acquired these skills. The answer to this is a pressing one in mathematical pedagogy.

On the one hand it is essential to focus on the delivery of skills-based techniques and cumulative mathematical knowledge. On the other and equally important is the cultivation of maturity and mathematical integrity needed to deploy those very skills and techniques to real problems students may face in the future. For this reason, the transition from cumulative knowledge acquisition to assimilating, synthesizing, and applying knowledge should be a clear component in course experiences at the tertiary level.

In collating our evidence, we found substantial overlap between the group of expert[§§] research mathematicians and our advanced groups; but significantly, we found additional behaviors not witnessed in the expert group that were pertinent to the positive participatory actions of mathematicians-in-training. This may offer insights for not only being better informed about the habits of mathematicians-in-training, but also for revisiting the behaviors of experts to formulate expanded models of the behaviors and strategies that they exhibit and employ.

We therefore developed an adapted model in the spirit of Carlson and Bloom [1], which we modified to reflect this distinct sample set of participants. As a result we can begin to formulate research-informed teaching practices that draw upon the exhibited strengths of mathematicians-in-training and that also seek to further promote expert behaviors that were not prevalent during the study we describe here.

2. Contextual Background

[‡‡] This is the "advanced" to which we refer in the title of the paper. In particular, students enrolled in the course represent the most mathematically able undergraduates at our institution (and for whom this course is required).

[§§] This is the "expert" to which we refer in the title of the paper.

With the development of the students as problem solvers in mind, problem-solving sessions, as a component of an advanced calculus course, were enacted. The problem-solving sessions, 50 minutes in length, were held on a weekly basis[***]. Once registered for a problem-solving session, students self-selected into groups that could be of any size of five students or less. The majority of students settled into groupings of three or four. Groups were not fixed from week to week, but overwhelmingly remained the same. Groups were given a handout (for example, see Appendix 2), typically with two or three multi-part problems and were instructed to select one to focus on for the duration of the session. Groups were also given the direction that the last five minutes of the session would be spent presenting their results to the whole class. The mode of presentation was unique, however, in that the students were forbidden from presenting the mathematical details of their discussion. Instead, groups were asked to provide an overview about the general approach to the problem and to describe the techniques they did or would use without going through these in detail.

Tutors, typically two or three years more advanced in their mathematical education than the students[†††], circulated through the classroom, assisting groups and facilitating discussions that were taking place. In doing so, tutors aided in opening up pathways for exploration when a group as a whole found themselves in difficulty.

Students were given outlines of solutions at the end of each session; however these too reflected the general philosophy of the sessions. The solutions provided were general in form, giving not necessarily the specifics of that individual problem, but rather the necessary information that would be useful and crucial to finding a solution.

Finally, we note that the semester-long advanced calculus class was composed of two distinctly different topics: multivariate calculus was the emphasis during the first term and solving differential equations was the emphasis in the second term. As such, the design and type of the problems during each term were quite different.

3. Methodology

3.1 Data Sources

Identification and description were key goals of the research. As such, the study was primarily qualitative in design and nature. Two types of data were collected:

> (1) Observational data from 10 weeks of problem-solving sessions (50 minutes each; five sessions for each 6-week term) for each of four small groups; and
> (2) Focus-group interview data from six students representing three of the four problem-solving groups observed during the study.

A total of five problem-solving sessions for each term were observed and the observational field notes were transcribed. Also, due to the primary researcher's absence during one of the sessions students gave permission for the session to be

[***] The course experienced an enrollment of approximately 85 students during the first semester of 2009. Consequently, students registered for one of four problem-solving streams offered during the week.
[†††] Course instructors also served as tutors during the problem-solving sessions.

recorded and the recording was transcribed. The observational data for each session was transcribed in such a way that every "problem-solving move" by students (i.e., performing calculations, organizing information, seeking consensus, asking questions) constituted a new line of transcription.

The focus-group interview was also recorded and transcribed and was conducted very much like a conversation with participating students responding to a collection of questions. The interview data are not the focus of this paper and only very brief excerpts will be shared in the Discussion section.

3.2 Participants

In the present paper we report on several important albeit initial findings of incorporating problem-solving sessions in an advanced calculus course. We selected one group from the four to focus on in the preliminary data analysis. The group was chosen because the members remained consistent throughout the ten sessions[‡‡‡], which was not the case with the other three groups observed during the course of the data collection. Additionally, only eight of the ten sessions were used in the preliminary data analysis. The first problem-solving session was eliminated from consideration because it provided only 12 codes, compared to an average of 41 from the eight sessions that were included. The seventh session was eliminated from the current analysis because the primary researcher was unable to attend the session, which was audio recorded and later transcribed. As a result of having access to an audio recording, every student utterance was captured during the seventh 50-minute session, as opposed to the primary researcher capturing everything she could during the others, while taking field notes without the aid of a recording device. The transcription of the seventh session was 275 lines long, compared with an average of 74 lines per transcription[§§§] of the eight sessions used for analysis. We plan to incorporate the analysis of this extensive data set in the next iteration.

Data from eight sessions of the selected group were analyzed using the framework created by Carlson and Bloom (see Appendix 1) by using line-by-line coding of the problem-solving sessions' transcripts. Using Carlson and Bloom's taxonomy, we determined the degree to which the students exhibited expert behaviors and we looked for patterns that would aid in explaining students' tendency toward particular behaviors. Our initial investigation indicates that the advanced calculus students exhibited a large number of the expert problem-solving attributes. Moreover, the results signaled the identification of attributes not included in the taxonomy, which we discuss in the next section.

4. Results

Data analysis was carried out by first identifying each problem-solving move as one of the behaviors or strategies found in Carlson and Bloom's taxonomy. This quickly

[‡‡‡] The group was composed of four female and one male student during the first term and three of the original four female students and the same male student during the second term.

[§§§] The transcriptions of the other sessions were completed from field notes taken during each observation. As previously noted, the problems provided to students during term 1 were qualitatively different from those during term 2. This could have caused a difference in the average lines per transcription between the first four sessions (average of 58) and the last four sessions (average of 89). The increase in lines of transcription could have also been influenced by the researcher's developing relationship with the group members over time.

revealed that the taxonomy did not include all behaviors exhibited by the students during problem solving. Consequently, we amended the taxonomy to include five additional behaviors. Four of the newly identified behaviors dictated that a new code, "questioning"[****], be inserted in the Carlson and Bloom taxonomy. The four "questioning" categories include:

- Questioning for assistance: student asks for guidance, advice, direction of another (or the whole group);
- Questioning for clarification: student asks for verification or affirmation of a particular aspect of the problem-solving work;
- Questioning to regroup, evaluate, or request status of the group's progress; and
- Questioning about or suggesting a method, tool, or direction to take in the problem-solving process

One additional category was added to the code for "Control: Meta-cognitive behaviors during problem solving". We labeled this category as "evaluating progress", either of self or group, of the problem-solving attempt, including the assessment of a given idea – either in the affirmative or negative.

4.1 Initial Findings: Trends

Each student problem-solving move for the group of focus was coded according to the appropriate categories from the Carlson & Bloom taxonomy [1]. We were first interested in obtaining a distribution of the different codes from the taxonomy identified per session and over the eight sessions as a whole (see Table 1). A count of the frequency to which the individual category behaviors occurred was collapsed into the overarching code. For example, the categories of "Knowledge, facts, procedures"; "Conceptual understandings"; "Technology"; and "Written materials" were collapsed into the overarching code, "Resources".

Table 1. *Distribution of Behaviors Among Eight Problem-Solving Sessions*

Session →	2	3	4	5	6	8	9	10
Codes ↓								
Resources	0	0	6	2	1	3	5	6
Control: Initial cognitive engagement	4	3	4	5	6	7	8	13
Control: Cognitive engagement during PS[a]	0	2	6	2	0	0	2	2
Control: Meta-cognitive behaviors during PS	8	6	12	8	8	13	9	8
Methods	2	1	6	5	8	3	3	0
Heuristics	5	8	3	1	1	0	2	4

[****] Since the data analysis is still preliminary as of the submission of this paper, we have not yet concluded whether it is appropriate for some or all of the "questioning" categories should be inserted as part of an existing code in Carlson and Bloom's taxonomy.

Session	2	3	4	5	6	8	9	10
Codes								
Attitudes/Beliefs/Emotions	1	0	2	2	5	0	3	3
Questioning	3	5	16	13	13	15	15	18
Total	23	25	55	38	42	41	47	54

[a] Problem solving

Upon initial inspection of this distribution we conjectured that a smaller number of codes identified in the analysis of sessions 2 and 3 was a result of the group adjusting to working together. Since students did not have formal experience with working on mathematics problems[††††] (Transcript, focus-group interview, May 14, 2009) as a group prior to the advanced calculus course, they often reverted to working individually during first several problem-solving sessions.

Although no other obvious patterns[‡‡‡‡] emerged from constructing the distribution, a deeper examination revealed several interesting trends that we share here, and which we acknowledge require further investigation.

4.2 Under-utilization of Methods and Heuristics

Carlson and Bloom [1] included in their taxonomy a number of distinct methods (3) and heuristics (10) that expert problem solvers use in the course of solving problems. Tables 2 and 3 provide the distribution of the instances of students' use of the various methods and heuristics, respectively.

Table 2. *Distribution of Students' Use of Methods*

Methods categories	S 2	S 3	S 4	S 5	S 6	S 8	S 9	S 10
Construct new statements or ideas	1	0	0	0	1	0	0	0
Carry out computations	1	1	6	5	7	3	3	0
Access resources[a]	0	0	6	2	1	3	5	6

[a] We coded this within the "Resources" code of the framework, of which there were four distinct categories. For Table 2 we collapse them to a single category; however, this total will match the value in the "Resources" code in Table 1.

Table 3. *Distribution of Students' Use of Heuristics*

[††††] Particularly with respect to the definition of 'problem' as preferred by Schoenfeld [4, p. 74]: "A doubtful or difficult question; a matter of inquiry, discussion, or thought; a question that exercises the mind".

[‡‡‡‡] A short disclaimer is appropriate here. There is no expectation for a certain number of each code to be observed during any given session. For example, there are several factors influencing whether or not students need to draw upon certain resources or methods while solving a particular problem. There is, however, the expectation that any given problem requires that the solver utilize a number of different strategies and to exhibit various behaviors that contribute to working toward a successful solution.

Heuristics categories	S 2	S 3	S 4	S 5	S 6	S 8	S 9	S 10
Work backwards	0	0	0	0	0	0	0	0
Observe symmetries	0	3	0	0	0	0	0	0
Substitute numbers	0	1	0	0	1	0	0	2
Represent w/picture, graph, table	1	3	2	0	0	0	2	1
Relax constraints	0	0	0	0	0	0	0	0
Sub-divide problem	0	0	0	0	0	0	0	0
Assimilate parts into whole	3	0	0	0	0	0	0	1
Alter given problem (into easier)	1	1	0	0	0	0	0	0
Look for counter-example	0	0	0	0	0	0	0	0
Investigate boundary values	0	0	1	1	0	0	0	0

Two significant observations are worth noting with respect to the methods and heuristics the students used. First, carrying out computations was slightly more prevalent than other methods in the students' problem-solving practices. Furthermore, an increased use of various resources occurred when problems lent themselves to using a graphical representation (e.g., problems from Session 4 and 10), were situated in a modeling context (e.g., problem from Session 9), or proved to be significantly more difficult for the students (e.g., problem for Session 10).

Secondly, table 3 raises awareness of the students' seeming lack of ability or inclination to draw upon the many heuristics that they most likely have practiced in their mathematical career. Schoenfeld [4, p. 95] would contend that practicing heuristic use is not enough. He stated that, "the successful use of such strategies calls not only for "knowing" the strategies, but for good executive decision-making and an extensive repertoire of subskills". The question remains whether the application of a variety of heuristic strategies was necessary for each of the problems the group worked on during the eight sessions. It seems realistic that in the cases where the students experienced difficulty (which was true to some degree for each of the problems), they could have relied on "altering the given problem"[§§§§] or "subdividing the problem" more frequently.

4.3 "Cycling Back" Phenomenon

The second trend we identified was the group's ability to "cycle back", a notion described by Carlson, Bloom, and Glick [3, p. 281] as the process of cycling through the "phases of *orienting, planning, executing,* and *checking* [that] are linked in a cycle, a cycle…executed repeatedly until [arriving] at a solution or [abandoning] the problem". In particular, if we look at the problem-solving efforts during sessions 9 and 10, the number of instances in which the group engaged in initial cognitive control behaviors and strategies increased relative to the previous sessions. Although many of the initial cognitive control strategies were utilized early in each session – as might be expected – the more complex problems the group selected prompted group

[§§§§] Both times the group "altered the given problem", the strategy was suggested by a tutor.

members to “cycle back” and use strategies such as “establish and represent goals and givens” or to “devise, consider, and select strategies” in the middle or late in the session.

The group’s tendency to engage in the “cycling back” was also indicative of their persistence with the problem-solving process, an implied belief that was present during observations of their engagement during each session.

4.4 Conservative Show of Attitudes, Beliefs, and Emotions

Carlson and Bloom [1, p. 48] reviewed the research literature regarding affective variables’ “powerful influence on the behavior of the problem solver”. Only a small number of the affective variables from Carlson and Bloom’s taxonomy were explicitly observable. In fact, no particular attitude (e.g., enjoyment, motivation, and interest) was discernable.[*****] The beliefs and emotions for which we have evidence included pride, persistence, and multiple attempts needed in problem solving (beliefs) and frustration, anxiety, and joy (emotions). Even the instances of these were few in number, however. Considering the data from a broader perspective, however, we observed a strong belief in persistence (when solving problems) in each of the group members and in the group as one collective body. Indeed, the students held the belief that the problem they selected deserved their attention and effort to see it through to a solution. After learning early in the sessions that a solution may not be achievable in the 50-minute session, the group became more focused on the search for a potential solution as opposed to just an answer. Consequently, each student in the group held an individually strong belief in persistence and a collectively strong belief as they worked. In addition to the collective persistence observed during the problem-solving sessions, three of the six students participating in the focus-group interview admitted to revisiting incomplete solutions for the selected problem attempted in the sessions beyond the sessions themselves.

The strength of the students’ persistence during the problem-solving component of the advanced calculus course provides conflicting evidence for one concern held by Schoenfeld [5] and others. Schoenfeld found that students came to “expect the problems they were asked to solve to yield to their attempts in just a few minutes, if at all” and that “prior experience shapes the amount of time and effort that will be invested in this problem” [5, p. 341]. For this group, the students did not subscribe to this belief at all. There was no hint of an individual or the whole group abandoning their problem-solving efforts altogether; instead, persistence provided the fuel for them to consider alternative methods and in the case with difficult problems, “cycle back” to a more initial stage in their solution process.

4.5 Significance of Questioning

The need to include the questioning code with four distinct categories arose out of the identification of a behavior (or strategy) that each group member exhibited on several occasions. Examples include:

> Shelly[†††††]: Does anyone have a clue if I’m doing this right? (Transcript, April 3, 2009, questioning for clarification)

[*****] We intend on looking into the distinction between attitudes and emotions more closely when we complete the data analysis that includes the other three groups.

[†††††] All student names are pseudonyms.

> Greg: Have we written this in polar coordinates yet? That means we need two parameters. (Transcript, March 27, 2009, questioning for obtaining status of group work)
>
> Shelly: Does anyone know how to do Laplace transforms on Maple? (Transcript, May 15, 2009, questioning for assistance)

We found that there were subtle differences in the categories we needed to insert in the framework and the existing cognitive and meta-cognitive control behaviors and strategies. Thus, in Shelley's question from April 3, 2009 we decided that there was something else underlying the question than a need to verify results [1]. In her question, Shelley asked for not only a confirmation of her result, but she requested an affirmation of her work[‡‡‡‡‡] and an explanation of why her calculations were appropriate.

Thus, we expanded the framework to include a collection of "questioning for…" categories to highlight the strategies we observed during the problem-solving sessions, but which did not fit well in other categories. The importance of group members questioning each other – for a variety of reasons and through several phases of the problem-solving process – was a strong contributing factor to the collaborative atmosphere within which this group worked. Furthermore, we have only examined the presence of the questioning strategies with the group studied for the purposes of this paper. We note that additional confirmation (or rejection) of these behaviors will take place when we consider the remaining groups' transcripts. Also, after further analysis we will be able to determine whether the categories should be included within existing codes, e.g., as a resource (asking for confirmation from a peer) or meta-cognitive control behavior (similar to reflecting, for example).

5. Discussion

The definition of mathematical problem solving, and classroom practice that includes it, has been the topic of discussion and research endeavors for mathematicians, mathematics educators, and educational researchers for decades [1]. We set out to investigate ways in which students enrolled in an advanced calculus course also participated in weekly problem-solving sessions associated with the course. The initial results described here prompted the identification of two key findings that will apply in similar efforts undertaken at other institutions. Furthermore, we anticipate that continued data analysis efforts will support these findings and reveal others to contribute to undergraduate mathematics teaching and learning.

Much of the research on problem solving has focused on students for which mathematics poses many cognitive and learning obstacles. For example, Ebert and Mwerinde [6] conducted research on students in a College Algebra and Statistics course in which they identified the problem-solving behaviors of students both in individual and cooperative learning group contexts. Our research, however, was focused on examining academically-able (i.e., "advanced") students' participation in

[‡‡‡‡‡] At this point the group was working on parameterization and calculating quite complex partial derivatives.

problem-solving sessions. In particular, we are interested in relating the behaviors and actions we observed with those of “expert” problem solvers, as delineated in the appropriate research literature. Although these are only initial results, we highlight two lessons we have learned thus far.

5.1 The Role of Problems

We claim that the students’ limited use of various methods and heuristics is due in part to the nature of the problems given during the weekly sessions. Each week, groups selected one of either two or three problems to work on for the majority of the 50-minute session. Although several of the methods and heuristics identified in Carlson and Bloom’s [1] taxonomy were attended to by the group of students discussed in this paper, still other methods and heuristics were highly appropriate but not employed. We contend that to afford students increased opportunities to draw upon additional methods and heuristics is to consider the nature of the problems themselves. The problems selected or designed by course instructors must lend themselves to promoting the behaviors and actions of successful problem solvers.

5.2 The Role of Modeling Successful Behaviors

It is critical that instructors do not take for granted what academically-able or advanced students have acquired with regard to problem-solving skills and sub-skills, as well as the vital decision-making abilities to apply the methods and strategies within problem-solving contexts. One way to address this concern is to model the methods and heuristics identified by Carlson and Bloom [1] when including examples in course lectures. Although the use of the caliber of problems needed for a problem-solving component is not appropriate for course lectures, brief examples in which a different subset of the methods and strategies are highlighted is appropriate. Using the methods may not be sufficient, however. Instead, making the particular method or heuristic used in the example explicit could impact students’ future problem-solving experiences. As part of a research study to compare undergraduate students’ problem-solving abilities with those of experts, Bloom [7] promoted instructional strategies – during class discussions – that enabled undergraduates to engage in practices that were more expert-like.

We anticipate that subtle instructional changes that support an increased use of heuristics and methods will also impact the collaborative atmosphere that was vital to the group we reported on here. Furthermore, the changes have the potential to influence the use of efficient control strategies, which students need “in order to be able to use their heuristic resources” [4, p. 114].

6. References

[1] Carlson, M.P. and Bloom, I. (2005) The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educ. Stud. Math.* **58**:1, pp. 45 – 75.

[2] Biggs, J. (2003) *Teaching for Quality Learning at University*, Open University Press, Buckingham, UK.

[3] Carlson, M. P., Bloom, I., and Glick, P. (2008) Promoting effective mathematical practices in students: Insights from problem solving research. In M. P. Carlson

and C. Rasmussen (eds.) *Making the Connection: Research and Teaching in Undergraduate Mathematics* (pp. 275 – 288), Mathematical Association of America, Washington, DC.

[4] Schoenfeld, A. H. (1985) *Mathematical Problem Solving*, Academic Press, San Diego, CA.

[5] Schoenfeld, A. H. (1989) Explorations of students' mathematical beliefs and behavior. *J. Res. Math. Educ.* **20**:4, pp. 338 – 355.

[6] Ebert, C., and Mwerinde, P. (2002) Delineating the relationship between problem-solving behaviors and achievement of students in cooperative-learning groups. In M. H. Ahmadi (ed.) *Readings in Innovative Ideas in Teaching Collegiate Mathematics* (pp. 43 – 55), University Press of America, Lanham, MD.

[7] Bloom, I. (2002) *The problem solving behavior of mathematics majors*, Twenty-fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA.

7. Appendices

Appendix 1

Outline of Carlson & Bloom (2005) Taxonomy

Overarching Code	Categories Include:
Resources	Accessing knowledge, procedures, conceptual understandings, technology, or written materials
Control: Initial cognitive engagement	Read/understand problem, organize information, establish and represent goals
Control: Cognitive engagement during	Evidence of sense-making, effort to stay mentally engaged
Control: Meta-cognitive behaviors during	Reflect on efficiency and effectiveness of cognitive activities or selected methods, generate conjectures, verify processes and results, engage in internal dialogue
Methods	Construct new statements or ideas, carry out computations, access resources
Heuristics	Work backwards, observe symmetries, substitute numbers, relax constraints, sub-divide problem, alter given problem, look for counter-example, investigate boundary values
Attitudes, beliefs, emotions, values	Enjoyment, motivation, interest, self-confidence, pride, persistence, frustration/anxiety, joy/pleasure, impatience/anger, mathematical integrity

Appendix 2
Sample of Weekly Problems

Differential Equations – Ants
Problem 1
Solve the following equations by applying a change of variables (*i.e.* $y(x) = f(v(x))$)

$$\frac{dy}{dx} = \frac{y}{2} + \frac{x}{2y},$$

$$\frac{dy}{dx} = xy^2 - y,$$

$$\frac{dy}{dx} = 1 + xe^{-y}.$$

What other equations can you come up with that could be solved with by a similar method? Can you write down a general equation (and its solution) that you could solve using the same method?

Problem 2
Consider an ant walking across a factory floor in a straight line. For each scenario below, write down the differential equation that describes the ant's position and sketch how the position changes with time.

i. The ant walks at speed v_0.
ii. The ant gets tired the further it walks and its speed decays exponentially
iii. The ant steps onto a conveyor belt that oscillates backwards and forwards.
iv. Both ii and iii.

If the belt is of length L will the ant manage to get off the other end?

Problem 3
It is well known that where there are ants there will soon be more ants. Presuming ants give birth at rate *r* write down a differential equation for the number of ants *N(t)* at time *t*. Under this model what happens to the population as *t* gets large?

A more realistic model supposes that the environment can only sustain *K* individuals and as the population gets closer to this limit the birth rate decreases. If $\dot{N} = f(r, K, N)$, and at *N=0* and *N=K*, $\dot{N} = 0$ and between these two limits there is a positive birth rate, what is the simplest function *f* that satisfies these conditions? Can you solve your model? Does the solution do everything you expect?

Finally, an ant's life is not a secure thing and various factors contribute to their rapid decline. If average life expectancy of an ant is d days, how would you include this in your model? How does the solution change for different parameter values?

"It's not my job to teach them this stuff!" Development of school mathematics skills within the tertiary environment

TRACY S. CRAIG[§§§§§]

Academic Support Programme for Engineers in Cape Town (ASPECT), Faculty of Engineering and the Built Environment, University of Cape Town, South Africa

Abstract: This paper reports on a year-long initiative in a first-year engineering mathematics course to teach and assess school-level mathematics skills within the tertiary environment. The "school mathematics" teaching and assessing strand within the course was initiated in response to both anecdotal expressions of concern on students' school leaving mathematics skills and formal responses to a questionnaire completed by approximately half of the academic staff in the engineering faculty concerned. A weakness among the students in school-leaving mathematics skills coupled with a lack of lecturers willing or able to take responsibility for the development of such skills results in undergraduate students who continue to develop their mathematical knowledge into more advanced topics while retaining fundamental weaknesses in topics such as log laws and basic trigonometry. This paper reports on a series of lectures and assessment tasks within an otherwise standard first-year university mathematics curriculum to address and meet this need. Study materials as well as contact time with lecturers and tutors were provided prior to each assessment task. Multiple re-assessment opportunities were provided, with a high final grade required as a course requisite. Crucial and often startling weaknesses were perceived among a majority of the students during the year and the intervention went some way towards developing the necessary skills. The intervention achieved a pleasing degree of success, both diagnostic and constructive. 2009 sees the pilot of this exercise, which will be improved upon in future years, building on the observations reported here and elsewhere.

Keywords: mathematics education; foundation education; remediation; engineering mathematics

1. Introduction

It is a widely acknowledged challenge in the teaching of tertiary mathematics that students entering university are often weak in the mathematical skills and knowledge that are prerequisites for success at university level mathematics [1], [2], [3]. Such skills and knowledge include basic algebra, basic curve sketching, logarithm laws, basic geometric manipulations and trigonometric identities. These weaknesses, if not addressed, have a twofold effect: any mathematics developed at university which builds upon those skills (such as advanced curve sketching) is further weakened and advanced fields of study (such as structural engineering) which rely primarily on school mathematics are affected.

The aim of this paper is to discuss the tensions encountered in a course initiative directed at teaching and assessing school mathematics skills within a tertiary mathematics course. Is it possible to teach a standard first-year university mathematics course while simultaneously strengthening and developing mathematics skills supposedly taught during the students' pre-university schooling? Is it, in fact, at all necessary to teach school mathematics at university level? In the intervention

---

[§§§§§] E-mail: Tracy.craig@uct.ac.za

discussed here, “school mathematics” was defined as skills and knowledge expected to have been learned at school, specifically ones which can be clearly named and defined in terms of equations or algorithms, for example factorisation, log laws, trigonometric identities and geometric facts or theorems. Skills which are desirable yet not describable by an equation or algorithms, such as “critical thinking” or “problem-solving” were not included in the definition.

What role does a first-year mathematics course play in an engineering faculty? If that role is to supply the students with the necessary general mathematical skills to deal with future and concurrent studies (and the more advanced mathematics they will learn there) and a passing grade is seen as an indicator of satisfactory levels of prowess in those skills, then it is vital that the school mathematics skills required by the faculty either be satisfactory upon university entry or be advanced to a satisfactory level during this first year.

This paper contributes to the ongoing conversation by reflecting on the tensions observed and experienced in one intervention designed to develop and assess mathematical knowledge and skills expected to have been first encountered in the pre-university years. It is the thesis of this paper that any such intervention will not achieve maximal success without explicit attention paid to these tensions and others.

The rationale for the intervention is first discussed, specifically two activities run by the author in two consecutive years, namely a faculty survey and a weakly formulated course-integrated intervention. The context for the study is described, that is an academic development stream of an otherwise traditional first-year engineering mathematics course. The methodology is described as well as the topics covered in the activities. Results are examined and particular points of interest are made. The tensions discerned to exist in and around such an intervention are reflected on and discussed, and finally conclusions are drawn as well as implications for further study.

2. Rationale

In 2007 the author ran a survey in the Faculty of Engineering and the Built Environment (EBE), within which faculty she taught mathematics at the time and does at the time of writing. The survey asked lecturers of courses within the faculty what mathematical skills and knowledge were used in their courses and where they perceived mathematical weaknesses to be. 40 of the 75 lecturers approached responded to the survey [4]. Of those 40, 23 lecturers responded that their courses required use of mathematics understood and expected to have been taught at school. In decreasing order of extent of use in EBE the following were reported as employed in engineering studies: trigonometry, algebra, geometry, arithmetic, introductory differential calculus, ratios and proportion, graphs, logarithms and inequalities. In addition, 6 lecturers reported that weaknesses had been observed in the areas of similar triangles, ratios, logarithms, straight lines, trigonometry and algebra. In survey responses the criticism was levelled at first-year mathematics lecturers of giving a passing grade at the end of first-year to students who are not proficient in school mathematics. One of the several aims in running the survey was to discern what should be taught to the first-year students in order to meet the needs of their future studies and how those topics should be taught. It was clear that strengthening any potential weaknesses in school mathematics was an issue that needed to be addressed.

In 2008 the author ran an assessment strand in her course referred to as "Mini Tests". Every Friday a brief (20 minute) Mini Test (MT) would be run assessing work covered during that week. Twice during the year, instead of assessing current work, certain school mathematics skills were assessed, specifically factorization, log laws and trigonometric identities. The procedure followed in class in 2008 was to highlight any appearance of a clearly defined "school mathematics" skill, such as completion of the square or use of a log law. The skill, rule or concept was named, explained and its presence within the new material made obvious. The need to know this material was emphasised and, where possible, a page in the textbook would be given as reference. By the time of the first Mini Test of interest in this discussion, the class had covered the areas of functions, limits, continuity and differentiability, differentiation rules and some applications of differentiation, notably curve sketching. "School mathematics" knowledge and skills encountered included algebra, basic curves, trigonometry and logarithms as well as a small amount of geometry.

The students were clearly informed of which laws and identities would be assessed in the Mini Test and the relevant sections of the textbook were assigned for learning. Results were disappointing with as few as 20% of the class answering some fundamental questions correctly. In response to the results, a worksheet of exercises designed to strengthen the weaknesses which had been perceived was given to the students to work through and the same work was re-assessed a few months later. In the intervening time, the explicit procedure of emphasising the place and importance of previously encountered knowledge was continued. In the second Mini Test assessing school mathematics skills, the response was substantially worse than before. For example, in the question

Rewrite $\log_2 3$ in terms of logs of base 10 (Mini Test 5)

Rewrite $\log_2 3$ in terms of logs of base *e* (Mini Test 14)

88% of the class (66 of 75) answered correctly in MT5 and 13% (8 of 64) answered correctly in MT14. In the question

Factorise $(x^3 - 8)$ (Mini Test 5)

Factorise $(8 - x^3)$ (Mini Test 14)

79% (59 of 75) answered correctly in MT5 and 48% (31 of 64) answered correctly in MT14. The direction to "write out the three standard trigonometric square identities" was answered correctly by 92% and 78% of the class respectively in the two Mini Tests.

Two observations can be made from these results. First, certain basic skills and knowledge accepted by lecturers across the faculty are not known or understood as much as they might wish. Secondly, relying on the students to do their own revision either without assessment or for assessment which is not high-stakes (such as these Mini Tests) cannot be expected to meet with much success [5]. In short, the students do not value the specific learning being encouraged. In the author's experience, many undergraduate mathematics lecturers, upon realizing their students' lack of proficiency with certain school mathematics skills, urge the students to engage in private study aimed at shoring up those weaknesses. The lecturers value the actual mathematical knowledge and skills, but not to the extent of allowing the teaching and development of those skills to become their responsibility. Bahr [3] refers to lecturers finding remediation "demoralizing" (3, p. 422)

The students who were registered for the academic development course described here were students who entered the university with lower Grade 12 results than the mainstream students (in the same engineering department). It is vital to note, however, that weak mathematical skills upon university entry are ubiquitous across the faculty. The National Benchmark Test (NBT) results reveal that only a small percentage of university entrants display a degree of mathematical proficiency considered sufficient to succeed at university without support beyond a standard curriculum [6]. Since the academic development stream within the faculty comprises less than 10% of the student body, widespread mathematical weaknesses reported by the teaching staff in the 2007 survey are likely to have been observed within the mainstream student cohort. While the academic development stream undoubtedly does include students who enter university with mathematical weaknesses, they are not alone in this respect.

2.1. *Context*

The author is a lecturer of mathematics in an academic development programme (ASPECT) at the University of Cape Town, South Africa. The practicalities of this particular programme are such that we have approximately twice the contact time with the students in first-year mathematics and first-year physics, with the students doing certain first-year subjects in their second year of study. The subject content covered in the course is identical to that of the mainstream course for which the bulk of the first-year engineering students are registered, however the extra time allowed for by the ASPECT timetable provides us with the ideal opportunity for pursuing any issues of interest that are not explicitly part of the first-year curriculum, in this case school mathematics.

In 2009, the students in the class were initially 56 students who had not made the minimum registration requirements but were considered to have the potential to succeed at university studies. After the first quarter, students from mainstream engineering studies were allowed to transfer to the course bringing the numbers up to 80 at final registration. The university entrance requirements work on a “points” system based on school-leaving results. The students in the course being discussed include students who have a wide range of points; in addition, the different engineering disciplines place their ‘cut off’ at different point levels due to their physical capacities and the number of students applying. These two facts, taken in combination with the fact that school mathematics skills are found to be weak across the board, mean that the students in the course represent a typical student population, not one which can be considered to be especially weak in mathematics compared to other groups [3]. It is important to note, however, that the view of the lecturer is not necessarily the view of the students or of other lecturers in the faculty. The students are heard to describe themselves as less academically capable than the mainstream students and other lecturers in the EBE faculty refer to ASPECT students as academically inferior despite evidence that this view is insufficiently nuanced. It is into this landscape of disparate views and conflicting measures of value that the “school mathematics” intervention was launched.

3. Methodology

Nine school mathematics activities were run during the year. Each took the form of a class or workshop on a Monday morning with the students being presented with a summary of the necessary information on the topic as well as a series of exercises.

The nature of the class differed according to the topic. For instance, absolute values were essentially a new topic, so a formal class was first held introducing the concept and discussing the algebraic and graphical implications. Thereafter an interactive workshop began wherein the students could grapple with the concepts and exercises individually and in groups. Trigonometric identities, in contrast, were well known to the students. The students were able to immediately engage with the exercises and activities of the workshop.

The students had opportunity during that Monday class to engage with the topic, ask the lecturer for guidance and begin working on the exercises. During the week the student was expected to work through the full series of exercises, with the lecturer available the majority of the time in her office for help. On the Friday morning a 20 minute test was held on the topic, focussing explicitly on fundamental skills and knowledge, only exercises were included with no “problem-solving” involved or intended [7]. In order to not clash with other assignments in the course, the school mathematics activities were located in weeks 2, 4, 5, 7, 9, 11, 12, 15 and 18 of the 26 week course schedule. At the beginning of the year the students were presented with a course handbook which included all the school mathematics worksheets, thereby allowing the students to work ahead should they have the time and inclination.

The topics covered were algebra (factorisation, logarithms, absolute values), simple curves, trigonometry (basic concepts, identities, graphs) and geometry (triangles, parallel lines, circle geometry). Those familiar with the new National Senior Certificate curriculum in South Africa [8] will be aware that to call the topics listed above “school mathematics” is a misnomer in that absolute values are no longer included in the school syllabus and geometry is covered patchily, with some schools dealing with geometry in depth and others very shallowly. The previous school curriculum resulted in university lecturers having certain expectations of what school leavers could be expected to have covered and it is one of the challenges of the new school curriculum that those expectations have to change. It is not widely known among lecturers that absolute values are no longer covered in the school syllabus, as evidenced by private communication between myself and colleagues, therefore the topic certainly falls into the category “mathematics your lecturers will expect you to have covered at school” if not “mathematics you have covered at school”.

The school mathematics tests were high stakes in that an average result of 80% or more was required to obtain permission to write the final examination. The nine activities were planned for the first three quarters of the university year, with opportunities for returning to the different topics and rewriting the tests planned for the fourth quarter. Tests may be rewritten as many times as necessary.

4. Results and observations

This paper is being written after all the original tests have been run, but before any of the rewriting opportunities. The results before rewriting are summarised in the table below.

Table 1. Test topics and results

		Percentage of class		
Test no.	Topic	Above 80% for test	Above 50% for test	Average result

1	Factorisation	65	88	80
2	Logarithms	64	98	83
3	Absolute values	50	90	74
4	Basic curves	56	89	73
5	Basic trigonometry	39	85	71
6	Triangles, Parallel lines	29	76	65
7	Trigonometric rules	36	74	63
8	Circle geometry	15	80	60
9	Trigonometric graphs	55	81	76

The portion of the class with over 80% average on this section overall is 28% at time of writing, that is, only 28% of the class (22 students) currently have permission to write the final examination. This quantity should rise significantly over the final quarter of the teaching year.

It is perhaps unsurprising that the weakest results are clustered in geometry and trigonometry. The new South African school curriculum has weakened the presence of geometry primarily by placing the Grade 12 geometry assessment in a third and optional final paper [9]. The response of schools to this situation has been uneven, although Para [10] reports that schools not teaching geometry predates the new curriculum. Certain schools have continued to teach Grade 12 geometry while other schools have weakened the teaching of geometry from Grade 10, resulting in students in the same university classroom having wildly different levels of geometry preparedness. In a subject such as engineering mathematics with its significant geometric demands, such differential preparedness is a serious challenge. While the student performance on the different topics is of interest, it is beyond the scope of this paper. Results are given here to illustrate the need for the intervention. It is the aim of this paper to reflect on the tensions which came to light during the running of the intervention.

5. Discussion and reflection

Experience with the intervention discussed here has revealed three primary tensions. There is the tension of time: spend class time on topics encountered at school or spend time on new topics specific to the course? There is the tension of value: is the activity valued by the instructor? Is it valued by the students? How can these skills be explicitly defined as something to be valued without somehow stigmatising them? Closely linked to the issue of value is the tension of how deeply to integrate school mathematics development within the course. There are advantages and disadvantages associated with the two different approaches of integration within the standard course and separation from the course curriculum.

The first tension which is immediate and obvious is that of time. Time spent on school mathematics is time not spent on university mathematics. Arguments in favour of spending time on school mathematics are the ubiquity of such skills in advanced courses coupled with the diagnosed weaknesses in such skills. School mathematics is used by the majority (23 out of 40) of lecturers responding to the survey discussed above. If the bulk of the faculty uses school mathematics skills and those skills are not well developed upon university entry, then it is the job of the first-year mathematics lecturer to teach those skills. School mathematics is used within mathematics learned at university, for instance advanced curve sketching requires knowledge developed during elementary curve sketching, and integration problems require facility with trigonometric identities and completing the square, to name just a few examples. If university mathematics is to be built on top of skills which are not well-developed, it is time well spent to bolster those skills.

Arguments opposed to spending time on school mathematics are the full first-year curriculum and a natural reluctance to reteach topics which are understood to have already been covered in some depth. The first-year curriculum is extremely full, covering differential and integral calculus (techniques and applications), differential equations, vector geometry, linear algebra, complex numbers and finite and infinite series, all in only 96 lectures (in the case of first-year engineering mathematics). One aim of the survey was to determine whether there was anything in the first-year syllabus which was superfluous and this was found to not be the case, primarily because the course is taught to student engineers in all the major disciplines resulting in a wide range of necessary mathematical topics. There are lecturers of first-year students who simply do not see the teaching of school mathematics as part of their job. Reports of such reluctance do not seem to be prominent in the literature (however, see [3]), yet the author is familiar with this attitude and would hesitate to consider it specific to her institution alone. While one can find sympathy for a lecturer who is forced to choose between teaching course work and engaging in remedial activities, it is an unarguable fact that many students entering university are poorly prepared for the mathematical demands that will be made of them; if the necessary mathematical development is not to be done in the first year then when is it to be achieved?

An additional tension is that of perception and hence of value. There is tension between the perception of the lecturer and the perception of the student. There is tension between the perception of the activities as valued and perception of the activities as remedial and hence valueless. An interesting illustrative example is the situation in which the author finds herself. The author's course is an academic development course with a student body in part defined by their low entry level points to university. Many lecturers, including the author, have sufficient experience to know that entry level points are a weak predictor of success at university [11], yet the students can be forgiven for not realising this and for considering themselves in some way academically inferior to the students who were accepted for mainstream studies. The waters are further muddied by the fact that each engineering discipline has a different "point cut-off" responsive to number of applications and physical resources, resulting in a wide range of points achieved by the students in the class. Before introducing anything out of the ordinary into the course, there is a tension between the lecturer's perception of the students' ability and the perception of the students themselves. The opportunity to pursue the school mathematics strand of teaching and learning within the course without impacting heavily on the time available to teach the standard course syllabus was seen as a luxury by the lecturer and as something to be greatly valued. Seen from the students' point of view, however, particularly if they already regard themselves as academically inferior, the fact that this class is being subjected to (re)learning remedial mathematics while the mainstream students are not is further evidence of their mathematical ineptitude; the activities become something confrontational and possibly demoralising, definitely not something to be valued.

Non-inclusion of such an intervention in mainstream has more to do with timetable constraints than with knowledge and ability.

There is an inherent tension in the issue of whether or how much to integrate prior mathematics skills development into a tertiary curriculum. On the one hand, keeping the developmental activities separate from the rest of the course activities allows for certain topics to receive dedicated attention and focus. In addition, the topics can be easily assessed since they have been separately defined from other, related, topics and can be purposefully timetabled. The disadvantage of such separation is a lack of development of the necessary understanding that these fundamental skills are inseparable from the skills developed elsewhere in the first-year curriculum. Separation of activities can lead to the perception that there is some hierarchy of value with activities seen to be remedial being of least value. The disadvantage of integrating certain types of knowledge into the standard first-year curriculum (as indeed such knowledge is necessarily integrated) is that the students do not perceive its importance. It was the finding of the 2008 Mini Tests that making the presence, importance and need for study of certain types of knowledge explicit has negligible effect on student performance. To design a successful foundational intervention, it is necessary to balance the need for focus and directed assessment with the need for subject integration and a coherent learning experience.

6. Conclusions

The indications that some sort of remedial support is necessary within the traditional standard first-year syllabus are many and varied. The experience of other universities, as reported in the literature, suggests that weak mathematical grounding in first-year undergraduates is widespread. In 2007 staff of the EBE faculty reported weaknesses in skills expected to have been developed before university studies and in 2008 a weakly formulated intervention designed to make explicit the presence of school mathematics skills within the course topics failed to achieve any improvement in the necessary skills. It was and remains clear that some form of active engagement with clearly defined school mathematics topics is necessary to achieve any degree of improvement in those skills. Simply pointing out their presence and encouraging their study is insufficient.

The tensions inherent in developing such an intervention need articulation and addressing before the intervention can be fully successful. The issue of time, while easily articulated is less easily addressed. The standard syllabus in the traditional first-year engineering mathematics course is densely packed with new material. In order to find the time to fit more into the syllabus, however organised, would either require a decrease in the standard syllabus or an increase in the amount of time. The course described here fits the latter description. It is important that any intervention designed to develop and strengthen school mathematics skills be valued by both the lecturer and the students. Such value could be encouraged by an understanding by both parties that the skills are necessary, that they require further development beyond that experienced before university entry and that any observed weaknesses are likely to be apparent across the first-year cohort and not restricted to a small subsection of the cohort. The degree to which activities aimed at developing school mathematics skills are integrated into the standard curriculum is problematic. It is the conclusion of the author that the ideal level of integration falls between those imposed in 2008 and 2009, that is between entirely separate and entirely intertwined.

It is clear that activities within a standard first-year curriculum that develop mathematical skills and knowledge first encountered before university entrance have a role to play in general undergraduate mathematics education. This paper discusses one such intervention and reflects upon the tensions which were apparent in and around it. Future versions of this intervention will attempt to intertwine school mathematics instruction with course material while retaining emphasis on the fact that skills previously left undeveloped cannot continue in that state.

7. References

[1] J. Du Preez, T. Steyn and R. Owen, *Mathematical preparedness for tertiary mathematics – a need for intervention in the first year?* Perspectives in Education, 26, 1 (2008), pp. 49 – 62.

[2] R. Eiselen, J. Strauss and B. Jonck, *A basic mathematics skills test as predictor of performance at tertiary level*, South African Journal of Higher Education, 21, 1 (2007), pp. 38 – 49.

[3] P.R. Bahr, *Does mathematics remediation work? A comparative analysis of academic attainment among community college students*, Research in Higher Education, 49 (2008), pp. 420 – 450.

[4] T.S. Craig, *Mathematics discourse needed for developing engineers – a survey*, For Engineering and Science Educators, 11 (2007), pp. 18 – 19.

[5] P. Ramsden, *Learning to Teach in Higher Education*, Routledge Falmer, New York, 1992.

[6] National Benchmark Tests Project, Presentation to the Parliamentary Monitoring Group 19 August 2009. http://www.pmg.org.za/files/docs/090819hesa-edit.pdf, last accessed 31 August 2009.

[7] A.H. Schoenfeld, *Mathematical Problem Solving*, Academic Press, inc., Orlando, Florida, USA, 1985.

[8] South African Department of Education, *National Curriculum Statement Grades 10-12 (General) Mathematics*, Government Printer Pretoria, 2003, http://education.gov.za/Curriculum/Substatements/Mathematics.pdf, last accessed 31 August 2009.

[9] South African Department of Education, *Mathematics Examination Guidelines Grade 12 2009*, Department of Education, 2009, http://education.gov.za/Curriculum/Examination/Mathematics/Mathematics%Gr%2012%20Exam%20Guideline%202009%20(Eng).pdf, last accessed 31 August 2009.

[10] J. Para, *Crisis in mathematics education. Student failure: challenges and possibilities*, South African Journal of Higher Eduction 15, 3 (2001), pp. 66 – 73.

[11] C. Bohlmann and L. Fletcher, *Diagnostic assessment for mathematics in a distance learning context*, South African Journal of Higher Education, 22, 3 (2008), pp. 556 – 574.

Period Doubling and Not

TEMPLE H FAY[******]

Department of Mathematical Technology, Tshwane University of Technology, Pretoria

Abstract: Interesting motions and other phenomena including periodic steady state solutions (both harmonic and subharmonic), almost periodic solutions, aperiodic solutions are some of the features of the forced Duffing equation

$\ddot{x} + k\dot{x} - x + \varepsilon x^3 = F\cos\omega t$

that can be observed through numerical investigations (in our case, using the computer algebra system *Mathematica* version 5.1). We focus on solving numerically the differential equation for a variety of values of the forcing amplitude parameter F holding all other parameters fixed and holding the initial conditions to be the at rest values zero. Steady states of solutions are usually periodic having period a multiple of the period of the forcing. Period doubling is an effect that is prominent which has been called "a route to chaos"; however this "route" is usually in reference to discrete dynamical systems and not for an ordinary differential equation. Our investigation show period doubling and other period modifications, but little to indicate what one might call chaotic behaviour. By and large, the solutions are well behaved, numerically stable. Issues involving numerical solutions are discussed.

1. Introduction

This article is an extension of the investigation begun with [3] and thus may be considered as a companion article. The idea is to investigate the nonlinear dynamics of the Duffing equation $\ddot{x} + k\dot{x} - x + \varepsilon x^3 = F\cos\omega t$, and to indicate that not only the (perhaps expected) period doubling phenomenon occurs but changes from period 3P to period 5P occur as well and this change can be very abrupt and smooth. Wishing to not repeat much of what is reported in [3], we still include some preliminary material that is required for appreciation of numerical techniques in the hope that this article will be self contained and accordingly somewhat independent of [3].

The nonlinear equation $\ddot{x} + k\dot{x} + \mu x + \varepsilon x^3 = F\cos\omega t$ was introduced by G. Duffing in his thesis [2] in 1918. If μ is positive, then the equation can be thought of as representing a simple coil spring. The terms $-\mu x - \varepsilon x^3$ represent a cubic restoring force, $-k\dot{x}$ represents a viscous damping force (here k is assumed positive) and $t, x(t), \ddot{x}(t)$ and $\dot{x}(t)$ are interpreted as time, displacement, acceleration and velocity respectively. If the parameter ε is positive, then the spring is called *hard* as all solutions are bounded, oscillatory, and periodic; if ε is negative, then the spring is called *soft* and solutions can be unbounded, periodic or almost periodic [4], [5] and [6]. These spring equations when being solved numerically are generally numerically stable and well behaved.
On the other hand, if μ is negative, then Duffing's equation is more numerically sensitive and claims have been made that there is chaotic behaviour. In the article [3],

[******] E-Mail: thfay@hotmail.com

attention given to the phenomenon of period doubling for the Duffing equation $\ddot{x}+k\dot{x}-x+\varepsilon x^3 = F\cos\omega t$. The parameters $k, \varepsilon,$ and ω were held fixed and steady state solutions to the at rest initial value problem were investigated. Many (but of course not all) of the amplitude values F where a bifurcation occurs and the steady states change behaviour, say going from period $P = 2\pi/\omega$ to period $2P$ (period doubling) were identified.

In this article, we will indicate that there are many F-intervals over which the steady states have periodic response and that occasionally the transition between one interval of periodic response and another is accompanied by a transition interval over which the response appears to have no steady state; that is, the steady state may be almost periodic or appear to be aperiodic. The transition from predictable behaviour to unpredictable behaviour may be very abrupt and the transition back to predictable behaviour may be equally as abrupt, often such changes occurring for variations in the value of F in the third decimal place.

There appear to be several definitions of chaos (see [1]) and many textbooks devoted to chaos really do not define the term (see [12]). There seems to be a general agreement that sensitivity to initial conditions is one requirement, but not the only one. However, in many expositions this is exactly what is cited to declare solutions to a dynamical system to be chaotic. So the term chaos appears to be often used when observations (trajectories in the phase plane) are unpredictable and there is no apparent regularity or order. Our investigation finds generally smooth transitions from one periodic response to another, and we have no prejudice against aperiodic or almost periodic responses whose trajectories can appear very messy.

2. What is zero?

When one deals with numerical algorithms, one must decide how many decimal places of accuracy yield a satisfactory result; seldom does a computer return the result of 0 but rather returns a decimal. If this decimal result is less than 5×10^{-n}, then we say the result is zero to $n-1$ decimal places (rounding rather than truncation).

All the figures and solutions in this article were produced using *Mathematica* (version 5.1). *Mathematica*'s **NDSolve** routine was used for solving initial value problems and producing phase plane trajectories. Within this state-of-the-art program one can set the working precision to attempt to control the local round-off and truncation error. Obviously, when solving an initial value problem over a long time interval, the local error accumulates and the algorithm can eventually produce a false solution rather than the true numerical solution.

When solving an initial value problem (IVP), theoretically (under mild hypotheses) there is a unique solution. However, when solving an IVP numerically, the solution to the IVP does not exist; there are many solutions depending upon the desired accuracy, hence a user must worry about the accuracy of the solution being generated. But how does one answer the question *to how many decimal places is your solution accurate and over what time interval does it maintain this accuracy?* This question is addressed in [8]. Generally, high precision solutions are easily generated and trajectories are accurate to at least 6 decimal places over the time interval $[0,1500]$. This Duffing equation is one that can give the solver fits if care is not taken, see for example [KW].

We must pay attention to this accuracy for we must decide if a response is periodic, say period P over a time interval I. One way to see if the response has period P

is to plot $x(t)-x(t-P)$ over I. If this plot were bounded in absolute value by 5×10^{-8} and we would declare that the response has period P at least to 7 decimals over the time interval I. Usually this would be enough to settle the issue, but in some cases a plot of $x(t)-x(t-2P)$ would be bounded by 5×10^{-13} so the response is periodic of period $2P$ to 12 decimals. Which period does one choose?

3. Working Precision

Generally speaking, a PC's default working precision is 16. Working precision N means that at each computation, the algorithm attempts to perform each calculation with N digits and the user may expect the calculation to be accurate to $N-10$ digits. Thus at machine default one expects 6 digits of accuracy.

For nonlinear equations, such a low precision may be insufficient to obtain accurate numerical solutions over a long time interval. To obtain accurate results, often one has to resort to higher precisions of 20, 26, or even 36. But this is a bit misleading, because this accuracy is only for a single calculation, and accumulated local round-off error, truncation error and algorithm error mean that as an algorithm executes more steps, the inputs to the accurate computation may be inaccurate. For more details on this see [13] and [8] (see also [7] and [9]).

4. Accuracy

As is customary, to solve Duffing's equation as an at rest initial value problem, we solve the 2×2 system

$$\dot{x}=y$$

$$\dot{y}=-ky+x-\varepsilon x^3+F\cos\omega t$$

with $x(0)=0,\dot{x}(0)=y(0)=0$ and we solve for $0\le t\le1500.$ Plots of the phase plane trajectories obtained by solving at machine default precision 16 and precision 26 might very well show different trajectories. The two solutions obtained might agree up to about $t=50$ say, and there after diverge rather quickly. So which is "the" solution? We suggest that accumulated round off error has crept into the working precision 16 solution so that around the step when $t=50$ things are going off. The working precision 26 solution is not immune from this creeping error; it just takes longer for it to happen.

So how do we decide over what time interval the two solutions are accurate and to what accuracy? If we had a closed form solution to the equation, we could compare the numerical solutions with it. But for nonlinear equations, seldom is such a closed form solution available and when it is, it often involves nonelementary functions which have to be evaluated with specific numerical algorithms and we would be comparing the results of one algorithm against another.

One way to get a hold on this is to solve not the 2×2 system, but rather the equivalent 3×3 system

$$\dot{x}=y$$

$$\dot{y}=z$$

$$\dot{z}=-kz+y-3\varepsilon x^2y-\omega F\sin\omega t$$

where $x(0)=0, \dot{x}(0)=y(0)=0,$ and $\ddot{x}(0)=z(0)=F.$ This system is obtained by differentiating $\ddot{x}$ and introducing the auxiliary variable $\ddot{x}=\dot{y}=z.$ When solving in this manner, we have a numerical computation of the second derivative $\ddot{x}$. This permits substituting the solution and its two derivatives back into the original equation written in the form

$$\ddot{x}+k\dot{x}-x+\varepsilon x^3-F\cos\omega t=0.$$

Plotting the result of the substitution one obtains an estimation of the accuracy of the solution and over what interval this accuracy is maintained. We call this calculating the *residual* of the solution. For more details on this technique see [8].
For representative example where the behaviour is good, consider the equation

$$\ddot{x}+\frac{1}{5}\dot{x}-x+x^3=\frac{1}{3}\cos\frac{7}{5}t$$

First, for working precision 26 , solving the corresponding 2×2 system and the 3×3 system and subtracting the two solutions, one sees an agreement to 7 decimal places. A plot of the full trajectories for $t\varepsilon[0,1500]$ is shown in Figure 1.

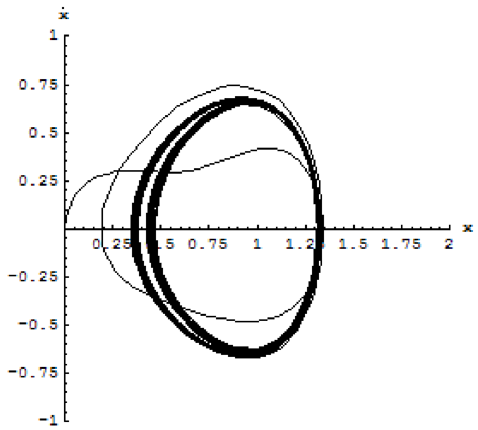


Figure 1. Full trajectory for precision 26

The trajectories for the 2×2 system and the 3×3 system are visually indistinguishable. In Figure 2, the difference between the solution calculated from the 3×3 system and that calculated from the 2×2 system is shown bounded by 2×10^{-8} indicating 7 decimal place agreement.

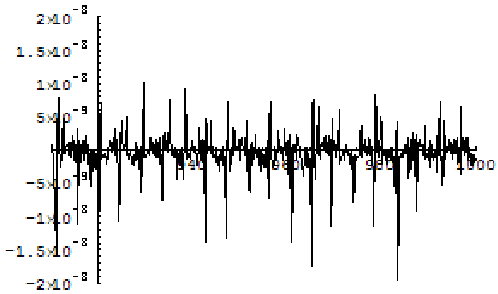


Figure 2. Difference between the 2×2 system and the 3×3 system solutions

Substituting the 3×3 system results back into the equation we obtain a residual uniformly bounded by 3×10^{-8} ; this is shown in Figure 3.

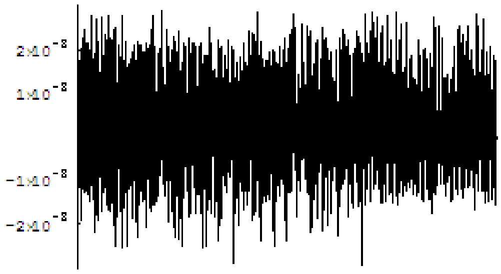


Figure 3. Residual for Precision 26

The steady state of the response has period $2P$ where $P=2\pi/\omega,$ the period of the forcing. Using a precision 26 solution (either one), a plot of $x(t)-x(t-2P)$ over the time interval $[1500-4P,1500]$ is bounded by 6×10^{-7} suggesting period $2P$ to very nearly 6 decimals. We call this plot the *periodic residual*.

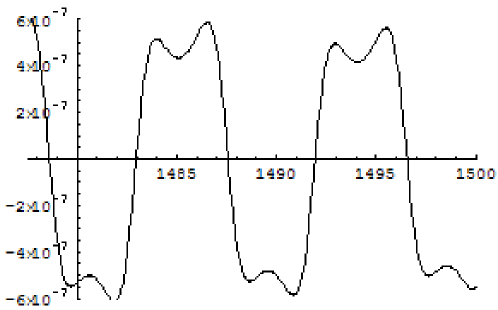


Figure 4. The $2P$ residual

5. Periodic response

Based upon the results of the representative example above, we solve all our initial value problems at working precision 26 and expect at least 6 decimal place accuracy. The steady state for the representative example above having $F=1/3$ has period $2P.$ For small positive values of F, keeping all the other parameters fixed , the steady state has period $P;$ this is the case for F up to $0.330.$ From 0.331 to 0.333 there is a smooth change in the steady state from period P to period $2P.$ For $F=0.334,$ the steady state has period P only to one decimal place while it has period $2P$ to seven decimal places. This change is smooth, but abrupt in the sense that the change is determined by the third decimal place of F. Thus we say we have a *period doubling*.

For $F = 0.364,$ the trajectory appears to be periodic of period $2P.$ A plot of $x(t-2P)-x(t)$ over the time interval $[800,1500]$ shows period $2P$ to 5 decimal places (the plot is bounded in absolute value 5×10^{-6}) . For $F = 0.365,$ the steady state has period $2P$ to 2 decimal places and shows a period of $4P$ to 4 decimal places. For $F = 0.366,$ the steady state does not have period $2P$ but is periodic of period $4P$ to 7 decimal places. Thus another period doubling is occurring. The steady state for $F = 0.370$ is periodic of period $4P$ to 5 decimal places over the interval $[800,1500]$ but for $F = 0.371,$ period $4P$ is essentially lost and period $8P$ is sustained to 7 decimal places. Yet another period doubling.
Increasing F from 0.372 to $F = 0.376$, period $8P$ is lost and there appears to be no steady state whatsoever. The transition from *4P* to *8P* to no steady state been rapid and smooth.

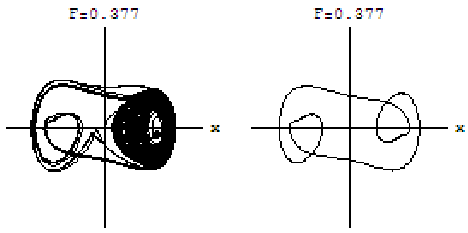


Figure 5. Full trajectory and steady state

From $F = 0.376$ to $F = 0.380$ there is a transition from having no steady state to having a steady state of period $5P$. Over this F -interval, solutions appear to be very sensitive to the working precision chosen. We encounter a region of F -values where the Duffing equation is extremely numerically sensitive. Could this be called a chaotic region?
From $F = 0.380$ to $F = 0.451$ reasonable steady states of period $5P$ are obtained, but from $F = 0.452$ to $F = 0.586$ there is no periodic steady state. However, by $F = 0.590,$ a periodic steady state of period $3P$ has established itself. This is the case until $F = 0.690,$ but from there to $F = 0.791$ again no steady state is obtained. When $F = 0.792,$ there is a periodic steady state of period P . So we have progressed from steady states of period P through period doublings to period $8P$ and then through numerically sensitive F -intervals to period $5P,$ then $3P$ and finally period P again.
The steady state of period P persists for quite a long range of F -values, up to $F = 12.480,$ where there begins a smooth change to steady state period $2P,$ a period doubling again. A period doubling from $2P$ to $4P$ occurs from $F = 13.920$ to $F = 13.950,$ and another doubling occurs between $F = 14.000$ and $F = 14.150$ going from period $4P$ to period $8P.$ Between $F = 14.160$ and $F = 14.170,$ the period doubles again to become $16P,$ but from $F = 14.171$ to $F = 16.070$ there is no steady state.
Steady state period $3P$ reappears at $F = 16.071$ and persists up to $F = 18.500,$ where thereafter a change to no steady state begins. At $F = 23.000,$ the steady state

has period $3P$ which remains the case up to $F = 32.000$. For $F = 33.000$ to $F = 40.000$, the steady state has returned to period P. We did not investigate the behaviour for $F > 40.000$

Certainly steady states of period P appear to be "preferred" as the F -intervals over which period P response occurs are longer than the F -intervals for other periodic responses. Period $3P$ seems to be also a "preferred" response, while period $5P$ less so.

There are lots of unanswered questions here. What happens for $F > 40$? Are these transition F -intervals predictable? Why are certain $F-$ intervals so numerically sensitive? It would appear that numerical investigations of this sort, while tedious, bring to light interesting behaviour and may indeed indicate chaotic behaviour - whatever that means. Certainly such investigations are suitable for individual and small group laboratory exercises and students can learn much about the vagaries of numerical analyses.

6. Chaos or not

As mentioned in the introduction, what chaos is seems to be in the eye of the beholder. This Duffing equation certainly can produce complicated trajectories and many students confuse chaos with a complicated trajectory. One might argue that an aperiodic initial value problem response is chaotic as its trajectory may very well appear to be wild, but in the face of a lack of clear definition this surely is a matter of personal preference. Perhaps what chaos needs is a non-smooth transition from one nice type of initial value problem response to another nice type of initial value problem response. In any event, this simple Duffing equation embodies many types of interesting behaviour and can serve as an excellent test-bed equation for numerical and theoretical investigations.

7. References

[1] R. Brown and L. O. Chua, (1996), ``Clarifying chaos: examples and counter examples", *International Journal of Bifurcation and Chaos* 6 219 - 249.

[2] G. Duffing, 1918.)*Erzwungene Schwingungen bei veränderlicher Eigenfrequenz*, Ph.D. Thesis (Sammlung Vieweg, Braunschweig,

[3] T. H. Fay, 2008. "Remarks on Duffing's equation", *Buffelspoort TIME2008 Conference Proceedings,* 22 - 26 September, pp 83 - 92.

[4] T. H. Fay, (2006) ``Nonlinear resonance and Duffing's spring equation", *International. Journal of Mathematical Education in Science and Technology,* 3**7** 593 - 599.

[5] T. H. Fay, (2006) ``The forced soft spring equation", *International. Journal of Mathematical Education in Science and Technology,* 37 707 - 720.

[6] T. H. Fay, (2006), The forced hard spring equation, *International. Journal of Mathematical Education in Science and Technology* 37 187 - 200.

[7] T. H. Fay & S. V. Joubert, (2000) Square waves from a black box, *Mathematics Magazine* 73 393 - 396.

[8] T. H. Fay & S. V. Joubert, (2003), Postanalysis of numerical solutions to ODEs, *New Zealand Journal of Mathematics* 32 (Supplementary Issue, November 2003) 67 - 75.

[9] R.Knapp & S. Wagon, (1999), ``Orbits worth betting on" *C ⌛ODE ⌛E* Winter 1999, 8 - 13.

[10] J. Guckenheimer & P. Holmes, **(**1983), *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, 1983.)

[11] F. C. Moon & P. J. Holmes, (1979) ``A magneto elastic strange attractor", *Journal of Sound Vibration* 35: 285 - 296.

[12] E. W. Weinstein, nd. ``Chaos". *MathWorld***--***A Wolfram Web.* http://mathworld.wolfram.com/Chaos.html

[13] S. Wolfram, (1996). *The Mathematica Book***,** *3rd ed.* Wolfram Media /Cambridge University Press, New York.

[14] M. A. van Wyk & W. -H. Steeb, (1997), *Chaos in Electronics*, Kluwer Academic Press, Boston.

On Generating Primitive Pythagorean Triples

TEMPLE H FAY[††††††] AND WILLIAM F SANFORD

* *Tshwane University of Technology, Pretoria, South Africa*
The Anchor Works, Hattiesburg, MS.

Abstract: Professional and amateur mathematicians have been fascinated over the ages by Pythagorean triples (a,b,c): triples of positive integers enjoying the constraint that $a^2+b^2=c^2$. We investigate the question of given a positive integer a (equivalently b), how many Pythagorean triples (a,b,c) are there and the slightly deeper question of given a positive integer c, how many Pythagorean triples (a,b,c) are there? Our answers to these simple questions are interesting, do not appear in any text we consulted nor on any website we found on the world wide web. Although we doubt our results are new, our investigations are suitable for an introductory part of a beginning abstract algebra or number theory course and quite possible would be appealing in course designed for teachers. Helping students to bridge the intelectual "hurdle" between computational thinking and such problem solving and more theoretical mathematical thinking is our aim and the historical and elementary nature of Pythagorean triples, we feel, makes our study suitable in assisting students over the "hurdle". Most of our arguments are elementary dealing with parity, unique prime factorizations and the like and thus may be appealing to the "amateur" mathematicians participating in such courses.

1. Introduction

Lost in antiquity are the origins of Pythagorean triples (a,b,c): triples of positive integers enjoying the constraint that $a^2+b^2=c^2$. These triples have fascinated professional and amateur mathematicians over the ages. If one does a *Google Search* with keywords "Pythagorean triples", one will get in excess of 132,000 hits. The Babylonian clay tablet Plimpton 322, translated by O. Neugebauer, containing lists of Pythagorean triples, some quite large suggesting that formulas for producing them were known, and this tablet predates Pythagoras by 1000 years. But interest in Pythagorean triples and related matters has certainly not ceased, see [1] and [2] for example.

We are interested in the question of given a positive integer a (equivalently b), how many Pythagorean triples (a,b,c) are there and the slightly deeper question of given a positive integer c, how many Pythagorean triples (a,b,c) are there?

Inarguably, the most famous such triple is $(3,4,5)$, the only such triple consisting of consecutive integers, taught to every school child at an early age, and perhaps only slightly less well known is the triple $(5,12,13)$. Note here that the integers $a,b,$ and c in these triples are coprime and Pythagorean triples of this sort are called *primitive*; we shall abbreviate this as PPT. Of course, the only interesting Pythagorean triples (PTs) are the PPTs. And also, it is well known that there are infinitely many such triples which can be derived with the Euclid formula: if (a,b,c) is a Pythagorean triple, then there are positive integers n,m with $n<m$, so that $(a,b,c)=(m^2-n^2,2nm,m^2+n^2)$ or possibly with the reversal

*E-mail: thfay@hotmail.com

$(a,b,c)=(2nm,m^2-n^2,m^2+n^2)$ depending upon the parity of a. All Pythagorean triples are obtained this way, and if n and m are relatively prime (coprime), the resulting triple is a PPT.

Our interest began naively with the question of how many PPTs are there with a given fixed a? Our answers to this simple question are interesting, do not appear on any posting we opened, nor in any text we consulted, however, the authors plead guilty to not checking out all of the 132,000+ internet postings and accordingly there may be texts or posting from which our results may be deduced; some related (and deeper) results may be found in [3]. But our investigations are suitable for an introductory part of a beginning abstract algebra or number theory course and quite possible would be appealing in course designed for teachers. Most arguments are elementary dealing with parity, unique prime factorizations and the like and thus may be appealing to the "amateur" mathematicians participating in such courses.

In the following table, we list a few examples of PPTs for easy reference and to whet one's appetite for what is to come. We have listed the PPTs with the order $a<b$, but there really is no need for this, the PPT $(3,4,5)$ is as good as the PPT $(4,3,5)$; the only constraint is that $a^2+b^2=c^2$.

$(7,24,25)$	$(9,40,41)$	$(13,84,85)$	$(36,77,85)$
$(8,15,17)$	$(12,35,37)$	$(16,63,65)$	$(20,99,101)$
$(84,187,205)$	$(88,105,137)$	$(140,171,221)$	$(204,253,325)$
$(111,680,689)$	$(111,6160,6161)$	$(60,899,901)$	$(60,221,229)$

In the first row of the table, the first three entries have the difference δ between c and b being 1, and a is odd. The last two entries of the first row have the same c value. In the second row, $\delta=c-b=2$ and a is even for all the entries; in the third row, each a is even and the deltas are respectively, $18, 32, 50,$ and 72 ; finally in the bottom row the first two entries have the same odd a and the next two entries have the same even a . From this table, we see that an a can appear in several PPTs regardless of its parity, there can be a variety of δ' s and the same value of c may appear in different PPTs. Of course all of this behaviour can be explained and it is the point of this article to do so.

As a final introductory remark, we mention that there is one degenerate PPT $(1,0,1)$, degenerate because it arises naturally in some formulae but violates the definition of PT with the zero entry. Fortunately this does not arise often and really causes no problems, so it is a matter of mathematical taste to include this triple in one's list of PPTs.

2. Some basic observations

a and b always have opposite parity

If (a,b,c) is a PPT, then not all of a,b,c can be even else we do not have a PPT. Both a and b can not be even, for then so would be $c^2=a^2+b^2$, and hence c would be even. From Euclid's formula, there exist positive coprime integers m and n with $n<m$ so that $(m^2-n^2,2mn,m^2+n^2)=(a,b,c)$ if a is odd, else

$(2mn, m^2 - n^2, m^2 + n^2) = (a,b,c)$ if a is even. So b is even or odd according to a being odd or even; note that m and n have to have opposite parity as well.

c is always odd and is the sum of two squares
If (a,b,c) is a PPT, then $a^2 + b^2 = c^2$ is odd and hence c is odd.

if a is even then it is divisible by 4
If (a,b,c) is a PPT, and $a = 2mn,$ then one of m and n is even hence a is divisible by 4.

if a is even, then $\delta = 2n^2$ and n divides a
If (a,b,c) is a PPT,with a even, then $(a,b,c) = (2mn, m^2 - n^2, m^2 + n^2)$ as above. Clearly $\delta = c - b = 2n^2$ and n is a factor of $a/2$.

if a is odd then $\delta = n^2$ and n divides a
If (a,b,c) is a PPT,with a odd, then $(a,b,c) = (m^2 - n^2, 2mn, m^2 + n^2)$ as above and plainly $\delta = c - b = (m-n)^2$ and $(m-n)$ is a factor of a.

3. Maximal PPTs

We are interested in knowing all the PPTs (a,b,c) with a given a. So one would want to know that there are only a finite number of them and how to find them. In this section we show that there are only a finite number for each a and give an upper bound on the values of b and c. The PPTs having these upper bounds will be called maximal. Not surprisingly, the descriptions depend upon the parity of a.

a is odd
Suppose that $a > 1$ is odd. Set $b = (a^2 - 1)/2$ and $c = (a^2 + 1)/2.$ Then it is easy to check that (a,b,c) is a PT. That it is a PPT follows from b and c being relatively prime, for if b and $b+1$ have a common factor say $p,$ then $b = pq$ and $b+1 = pt$ for some positive integers q and $t.$ Then $pq + 1 = pt$ so p divides 1 and hence $p = 1.$

We shall show that if (a,b',c') is any other PPT, then $b' < b, c' < c,$ and $\delta = c' - b' > 1;$ in this sense, the triple (a,b,c) above is called *maximal*. To that end, suppose $a = 2k+1,$ and note that the above triple (a,b,c) becomes $(2k+1, 2k(k+1), 2k(k+1)+1).$

Suppose that (a,b',c') is a PT with $b' = 2m$ and c❖■$2m$ ✉ where $\delta > 1$ (if $\delta = 1,$ then $b' = b$ and $c' = c$). Since we have a PT, it follows that
$4k^2 + 4k + 1 + 4m^2 = 4m^2 + 4m\delta + \delta^2,$

so

$$b = 2k^2 + 2k = 2m\delta + \frac{\delta^2 - 1}{2} > 2m + 1 > 2m = b'.$$

It also follows that $c > c'$ since for all $\delta \geq 2$, $\delta^2 + 1 > 2\delta$ and

$$c = 2k^2 + 2k + 1 = 2m\delta + \frac{\delta^2 + 1}{2} > 2m + \delta = c.$$

We formalize these facts in

Proposition 1. *If* $a = 2k + 1$, *then* $(a,b,c) = (2k+1, 2k(k+1), 2k(k+1)+1)$ *is a PPT with* $\delta = c - b = 1$, *and if* (a,b',c') *is any other PPT, then* $b' < b$ *and* $c' < c$.

a **is even**

First suppose $a = 2k$ and (a,b,c) is a PPT with $\delta = c - b = 2$. Then $4k^2 + b^2 = b^2 + 4b + 4$ so $k^2 = b + 1$. Since a is even, b is odd and so $b+1$ is even, hence k^2 and consequently k is divisible by 2 . Thus $a = 4q$ for some positive integer q. It follows that $b = 4q^2 - 1$ and $c = 4q^2 + 1$ and $(4q, 4q^2 - 1, 4q^2 + 1)$ is a PPT.

If (a,b',c') is any PPT, then b is odd hence $b' = 2m+1$ and $c' = 2m + \delta$ where $\delta \geq 3$ (recall c' must be odd). Then since $a = 4q$, we have

$$16q^2 + 4m^2 + 4m + 1 = 4m^2 + 4m\delta + \delta^2$$

and cancelling, rearranging and factoring,

$$4q^2 = m(\delta - 1) + \frac{\delta^2 - 1}{4}$$

where $(\delta^2 - 1)/4 \geq 2$ and thus

$$4q^2 - 1 = m(\delta - 1) + \frac{\delta^2 - 5}{4}$$

where $(\delta^2 - 5)/4 \geq 1$. Now it is clear that $b' = 2m + 1 \leq m(\delta - 1) + (\delta^2 - 5)/4 = 4q^2 - 1$. It also is a simple computation to see that

$$c'^2 = a^2 + b'^2 \leq 16q^2 + 16q^4 - 8q^2 + 1 \leq (4q^2 + 1)^2 = c^2$$

hence $c' \leq 4q^2 + 1 = c$. We have proved the following proposition.

Proposition 2. *If* $a = 2k$ *and if* (a,b,c) *is a PPT with* $\delta = c - b = 2$, *then* $a = 4q$, $(a,b,c) = (4q, 4q^2 - 1, 4q^2 + 1)$ *and if* (a,b',c') *is any other PPT, then* $b' < b$ *and* $c' < c$.

Definition. We call the PPTs $(2k+1, 2k(k+1), 2k(k+1)+1)$ and $(4q, 4q^2 - 1, 4q^2 + 1)$ maximal.

4. Generating all PPTs for a given a

The maximal PPTs were determined by the facts that for a odd, $c-b=\delta=1$ and for a even, $c-b=\delta=2.$ Hence for any odd positive integer $a,$ there is a PPT (a,b,c) and for even a , only those divisible by 4 can appear in a PPT. To get a feel for how to generate other PPTs, it is useful to examine cases for where δ is larger than $2.$

For example, suppose (a,b,c) is a PT with $c-b=3.$ Then $a^2+b^2=b^2+6b+9$ so that $a^2=3(2b+3)$ and consequently a is odd and divisible by 3 . So $a=3q$ where q is odd and it follows that $3q^2=2b+3$ and hence $b=3(q^2-1)/2$ and $c=3(q^2+1)/2$ and both b and c are divisible by $3.$ The PT (a,b,c) is not a PPT, but $(a/3,b/3,c/3)$ is a maximal PPT and we can generate maximal PPTs by allowing various values for q . A representative sample is given in the following table.

a	q	(a,b,c)	$(a/3,b/3,c/3)$
3	1	$(3,0,3)$	$(1,0,1)$
9	3	$(9,12,15)$	$(3,4,5)$
15	5	$(15,36,39)$	$(5,12,13)$
21	7	$(21,72,75)$	$(7,24,25)$

Note the degenerate PPT $(1,0,1)$ arises here.

One can experiment with setting δ to be other values. For example, an interesting case arises for $\delta=9.$ In this case, $a=3q, b=(q^2-9)/2, c=(q^2+9)/2$ and (a,b,c) is a PT which reduces to a maximal PPT with $\delta=1$ when q is divisible by 3 and (a,b,c) is a PPT with $\delta=9$ otherwise.

While this approach is interesting and produces many PPTs, it does not answer the question of when a is given and fixed, what are all the PPTs of the form (a,b,c)? We know there is only a finite number because b is bounded above by Propositions 1 and 2.
Our first observation along this line of investigation is if (a,b,c) and (a,b',c') are two PTs with $c-b=\delta=c'-b',$ then $b=b'$ and $c=c'.$ For from $a^2+b^2=c^2$ and $a^2+b'^2=c'^2$ we have $b^2-b'^2=c^2-c'^2$ so that $c^2-b^2=c'^2-b'^2.$ Factoring $(c+b)\delta=(c+b')\delta$ so $c+b=c'+b',$ and thus from our assumption it follows that $b=b'$ and $c=c'.$ Hence the two PTs are the same.

a is odd

Suppose that a odd is given. Then $\delta=c-b=q^2$ where q is an odd factor of a and it follows that $a=q(2k+1)$. Furthermore,

$$(q(2k+1), 2k^2+2k+\frac{1-q^2}{2}, 2k^2+2k+\frac{1+q^2}{2})$$

is a PPT with the appropriate $\delta.$

Example. Suppose $a = 63 = 1 \cdot 3^2 \cdot 7$. Then the following are all the PPTs (a,b,c).

q	$2k+1$	k	δ	(a,b,c)
1	63	31	1	$(63,1984,1985)$
3	21	10	9	$(63,216,225)$
7	9	4	49	$(63,16,65)$
9	7	3	81	$(63,-16,65)$
21	3	1	441	$(63,-216,225)$
63	1	0	3969	$(63,-1894,1895)$

Note that since we only consider positive integer values, we would discard the last three rows of the above table. The above formula for the PPTs generates pairs $(a,\pm b,c)$ by considering all possible odd factors of a.

a is even

Suppose that a even is given. Then $\delta = c - b = 2q^2$ where q is a factor of a. Then $a = 2qk$ and

$(2qk, k^2 - q^2, k^2 + q^2)$

is a PPT with the appropriate δ.

Example. Suppose $a = 20 = 1 \cdot 2^2 \cdot 5$. Then the following are all the PPTs (a,b,c) with $a = 2qk$ and $\delta = 2q^2$.

q	k	δ	(a,b,c)
1	10	2	$(20,99,101)$
2	5	8	$(20,21,29)$
5	2	50	$(20,-21,29)$
10	1	200	$(20,-99,101)$

Note again, we would discard the last two rows of the above table. The above formula for the PPTs generates pairs $(a,\pm b,c)$ by considering all possible factors of $a/2$.

5. PPTs for a given c

The question of given a value of c, how many PPTs are there of the form (a,b,c) is a deeper question and indeed when an integer is the sum of two squares is a question that has a history dating back to the tenth century. A list of names associated with these questions reads like a *Who's Who* of mathematicians: Fibonacci, Bachet, Girard, Fermat, Euler, Jacobi, Hardy, Wright, Liouville, Eisenstein, Ramanujan and many more. Further references and more information may be found at http://mathworld.wolfram.com. It is beyond the scope and intent of this article to provide all details on this, so we will merely quote two results and suggest that this would be a good topic for further student or small group investigation

Theorem 1. *A positive integer* n *can be expressed as the sum of two squares if and only if, when uniquely factored into a product of prime powers, the power of prime factors of the form* $4k+3$ *is even.*

This result doesn't quite answer our question of given a c for we are interested in expressing c^2 as the sum of squares. Any a^2 is congruent to either 0 or 1 $\bmod 4$ and a^2+b^2 is congruent to $0,1,$ or 2 $\bmod 4.$ If (a,b,c) is a PPT, then c is odd and thus c^2 is congruent to 1 $\bmod 4.$ Consequently, the unique prime power decomposition of c is of the form $\{q_1^{\mu_1} q_2^{\mu_2} \cdots q_k^{\mu_k}\}$ where each odd prime q_i is of the form $4m+1.$

Theorem 2. *The number of ways an odd positive integer* c *can appear in a PPT* (a,b,c) *is* 2^{k-1} *where* k *is the length of the prime factors* q *of the form* $4m+1$ *and* c *has no prime factors of the form* $4k+3,$ *and the number is* 0 *otherwise.*

Examples. *PPTs with the same* $c.$

$(16,63,65),(33,56,65);$
$(36,77,85),(13,84,85);$
$(87,416,425),(297,304,425);$
$(2277,31964,32045),(8283,30956,32045),$
$(17253,27004,32045),(21093,24124,32045).$

We offer a "quick and dirty" way to calculate all the PPTs (a,b,c) for a given $c.$ Let us assume without loss of generality that a is odd and b is even. The idea stems from Euclid's formula $(m^2-n^2, 2mn, m^2+n^2).$ We need to know the different ways to express c as the sum of two squares. As we have noted, $c-m^2=n^2$ and $c-a=2n.$ So we need only to begin subtracting squares from c and seek square integer results. Perhaps naturally the place to start is with $F = Floor(\sqrt{c}),$ that is, F is the integer part of $\sqrt{c}.$ Iteratively we examine $c-(F-i)^2$ for $i=0,1,\cdots,F.$ If $c-(F-i)^2$ is a perfect square, say $n_i^2,$ then set $m_i = F-i,$ and we have $m_i^2+n_1^2=c, a=m_i^2-n_1^2,$ and $b=2m_i n_i.$ Also note that one need not iterate completely for $i=0,1,\cdots,F,$ but only until $(F-i)^2 \geq c/2.$ A couple of examples illustrate this.

Example. *Find all the PPTs* (a,b,c) *having* $c=7085=5\cdot 13\cdot 109.$
From Theorem 2 that we can expect to find four PPTs. For this $c,$ *F = 84,.* and it follows that

$c-(F-1)^2=14^2 \quad m_1=83 \quad n_1=14$
$c-(F-2)^2=19^2 \quad m_2=82 \quad n_2=19$
$c-(F-7)^2=34^2 \quad m_7=77 \quad n_7=34$
$c-(F-23)^2=58^2 \quad m_{23}=61 \quad n_{23}=58$

and the four PPTs are

$(6693, 2324, 7085)$
$(6363, 3116, 7085)$
$(4773, 5236, 7085)$
$(357, 7076, 7085)$

Example. *Find all the PPTs* (a,b,c) *having* $c = 1325 = 5^2 \cdot 53.$
From Theorem 2, we can expect to find two PPTs. For this c, $F = 36.$ and it follows that

$c-(F-1)^2 = 10^2 \quad m_1 = 35 \quad n_1 = 10$
$c-(F-2)^2 = 13^2 \quad m_2 = 34 \quad n_2 = 13$
$c-(F-7)^2 = 22^2 \quad m_7 = 29 \quad n_7 = 22$

Note that m_1 and n_1 are not relatively prime and thus do not generate a PPT, only a PT. The two PPTs

$(987, 884, 1325)$
$(357, 1276, 1325)$
are derived from the remaining two results.

We have only scratched the surface of the theory about PPTs. There are many other interesting features of PPTs and their connections to geometry, points on the unit circle with rational coordinates, Fibonacci sequences, and other ideas. Given the 132,000+ hits on the internet, albeit with a great deal of overlap, there is no shortage of material on the subject and of course, there is a plethora of elementary number theory texts available in any good mathematics library. All three of our references have extensive bibliographies yielding invaluable sources for further study.

6. References

[1] Ming-Sun Li, Katheryn Robertson, Thomas J. Osler, Abdul Hassen, Christopher S. Simmons, and Marcus Wright, "On numbers equal to the sum of two squares in more than one way", *Mathematics and Computer Education* Vol. 43, No.2, pp. 102 - 108, (2009).
[2] Darryl McCullough, "Height and excess of Pythagorean triples", *Mathematics Magazine*, Vol. 78, N0. 1, pp. 26 - 44, (2005).
[3] Eric W. Weinstein, "Pythagorean triples", from *Math-World* -A Wolfram Web Resource http://mathworld.wolffram.com/PythagoreanTriple.html

Tertiary mathematics students' achievement and perception of their learning environment: some gender differences.

*M.-R. FREISLICH‡‡‡‡‡‡ AND A. BOWEN-JAMES

*School of Mathematics and Statistics, University of New South Wales, Sydney, Australia;
Cognitive Systems Pty. Ltd, Sydney, Australia*

Abstract: The purpose of this study was to investigate perceptions of the learning environment among tertiary mathematics students in two groups of different sizes, enrolled in a first year mathematics subject at an Australian university. The smaller group received additional teaching material intended to clarify the scope and level of work required. Scales adapted from the SLEI were used to compare groups and to examine links between perceptions and achievement. The perception scales discriminated among groups, with some strong interaction with gender. Perceived learning environment factors were linked to achievement, again with some interaction with gender. Overall, confidence about handling the subject load, and perceived teaching support were significantly related to achievement. Good integration of teaching and clarity of requirements were important for women.

Keywords: undergraduate mathematics, learning environments, gender.

1. Introduction

The quality of learning outcomes among university students has been studied extensively over recent decades. Early work in Britain ([1], [2]) and Australia ([3-5],) converged closely to a common consensus. This was that high quality of learning was associated with an orientation to real understanding, pursued using active study methods that were diligent without being concerned purely with isolated detail. The approach has continued to be fruitful (see, for example, [6]) In mathematics, in particular, action is essential, because working on problems is how most people both develop and test their understanding of mathematical material. It follows that university teachers of mathematics would give high value to any means of influencing their students to take action.

The study of students' perceptions of their learning environment, with reference to factors involving the social, academic and physical environment in classes, is a well established field of research at secondary school level. Its origins in the field of science learning are to be found in important Australian work, mainly by Fraser and Fisher (surveyed, for example, in [7]) who constructed and validated several scales, of which the *Science Learning Environment Inventory* is the best known. Their work established the scale's internal consistency and ability to discriminate among classes, and also provided evidence that environmental factors had the potential to facilitate better learning outcomes. Continuing work has validated both discriminative and predictive power of this and similar instruments over a wide variety of cross cultural settings (see, for example, [8-12]).

‡‡‡‡‡‡ E-mail: mrfreislich@hotmail.com

A previous study by one of the present authors [13] used a similar approach, with modified scales, to examine university mathematics students' learning environment perceptions. In this study, an additional factor not adapted from previous research on learning environments was included, which involved students' confidence about coping with the subject load in the time available. This variable relates to confidence about the mathematical task. Such confidence has well established links to higher quality mathematics learning. Confidence levels have historically shown gender differences at secondary level [14], and are the subject of ongoing research (see, for example the overview in [15]) and so were considered important to the study. The findings showed clear discrimination between classes, and evidence that deliberate intervention to enhance the environment by providing extra teaching materials could foster higher achievement. The groups studied consisted of relatively capable second year mathematics students. The present work investigates similar factors with students from the same university, in a first year terminating subject, done by students who require only one year of mathematics, which means that the group has lower entrance qualifications and much lower commitment to mathematics than those studied previously.

2. Method

2.1 Target populations

Enrolment groups of different sizes in a first year mathematics subject at a large Australian university were studied. The larger group was taught in Semester 1, which was the standard enrolment pattern, whereas the smaller group was taught in Semester 2, in blocked sessions in the late afternoon, and included a higher percentage of students who were repeating the subject. For convenience the two groups are labeled day and evening groups.

Syllabus material was identical, but the evening group received some extra teaching material designed, consisting of pairs of similar problems, one with a solution and one without. This difference in teaching material had two purposes. First, the solved problem was intended to provide scaffolding for students' work. Second, the style of problem was intended to make the scope of the subject and the level of performance required clearer to the students. So the learning environment of the two groups differed, because teaching contact with the small group was closer, and some materials different. There were therefore observable differences in the actual learning environments of the two groups, but the use of questionnaire data (see below) implies that the study is concerned with students' perceptions of their environments.

The smaller group contained more repeating students, which means that the groups differed in composition, and so might react differently to environmental factors.

2.2 Samples

Questionnaire data, containing background information, marks from class tests, and responses to items about perceptions of the learning environment, were collected from 155 day and 25 evening students. Further achievement data, from the final examination, was obtained from a random sample of 50 day and all 41 evening students. Sample breakdowns are given below (Table 1). In the sample for examination achievement, 37% (5% were female, 32% male) of the evening group were repeating the subject, but only 4% (all male) of the day group.

Table 1. Sample

	Questionnaire data		Examination data	
	Day	Evening	Day	Evening
Women	53	12	18	15
Men	103	13	32	26
Total	156	25	50	41

2.3 Instruments and examination achievement scores

Learning environment scales similar to those adapted from the *Science Learning Environment Inventory* in [13] were used. Students were asked to fill in levels of agreement with items, on a five-point scale. Negatively worded items had their scores reversed, so that high scores indicated attitudes hypothesized as favourable. Factor analysis gave 7 groupings of items. Group items were combined to give scale scores. The scales were labeled Subject Load (enough time to cope with syllabus material), Integration (different aspects of teaching well related), Task (task orientation in teaching time), Physical (comfortable classes), Support (teachers' accessibility and helpfulness), Materials (equipment and course information) and Clarity (clear requirements of scope and level of the subject). Scale data were collected in class, anonymously, but with a record of students' gender and their class test marks so far in the subject.

Examination achievement was assessed using a method similar to that developed in [16], and used in [13]. A complete description of the method is available in either document. Examination scripts were scored according to a coarse ranking, on lines originally suggested by the SOLO taxonomy [17] This was designed to provide a common standard for performance on different tasks involving the same subject material and level of difficulty, as assessed by university mathematics staff. Scoring was done by the same people who had marked the original examination, using the following criteria. Each logically self contained task was given a score from 0 to 5. Score meanings are described below (Table 2).

Table 2. Achievement scores

Score	Meaning
5	A completely correct solution
4	Correct except for minor slips, not showing misunderstanding.
3	A substantial part of a solution, but with evidence of misunderstanding.
2	More than one relevant idea, but far from a solution.
1	One relevant idea written down
0	Nothing correct

Scores for separate tasks were weighted according to the examination mark allocation, and then combined.

2.4 Analyses

Factorial analysis of variance by group and gender was used to compare groups on each of the achievement scores.

A multivariate factorial analysis of variance, by group and gender, was used to compare group perceptions of the learning environment, including all the perception scales.

Scales were correlated with achievement on class tests, and a multiple regression was done using the four scales that showed significant individual correlations with achievement.

3. Results

3.1 Descriptive statistics

Means and standard deviations for all variables are given below (Table3).

Table3. Descriptive statistics

		Questionnaire data						
Group	Day				Evening			
N	Women		Men		Women		Men	
Variable	55		109		12		15	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Test scores	70.88	18.62	72.33	26.86	71.08	19.18	79.07	20.84
Scales								
1 Subject load	2.87	0.83	3.20	0.84	2.69	0.63	3.48	0.52
2 Integration	3.79	0.47	3.71	0.66	3.50	0.58	4.05	0.33
3 Task	3.46	0.58	3.54	0.62	3.48	0.43	3.70	0.71
4 Physical	3.64	0.77	3.57	0.71	3.19	0.71	3.53	0.84
5 Support	3.38	0.69	3.48	0.61	3.02	0.70	3.73	0.68
6 Materials	3.50	0.70	3.62	0.69	3.19	0.46	3.51	0.82
7 Clarity	3.70	0.70	3.68	0.81	3.70	0.80	3.75	0.56
		Examination data						
N	18		32		15		26	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
	50.13	23.88	49.23	18.54	51.63	25.37	43.36	25.02

The descriptive information for the scales indicates that most means are above the hypothesized neutral score of 3. The exception is the Subject load scale, where both means for women are below this, indicating that the women tended to feel less able to handle the subject load.

But the most notable aspect of the scale scores is the frequency of different relativities of means for women and men in the two groups. For all scales, the evening group women have lower scores than the evening group men, and, in the day group, differences are either in the opposite direction or in the same but smaller.

3.2 Correlations

Significant and near significant correlations between test achievement and scales 1, 2, 5 and 7 are given below (Table 4). The near-significant correlation for Scale 2 is included because the corresponding coefficient calculated for women only was highly significant, and that for women in the day class even more so. Similarly, the Clarity variable correlated significantly with achievement among the evening students. Both coefficients are therefore included.

Table 4. Correlations with test achievement

Scale	1. Subject load	2 Integration	5 Support	7 Clarity
Correlation coefficient	0.34	0.10	0.21	0.12

Probability	0.000	0.089	0.002	0.0525
N	180			
		Women		Evening class
Correlation coefficient		0.26		0.36
Probability		0.022		0.038
n		63		27

Results of the multiple regression relating test achievement to the four scales above are given below (Table 5). It is clear that, in the presence of the subject load scale, the others do not afford any further significant contribution to the multiple correlation coefficient.

Table 5. Multiple regression

	Coefficients			
	Unstandardised		Standardised	
	B	Std. error	Beta	Probability
(Constant)	38.89	10.79		0.00
Subject load	7.32	1.83	0.31	0.00
Integrated teaching	-0.68	2.85	-0.02	0.81
Teaching support	3.29	2.42	0.11	0.18
Clear requirements	0.59	2.07	0.02	0.79
Multiple R	0.36			0.00

3.3 Achievement comparisons

The analysis of variance tables below relate to test achievement (Table 6), and examination achievement (Table 7). Neither showed any significant results.

Table 6. Test achievement comparison

Source	Sum of squares	df	Mean square	F	Probability
Group	247.08	1	247.08	0.61	0.36
Gender	191.47	1	191.47	0.69	0.31
Interaction	278.75	1	278.75	0.69	0.31
Residual	7098.91	1	405.62		
Total	7107.21	178			

Table 7. Examination achievement comparison

Source	Sum of squares	df	Mean square	F	Probability
Group	110.62	1	110.62	0.22	0.64
Gender	613.42	1	613.42	1.19	0.28
Interaction	765.28	1	765.28	1.49	0.23
Residual	44761.66	87	514.50		
Total	46250.97	90			

3.4 Environment scales comparisons

The results of the multivariate analysis of variance by group and gender are given below (Table 8). The direct comparisons between classes were not significant, but there was evidence for interaction. The results involving individual variables showed significant interaction for the scales dealing with integration, clarity and support. The multivariate tests did not reach significance, but the probability of the result was p=0.089, indicating some evidence in the same direction. The multivariate tests showed highly significant differences associated with gender, and the gender comparisons for individual variables gave significant results, favouring men, for subject load, integration, clarity and support, which were the scales that correlated significantly with test achievement. Inspection of subgroup means indicates that the pattern of the individual results is the revealing one, because the mean scores of day class women and men are very close, for integration, support and clarity, so that it is the interaction that counts for these, and the overall importance of gender is most clearly associated with Subject Load.

Table 8. Multivariate analysis of variance by group and gender

Between-subjects factors

		N
Group	Day	154
	Evening	25
Gender	Women	63
	Men	116

Multivariate tests

Effect		Value	F	Hypothesis df	Error df	Significance
Intercept	Pillai's Trace	.974	898.543	7	169	0.000
	Wilk' Lambda	.026	898.543	7	169	0.000
	Hotelling's Trace	37.22	898.543	7	169	0.000
	Roy's Largest Root	37.22	898.543	7	169	0.000
Group	Pillai's Trace	0.04	0.998	7	169	0.434
	Wilk' Lambda	0.96	0.998	7	169	0.434
	Hotelling's Trace	0.04	0.998	7	169	0.434
	Roy's Largest Root	0.04	0.998	7	169	0.434
Gender	Pillai's Trace	0.11	2.982	7	169	0.006
	Wilk' Lambda	0.89	2.982	7	169	0.006
	Hotelling's Trace	0.12	2.982	7	169	0.006
	Roy's Largest Root	0.12	2.982	7	169	0.006
Group x Gender	Pillai's Trace	0.07	1.806	7	169	0.089
	Wilk' Lambda	0.93	1.806	7	169	0.089
	Hotelling's Trace	0.75	1.806	7	169	0.089
	Roy's Largest Root	0.75	1.806	7	169	0.089

Tests of between-subjects effects

Source	Dependent	Sum of squares	df	Mean square	F	Significance
Corrected model	Subject load	9.924	3	3.308	4.990	0.002
	Integration	2.197	3	0.732	2.259	0.083
	Clarity	3.933	3	1.311	2.312	0.078
	Materials	2.095	3	0.698	1.478	0.222
	Support	5.35	3	1.786	4.280	0.006
	Physical	2.257	3	0.752	1.395	0.246
	Task	2.127	3	0.709	1.962	0.121

Intercept	Subject load	790.088	1	790.088	1191.867	0.000
	Integration	1201.445	1	1201.445	3706.065	0.000
	Clarity	1137.316	1	1137.316	2005.883	0.000
	Materials	1022.622	1	1022.622	2163.514	0.000
	Support	996.385	1	996.385	2387.687	0.000
	Physical	1053.914	1	1053.914	1954.018	0.000
	Task	1068.974	1	1068.974	3012.950	
Group	Subject load	0.335	1	0.335	0.505	0.478
	Integration	0.020	1	0.020	0.063	0.803
	Clarity	0.060	1	0.060	0.106	0.746
	Materials	0.659	1	0.659	1.394	0.239
	Support	0.059	1	0.059	0.141	0.707
	Physical	0.672	1	0.672	1.247	0.266
	Task	0.836	1	0.836	2.312	0.130
Gender	Subject load	8.316	1	8.316	12.545	0.001
	Integration	1.498	1	1.498	4.622	0.033
	Clarity	3.469	1	3.469	6.118	0.014
	Materials	1.317	1	1.317	2.787	0.097
	Support	5.234	1	5.234	12.542	0.001
	Physical	0.941	1	0.941	1.745	0.188
	Task	1.308	1	1.308	3.618	0.059
Group x Gender	Subject load	1.240	1	1.240	1.870	0.173
	Integration	2.034	1	2.034	6.273	0.013
	Clarity	3.095	1	3.095	5.459	0.021
	Materials	0.445	1	0.445	0.941	0.333
	Support	3.127	1	3.127	7.493	0.007
	Physical	1.727	1	1.727	3.201	0.075
	Task	0.652	1	0.652	1.803	0.171
Error	Subject load	116.007	175	0.663		
	Integration	56.732	175	0.324		
	Clarity	99.223	175	0.567		
	Materials	82.717	175	0.473		
	Support	73.028	175	0.417		
	Physical	94.388	175	0.539		
	Task	63.250	175	0.361		
Total	Subject load	1815.875	179			
	Integration	2585.500	179			
	Clarity	2547.750	179			
	Materials	2354.455	179			
	Support	2191.375	179			
	Physical	2401.748	179			
	Task	2298.563	179			
Corrected total	Subject load	125.931	178			
	Integration	58.929	178			
	Clarity	103.156	178			
	Materials	84.812	178			
	Support	78.386	178			
	Physical	96.644	178			
	Task	65.378	178			

4.Discussion

One should first note that the scale scores for task orientation, physical environment and materials gave no significant results, either in correlations or comparisons. Classrooms and materials for the two groups were very similar, so the lack of differences seems reasonable. In university mathematics teaching, it is also reasonable to expect that classes are run with a high level of task orientation. In studies ranging over a variety of institutions, such factors have been found to contribute to discrimination, and for this reason items relating to them were included in the present study's instrument.

It follows that the rest of the discussion needs to deal only with factors that did relate to achievement. Of these, it was noted above that Integration, Support and Clarity discriminate between classes. In the multivariate analysis of variance, they showed significant interactions between group and gender, as well as overall gender differences. If one examines the mean scores for subgroups, it is clear that it is the interaction that is important. That is, the pattern indicates that the overall gender difference found is likely to stem from the difference in the evening group. Subject load, on the other hand, did not show significant interaction in the multivariate analysis of variance, but the gender difference was clear for this scale. A high score on this scale indicates a feeling that the load is manageable, and thus reflects confidence with respect to the particular subject. As noted above, confidence has been shown to be an important factor in higher quality mathematics learning. Also, the Subject Load scale dominates the multiple regression result in the present study, so that the difference may indeed be considered favourable to men. But the subgroup correlations indicate that integrated and well organised teaching were useful to the women in the day class, and clarity of expectations to the night class. Two conclusions are relevant. First, the effects of lower confidence depend on how it is worked out in context. It seems that the women in the day and evening classes had different coping strategies. Second, both integrated teaching and clear requirements are accessible to intervention, so that the teaching team may be able to provide a more favourable environment for women students, even though this may be more relevant for groups not at the highest level of commitment to, or performance in mathematics. Turning now to the achievement comparisons, the lack of significant achievement differences among subgroups is, in the present context, informative rather than neutral. If one recalls that half of the men in the evening group had a record of failure, the lack of differences seems to indicate that this subgroup flourished in the smaller class with some extra teaching materials.

It seems, therefore, that the results of the present study confirms the importance of the perceived learning environment for tertiary mathematics students, and suggests that some interventions in the learning environment may enhance quality of outcome.

5. References

[1] P. Ramsden and N.J. Entwistle (1981) *Effects of academic departments on students' approaches to studying.* Brit. J. Educ. Psychol. 51 (1981) pp. 368-383.

[2] F. Marton, D. Hounsell, and N.J. Entwistle, N (1984) *The experience of learning*. Scottish Academic Press, Edinburgh, 1984.

[3] J.B. Biggs, *Student motivation and study strategies in university and college of advanced education populations.* Higher Educ. Res. .Dev., 1 (1982), pp. 33-55.

[4] J.B. Biggs, *Student approaches to learning and studying*. Australian Council for Educational Research, Melbourne, 1987.

[5] J.B. Biggs, *What do inventories of students' learning processes really measure? A theoretical view and clarification*, Brit. J. Educ. Psychol., 63 (1993) pp. 3-19.

[6] M. Prosser, P. Ramsden, K. Trigwell, and E. Martin, (2003) *Dissonance in experience of teaching and its relation to the quality of student learning*. Studies in Higher Educ., 28 (2003), pp. 37-48.
[7] B.J. Fraser, (1998) *Classroom environment instruments: development, validity and applications.* Learning Env. Res., 1 (1998), pp. 7-33.
[8] D.L. Fisher,and B.G. Waldrip, *Cultural factors of science classroom learning environments, teacher-student interactions and student outcome,* Res. Sci. Tech. Educ., 17 (1999), pp. 83-96
[9] P. Den Brok, D.L. Fisher, T. Wubbels, M. Brekelmans, and T. Rickards, *Secondary Teachers' interpersonal behaviour in Singapore, Brunei and Australia: a cross-cultural comparison*. Asia Pacific J. Educ., 26 (2006), pp. 79-95
[10] B.J. Fraser and S.S.U. Lee,. *Science laboratory classroom environments in Korean High Schools*. Learning Env. Res., 12 (2009), pp. 67-84.
[11] V. Chandra and D.L. Fisher, *Students' perceptions of a blended web-based learning environment.* Learning Env. Res., 12 (2009), pp. 31-44.
[12] W. Wu, H. Chang, and C. Guo, *The development of an instrument for a technology-integrated science learning environment.* Int. J. Sci. Math. Educ., 7 (2009), pp. 207-233.
[13] Bowen-James, A.(2002) *The perception of learning environments among tertiary mathematics students*. Sc. Ed. D thesis, Curtin University, 2002.
[14] E. Fennema, *Explaining sex-related differences in mathematics: theoretical models.* Educ. Stud. Math., 16 (1985), pp. 303-21.
[15] J. Watson and K. Beswick, *Mathematics: essential research, essential practice*. Mathematics Education Research Group of Australia, Adelaide, South Australia, 2007.
[16] M.R. Freislich, *A comparison between the effects of Keller Plan and traditional teaching methods on structure of learning outcomes among tertiary mathematics students*. Sc. Ed. D thesis, Curtin University, 1998.
[17] J.B. Biggs and K.F. Collis, *Evaluating the quality of learning: The SOLO taxonomy*, Academic Press, New York, 1982.

The language of linear algebra

JOHN HANNAH[§§§§§§]

University of Canterbury, New Zealand.

Abstract: Most students learn the techniques of linear algebra fairly readily. However, because the techniques can be applied without thinking, such success can hide a failure to understand the underlying ideas. One way to give the techniques some meaning is to emphasize geometric interpretations of these ideas but, even with this, students still struggle to articulate any understanding. This paper is a preliminary report on an attempt to improve students' understanding by developing their powers of self-expression.
This experiment took place in a second year linear algebra course aimed at science and engineering students. In addition to the usual features of such a course, students were set true-false questions and mini-essays. True-false questions encourage careful reading and critical thinking. Essays, particularly the sentence structure and the connections between these sentences, encourage the recognition of connections between the key ideas of linear algebra. Students' ability to express themselves clearly was surprisingly good. The clear expression revealed a good understanding of concepts such as *linear combination* and *span*, but it also revealed a widespread failure to grasp concepts such as *subspace*.

1. Introduction

Two years ago, the author took over a second year linear algebra course aimed at middle to lower ability engineering and science students. The course begins with an introduction to the basic concepts of linear algebra, from *span* and *linear combination* through to *basis* and *dimension*, all within the context of $\mathbf{R}^n$. After making enquiries, it became apparent to the author that it was unlikely that these students would ever need to find a basis for a column space (for example) even in a later mathematics course. What they would need, it seemed, was a facility with the language of vector spaces. So the author made this a major goal for the first part of the course. This paper looks at whether focusing on the language of vector spaces has helped the students to understand those basic concepts.

From a theoretical perspective, we can get a useful description of the student's introduction to linear algebra by looking at David Tall's three worlds: the embodied world, the symbolic world and the formal world [7]. The embodied world is based on the perception of, and reflection about, properties of objects which might be either experienced in the real world or imagined in the mind. The symbolic world is the world of symbols, algebra and algorithms, while in the formal world, concepts are introduced by definitions, and their relationships delineated by stating and proving theorems (see Table 1). Although the symbolic world may initially grow out of the embodied world within school mathematics [8, p. 9] it can become an apparently self-sufficient world for many university students, a world where equations appear out of nowhere and are solved by familiar algorithms such as Gaussian elimination. Furthermore, it is often a silent world, where problems are solved simply by calculations, with no accompanying explanation. Tall suggests that mathematical maturity comes through an integration of the three worlds [8, Figure 2] and recent

[§§§§§§] E-Mail: J.Hannah@math.canterbury.ac.nz

texts, such as [3] or [4], encourage this by linking the embodied and symbolic worlds through an emphasis on geometry. In the matrix-based introduction to linear algebra used in these texts, students meet the formal world in the supposedly simpler setting of $\mathbf{R}^n$ but, as Hillel and Mueller point out [2, p. 58], students still have to face the notions and language underpinning the general theory of vector spaces.[*******] It is this language that I hoped my students would learn to use.

Table 1: Linear algebra as seen in Tall's three worlds.

Embodied World	**Symbolic World**	**Formal World**
Mental perceptions of real world objects (e.g. pens, sheets of paper, rods, walls) and internal conceptions involving spatial imagery (e.g. points, lines, planes and maybe even hyperplanes)	Symbolic algebra with scalars, vectors and matrices. Processes and algorithms for solving equations (e.g. Gaussian elimination), and for transforming vectors (e.g. Gram-Schmidt process) or matrices (e.g. LU factorization).	Defined concepts, examined axiomatically through formal proofs (e.g. vectors, scalars, span, linear independence, basis, dimension)

2. Method

The subjects discussed in this paper come from a second year linear algebra course, taught by the author, and consisting of 118 middle to lower ability, science and engineering students. The author delivered all the lectures and ran one of the eight tutorials. The set text for the course was [3] but the author has also been influenced by books like [4].

Typically students in this course have already met several basic aspects of linear algebra in their first year courses: solving systems of linear equations by Gaussian elimination, an introduction to matrix algebra, vector geometry, and an introduction to determinants (viewed as the magnification factor of the transformation of n-space given by multiplication by a matrix). However, this second year course is the first time they have met the technical language of vector spaces (linear combination, span, linear independence, and so on).

Most of the data discussed here was collected from the final exam for the course. While such data lack some of the fine detail and objectivity of formal research projects, such as [5] and [6], it also has some compensating features, such as large numbers of subjects who are strongly motivated both to learn the material and to convey their grasp of the material to the examiner. Thus there is the potential for the data to reveal large-scale trends, and to suggest fruitful lines of inquiry for future research projects.

[*******] For the purposes of this paper, Tall's embodied, symbolic and formal worlds correspond roughly to Hillel's three modes of description (see [1, p. 192] or [2, p. 58]): the geometric mode, the algebraic mode, and the abstract mode, respectively. I have preferred the idea of worlds because students can live in these worlds, speaking the language appropriate to each world. For Hillel, though, the matrix based approach to linear algebra counts as operating in algebraic mode (as opposed to the abstract mode where general vector spaces are discussed.)

Students were given the opportunity to engage with each of Tall's three worlds. For the embodied world, geometric interpretations were emphasized throughout the course, with key concepts often illustrated during lectures using the pointers and flat surfaces common to most lecture theatres. Traditionally, much of a student's work in a linear algebra course takes place in the symbolic world. To help dispel the silence which usually accompanies this work, the need for written explanation was emphasized through feedback in fortnightly homework. Contact with the formal world of linear algebra came mainly through the introduction of key concepts in the context of $\mathbf{R}^n$. Students were encouraged to treat the new vocabulary associated with these concepts as part of a language which needed practicing, both orally and in writing. In addition to the usual tasks involving calculations, students were also given true-false questions where they were expected to explain their answers. The answers to such questions often depend on both careful reading and critical thinking, and so they indirectly encourage students to pay more attention to language. A mini-essay task was set in each exam. This task asked students to describe the relationships between certain key concepts, and a tight word limit was imposed to prevent the students from simply repeating the relevant definitions. The aim was to assess the students' understanding of key ideas and their relations to one another, and to see whether they had picked up some of the new language.

3. Summary of results

As part of their final examination, students were asked several short questions (see Figure 1) which were intended to test their understanding of some of the basic concepts of linear algebra.[†††††††]

3. (a) Does the vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

belong to the span of the vectors

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}?$$

(b) Give an example of a subspace of $\mathbb{R}^4$ which has dimension 3.

(c) Give an example of a nonempty subset of $\mathbb{R}^2$ which is not a subspace of $\mathbb{R}^2$.

Figure 1: Questions about span and subspace, from final exam.

3.1 Responses to Question 3(a): Span (and linear combination)

[†††††††] To save space in what follows, I have transcribed students' vectors and matrices using MATLAB notation. Thus vector or matrix entries are listed in square brackets, row by row, separated by spaces, with semicolons indicating the end of a row. For example, the first column vector in Question 3(a) would be written as [0;0;1] while the 3x2 matrix whose columns are the second and third vectors in Question 3(a) would be written as [2 3;3 4;0 0].

The class as a whole seemed to have a good grasp of the concept of the span of a set of vectors. Thus 99 students (or 84% of the class) gave correct answers to Question 3(a) complete with a well articulated explanation. Sometimes, the explanation was in terms of a symbolic equation. For example, Student A said:

> No, for it to be in the span of these vectors, call them $\mathbf{x}_1$ and $\mathbf{x}_2$, there must exist c_1 and c_2 such that $\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, however the equation that would describe the bottom row is
>
> $c_1.(0) + c_2.(0) = 1$
>
> which is clearly not the case.

Most students (60%) made an explicit link to the concept of *linear combination*. For some students the relevant calculations were implicit, as in Student B's answer:

> No, since the vector has a 1 in the third row, it cannot be a linear combination of the others because they both have a zero in the third row.

Others made a similar statement after first setting up the symbolic equations corresponding to the linear combination (see Student N below) but a small number of students felt the need to carry out a full Gaussian elimination, perhaps seeking the comfort of the symbolic world they are familiar with. For example, Student C said

> Is [0;0;1] a linear combination of [2;3;0] and [3;4;0]?
>
> [2 3;3 4;0 0] $\mathbf{x}$ = [0;0;1] → [2 3 0;3 4 0;0 0 1] $3R_1 - 2R_2 \rightarrow R_2'$
>
> [2 3 0;0 1 0;0 0 1]
>
> The system is inconsistent as seen from row 3.
>
> Thus [0;0;1] is not in the span of vectors [2;3;0] and [3;4;0].

Finally, a very small number of students (3 out of 118) showed a preference for a geometric interpretation, linking span with the embodied world. For example, Student D said:

> No, the vectors [2;3;0] and [3;4;0] both have zero components in the z direction (their third entry).
>
> There is no way [0;0;1] can be in the plane spanned by these two, as it only has components in the z-direction.

3.2 Responses to Question 3(b): Subspaces

Only 35 students (or 30% of the class) gave satisfactory answers to Question 3(b), although a further 69 (or almost 60%) showed enough understanding to get partial credit. Among those giving satisfactory answers was Student E:

> a subspace spanned by [the vectors] [1;0;0;0], [0;1;0;0], [0;0;1;0]
>
> this has 3 vectors in its basis (so dimension = 3) as the 3 given vectors are linearly independent so form a basis for that subspace in $\mathbf{R}^4$.

As the concepts become more complex, even good students do not recognize the need for careful use of language. Thus Student F, who eventually earned an A+ for this course, said:

> A subspace S = span{[1;0;0;0], [0;1;0;0], [0;0;1;0]} is a plane in $\mathbf{R}^4$ that passes through the origin. The span of S above is given by three linearly independent vectors. Therefore, {[1;0;0;0], [0;1;0;0], [0;0;1;0]} is also a basis for the subspace. The basis has 3 vectors therefore S has dimension 3.

I was inclined to ignore the misuse of the word *span* here, but when the students' use of language was sufficiently out of line with standard mathematical usage, I awarded them only partial credit. Among the students earning partial credit here, a common theme was confusion between matrices and the subspaces associated with them (especially row or column spaces). Occasionally the students made this identification explicit, as in Student G's answer:

> A = [1 0 0 1;0 1 0 2;0 0 1 0; 0 0 0 0]
> The columns of this matrix span the subspace A and with 3 columns in the basis has a dimension of 3 as required.

This was then followed by an explanation of why only three columns are needed in the basis. But many students simply put down a 4x3 or 4x4 matrix of rank 3 with no explanation at all. Of course, writing down such a matrix is not all bad: presumably the student has grasped some of the ideas involved in basis and dimension, even if they do not see the distinction between subspaces and the objects which gave rise to them.

3.3 Responses to Question 3(c): Sets which are not subspaces

Only 24 students (or 20% of the class) gave satisfactory answers to Question 3(c), and 70 students (almost 60%) received no credit at all, including 31 students (26%) who gave a matrix as their answer.

Some students sought examples among familiar subsets of $\mathbf{R}^2$. With the right choice, the familiar context carried with it an easy way of seeing that the set was not a subspace. Thus Student F offered:

> $\mathbf{v}$ = [0;2] + t[1;3] is the vector equation of a line in $\mathbf{R}^2$.
> All points on this line are a subset of $\mathbf{R}^2$ which is a nonempty set. The line does not pass through the origin. Therefore, it is a nonempty subset of $\mathbf{R}^2$ which is not a subspace of $\mathbf{R}^2$.

Sometimes the student had to look at the familiar object in an unusual way. Thus Student C found himself adding points on a parabola:

> An example would be [x; x^2]
> which fails the condition $\mathbf{u}+\mathbf{v}$ is in the subspace.
> Thus [2;4] and [3;9] vectors in subspace of type [x; x^2]
> [2;4] + [3;9] = [5;13] → fails as $5^2 = 25$ and not 13.

Also successful was Student D who continued the geometric theme we saw earlier:

> The square in the xy plane bounded by the lines $1<x<2$ and $1<y<2$.
> This subset contains all the values of x and y between 1 and 2, but is not a subspace because it does not contain the origin and some linear combinations of vectors within it extend beyond its limits.

Although none of these three students drew a picture of their essentially geometric examples, a few students did draw pictures equivalent to Student F's example. Interestingly, 17 (that is, 70%) of the 24 correct answers involved some geometric object, linking this question with the embodied world. The other seven gave sets of just two or three vectors (pointing out that closure under addition or scalar multiplication failed) suggesting that they may have been working in the formal world.

Some students came close to success, but were let down by their grasp of the language. Thus the meaning of *span* seems to have been extended by Student A:

> U = span{[x;y]: x+y=4}
> Not a subspace as it doesn't include **0**, i.e. not closed under multiplication.

Student H seemed to use *span* in the same way, but clarified this extended meaning:

> A subspace must contain the **0** vector
> [Therefore] any spanning set that doesn't contain **0** will satisfy this
> let U = [1,1] + t[1,0] be a subset of $\mathbf{R}^2$ (= span([1+t, 1])
> This is not a subspace as it does not contain **0**.

This is perhaps a situation where ordinary language is more flexible than mathematical language. A few students struggled with another mathematical convention that violated their understanding of how English works: the whole space is a subspace of itself. Thus Student I combined this idea with the confusion of matrices and subspaces that we saw earlier and said:

> [1 0;0 1] in which the vectors [1;0], [0;1] constructed the entire 2-space.

a theme repeated by Student J:

> [1 0;0 1] this spans the whole of $\mathbf{R}^2$, so is not a subspace of $\mathbf{R}^2$. Also written as {[1;0], [0;1]} as a basis for $\mathbf{R}^2$.

An unexpected error saw students giving answers which indicated that they thought that subspaces had something to do with linear independence (or dependence). For example, Student K said:

> [1 2;2 4] is a non-empty subset of $\mathbf{R}^2$, but is not a subspace of $\mathbf{R}^2$, as the columns of the matrix are not linearly independent.

and Student L said:

> U = {[1;1], [3;4]; 4;5]}
> These are not linearly independent. [Therefore] no basis. [Therefore] no subspace.

Most answers along these lines offered no further explanation, but one student gave a 'definition' which might offer a possible clue to the underlying thinking (although his actual example was neither dependent nor in $\mathbf{R}^2$). Thus Student M said:

> A subspace is a linearly independ[ent] set of vectors which span the subspace.
> So I'm going to choose vector is linearly depend[en]t.
> $\mathbf{u}_1$ = [1;1;0] $\mathbf{u}_2$ = [-3;0;4].

4. Progress through the hierarchy of concepts

The examples above show a general decline in understanding at the class level, as we go from Question 3(a) to 3(c). In the case of some of the more articulate students, it is possible to see this decline at the individual level too. For example, Student N produced a correct answer to Question 3(a):

> In other words, we are checking whether the vector is a linear combination of the other two vectors.
> We note that
> $s[2;3;0] + t[3;4;0] \neq [0;0;1]$
> as $0s + 0t \neq 1$
> where it shows that the system cannot be solved.
> Therefore the vector [0;0;1] does not belong to the span of the vectors [2;3;0] and [3;4;0].

but seemed to confuse matrices with their associated subspaces in Question 3(b):

> The example matrix of $\mathbf{R}^4$ which has dimension 3 is
> A = [1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 0]
> since A is a 4x4 matrix which is already in reduced row echelon form and the vectors are linearly independent set of vectors, they form a basis for subspace $\mathbf{R}^4$ with dimension 3.
> [Therefore] {[1;0;0;0], [0;1;0;0], [0;0;1;0]} span the subspace of $\mathbf{R}^4$.

and finally became completely lost in Question 3(c), where the confusion of matrices and subspaces extended to vectors too, and was linked further to linear dependence:

> Example of a nonempty subset of $\mathbf{R}^2$ which is not a subspace of $\mathbf{R}^2$
> A = [1 2;2 4]
> or the vectors {[1;2], [2;4]}
> They are not a subspace of $\mathbf{R}^2$ because
> [2;4] is a multiple of [1;2]
> [Therefore] only {[1;2]} forms the basis of $\mathbf{R}^2$.

5. Discussion

As remarked above, there was a clear decline in the students' understanding, as we go from Question 3(a) to 3(c). This decline was quantified in the exam, albeit rather crudely. Each of the questions was marked out of 2 points, and with this measure the average mark for Question 3(a) was 1.8, for Question 3(b) was 1.2, and for Question 3(c) was 0.6.

I was pleasantly surprised by the students' answers to Question 3(a). Students in the small case study described in [5] seemed confused and inarticulate when talking about span and linear combination. Admittedly, my students were not subjected to in-depth interviews as in the case study, but their clear expression seems to reveal a good understanding of not just the target concept of *span* but also the related concept of *linear combination*. With only three students explicitly referring to embodied notions, it is difficult to argue that this good performance is due to my emphasis on geometric interpretations, as seemed to happen in [5]. So perhaps my emphasis on verbal explanations has contributed to student understanding in this case.

On the other hand, my students' answers to Questions (b) and (c) were quite disappointing, revealing a widespread failure to grasp the concept of *subspace*. It seems, though, that the students' powers of self expression in some sense outrun their understanding of the concepts here. Thus we get answers where sentences look grammatically correct, but individual words are used incorrectly, at least from the strict mathematical point of view. There may be only one word slightly out of place, as with *span* in Student F's answer to Question 3(b), so that the general sense is clear and the student may even be thought of as still understanding what is going on. But there may also be glaringly incorrect words, as in Students K, L and M's answers to Question 3(c), revealing a fundamental misconception. Whether my emphasis on language has helped the students is hard to tell, but it does seem to help the diagnosis of student difficulties.

Why did the students do so much worse in Question 3(b) than in 3(c)? After all, both questions were about subspaces. We can only conjecture, on the basis of some general trends. There is little doubt that my students arrived in the course comfortable in the symbolic world of routine calculations with vectors or matrices. My guess is that it was in this world that most of them sought the desired examples for Questions 3(b) and (c). In (b), they were lucky that a suitable matrix (a symbolic object) came close to the right answer. However, for (c) it seems that success depended on having a good repertoire of subsets of $\mathbf{R}^2$. As most of the students who did well here used geometric subsets (lines, parabolas or even a square) this suggests that students performed better in (c) if they had good links with the embodied world. The remaining successful students in (c) all focused on violating the closure rules for a subspace, giving subsets of just two or three vectors. This suggests that they were working in the formal world. A reliance on the symbolic world may have been the reason so many students gave a matrix as their answer for (c) but, if so, using just the symbolic world was a recipe for failure here.

6. Conclusions

In conclusion, it is encouraging that most of the students understood some simpler concepts from linear algebra such as *span* and *linear combination* and were able to express this understanding clearly. Given the hierarchical nature of the key concepts of linear algebra, it is not too surprising that students struggled to grasp concepts, such as *subspace*, which are higher up the hierarchy. It is interesting that students' ability to express themselves sometimes outstripped their actual understanding, as in their confusion between *subspace* and *linear independence*. This mirrors (at a higher level) the common observation of small children mastering the names of numbers long before they understand what those numbers actually mean. For the future, further iterations of the process described in this paper may help students grasp the higher level concepts, so that basic concepts such as *span* and *linear combination* could be assessed, and feedback received, before assessment moves on to higher level concepts such as *subspace*.

7. References

[1] J. Hillel, *Modes of description and the problem of representation in linear algebra*, in *On the Teaching of Linear Algebra*, J-L Dorier, ed., Kluwer, Dordrecht, 2000, pp. 191-207.

[2] J. Hillel and U. A. Mueller, *Understanding multiple interpretations of the notion of solution in linear algebra*, Proceedings of Fifth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning, University of Queensland, 2005, pp. 57-64.

[3] D. C. Lay, *Linear Algebra and its Applications*, 3rd ed., Pearson, Boston, 2006.
[4] D. Poole, *Linear Algebra: A Modern Introductio*n, 2nd ed., Brookes/Cole, Belmont, 2006.
[5] S. Stewart and M. O. J. Thomas, 2007. *Embodied, symbolic and formal aspects of basic linear algebra concepts*, Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Vol. 4, pp. 201-208. Seoul: PME, 2007.
[6] S. Stewart and M. O. J. Thomas, 2009. *A framework for mathematical thinking: The case of linear algebra concepts*, International Journal of Mathematical Education in Science and Technology (to appear).
[7] D. O. Tall, *The three worlds of mathematics*, For the Learning of Mathematics, 23 (3) (2004), pp. 29-33.
[8] D. O. Tall, *The Transition to Formal Thinking in Mathematics*, Mathematics Education Research Journal, 20 (2) (2008), pp. 5-24.

sigma: providing university-wide mathematics support

M.C. HARRISON[‡‡‡‡‡‡‡] AND M.G PETRIE

Mathematics Education Centre, Loughborough University, Loughborough, UK

Abstract: Both Loughborough and Coventry Universities have excellent track records in the provision of mathematics support services to students from diverse backgrounds. In 2005, following a collaborative bid, they were designated as a Centre for Excellence in Teaching and Learning (CETL) in the provision of mathematics and statistics support by the Higher Education Funding Council for England (HEFCE). This CETL, known as **sigma**, has received recurrent and capital funding totalling £4.85m for the period 2005-10, so that it could further enhance these exceptional mathematics support services and disseminate them both nationally and internationally to the benefit of those in other Higher Education Institutions (HEIs).

This paper gives some background to the establishment of **sigma** before describing some developments in its activities made possible through the CETL funding. First, our drop-in facilities, which provide 1-to-1 staff support and an extensive range of paper-based resources, have been considerably enhanced. Second, a significant effort has been made into the development of comprehensive statistics support at both Universities. Third, we have invested time in investigating how new technologies can be used to enhance the teaching and learning of mathematics. Fourth, we have set out to address the problems of all students, rather than individuals, through a deliberate strategy of proactive teaching interventions. Fifth, the Eureka Centre for Mathematical Confidence at Loughborough has been set up to support students with mathematics anxieties. Pedagogic research supports all these activities. Finally, a major aim of **sigma** has been to broaden its impact via staff secondments from other universities to work on projects; as such it has been the catalyst for the development of support centres at three other English universities, and has helped to establish networks of support professionals in Scotland, the South-West of England and the Republic of Ireland. There are plans to expand this element of **sigma**'s activity to ensure that all staff working in the UK have access to a local support network and events. **sigma**'s dissemination strategy has focused on engaging the wider higher education community both in the UK and beyond. Evaluation of all these activities is under way and will be reported in due course.

Keywords: Mathematics support, statistics support, proactive teaching intervention

1. Introduction

Mathematics education is a crucial issue for Higher Education (HE) in the UK because mathematics underpins so many other disciplines and undergraduate courses, not only in the physical sciences and engineering, but also in fields such as economics, business and health and life sciences. The challenge has been highlighted by a number of reports, for example Roberts [1] and Smith [2]. In addition, the recognition of the importance of numerical skills as a core part of the employability skills of graduates underlines the value of university-wide provision of mathematics support.

There are widespread concerns expressed by government, educators and employers that so many students embarking upon higher education courses find that their mathematical backgrounds are insufficient to enable them to make satisfactory progress and succeed. A report by the National Audit Office in 2007 [3] stated,

[‡‡‡‡‡‡‡] E-mail: M.C.Harrison@Lboro.ac.uk

"Many students require some additional academic support, especially in mathematical skills". Too many young people in the UK have already become alienated and disengaged from the learning of mathematics before they reach university. The challenge for university educators is to encourage these students to overcome their difficulties and to want to succeed.

Loughborough University (LU) and Coventry University (CU) are acknowledged leaders in the field of university-wide mathematics support. They have impressive track records, both individually and collaboratively, of providing exceptional mathematics support services as well as UK-wide and international dissemination activities to benefit those in other institutions. This expertise was formally recognized by the Higher Education Funding Council for England (HEFCE) in the award of a collaborative Centre for Excellence in Teaching and Learning (CETL) in 2005. A five-year funded project, **sigma** CETL (see http://www.sigma-cetl.ac.uk/) is receiving recurrent and capital funding totalling £4.85m for the period 2005-2010, so that it can further enhance these exceptional mathematics support services and disseminate them both nationally and internationally to the benefit of those in other Higher Education Institutions (HEIs).

sigma's rationale is based upon two principles:

1. Mathematical and statistical tools are essential in many disciplines, and
2. Significant numbers of students have not developed the bedrock in mathematics and statistics required for more advanced study; they need substantial support if they are to meet their course requirements.

A key aim of **sigma** is to move the balance from reactive to proactive activities in order to start to address the problems for the *mass* of students in need. Thus, the primary purpose of **sigma** is to build upon the established excellence of LU and CU in the field of generic mathematics and statistics support to effect a culture change amongst staff and students so as to influence modes of teaching and learning within and beyond the two institutions, for students from a wide range of disciplines and backgrounds using mathematics and statistics. **sigma** also aims to stimulate and encourage similar proactive activity across the HE sector in the UK and beyond.

The work of **sigma** is split up into 6 core activities and other associated activities as detailed in Figure 1. Whilst recognising that much of this work is still in progress and not yet fully evaluated, this paper seeks to describe some developments in these activities and how we are attempting to extend our reach to the wider HE community.

2. Drop-in Centres

A core activity of **sigma** has been the enhancement of the mathematics and statistics drop-in provision at both institutions. As well as providing 1-to-1 support and advice, academic staff are available for consultation and specialist help at certain times, e.g on Maple, SPSS. A comprehensive range of paper-based resources is available to all students including the HELM (Helping Engineers Learn Mathematics) resources (see http://helm.lboro.ac.uk/) and online help is provided via mathcentre (see http://mathcentre.ac.uk/), the UK on-line Mathematics Learning Support Centre. Individual and group study space has been boosted by the creation of separate dedicated study rooms for 2nd and 3rd year mathematics students, clear evidence of **sigma**'s commitment to support students of all abilities improve

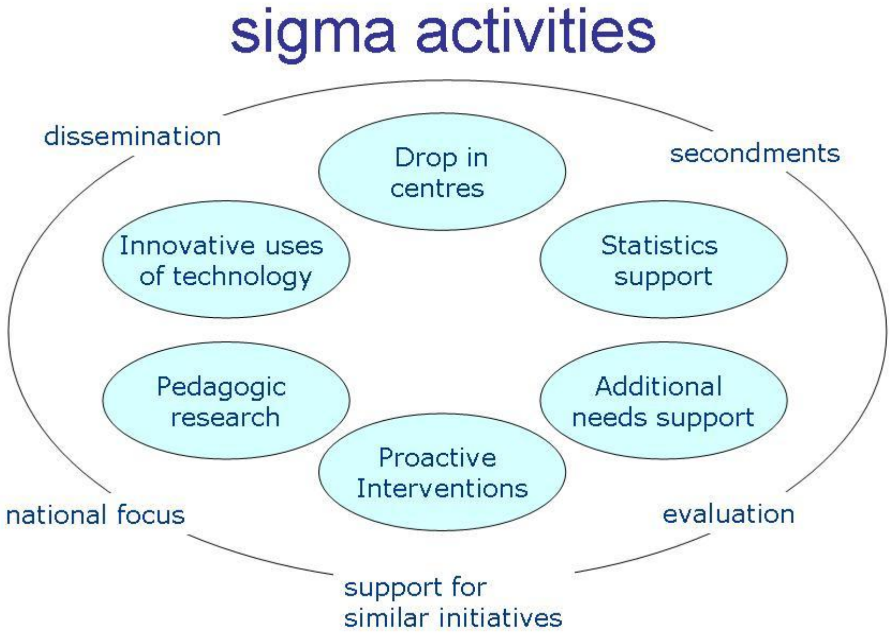


Figure 1. **sigma** activities.

Table 1. Student visits by location and year.

Number of visits	2004 baseline	2005/06	2006/07	2007/08	2008/09
Loughborough	4,376	3,926	4,617	6,490	8,023
Coventry	1,901	2,889	3,549	4,342	4,570
Total	6,277	6,815	8,166	10,832	12,593

Table 2. LU student visits by department and year (previously under-represented groups).

Number of visits (by department)	2005/06	2008/09
Business School	41	293
Chemistry	10	80
Computer Science	13	72
Economics	42	516
European and International Studies	2	83
Human Sciences	16	172
Social Sciences	4	30

their mathematical skills. Since 2005, there has been a year on year increase in the number of student visits across all locations, as illustrated by the usage figures in Table 1.

At LU capital funding was used to refurbish the existing centre as well as to establish a second site at the other end of campus. Across its two sites, the Mathematics Learning Support Centre (MLSC) offers 40 hours of drop-in support each week through term time. The MLSC is also open during the Resit Examination Period with drop-in support offered in the afternoons. Since the opening of the second site in 2006/07, there has been a significant increase in visits by students from previously under-represented departments, as illustrated in Table 2.

At CU the existing Mathematics Support Centre (MSC) centre was moved to a bigger room in order to accommodate the growth in the number of students supported. The new centre now has seating for 42 students at study desks and provides nine computer stations for student use. Support is offered for 31 hours per week during term time, with support provided by lecturers from departments within the Faculty of Engineering and Computing. MSC usage by students from other faculties has also increased since the inception of **sigma**, see Table 3. Note that the MSC also provides drop-in and revision workshops specifically for nursing and midwifery students, located within their own Faculty building. In 2008/09, over 150 nursing students were supported through this provision. In addition to the students attending the Drop-in Centre and 2nd/3rd Year Resource Room, in 2008/09, a total of 372 students visited the Outreach Desks in locations around the campus, including the Library.

Table 3. CU student visits by Faculty and year.

Number of visits (by Faculty)	**2006/07**	**2008/09**
Business and Environmental Science	**212**	**342**
Health and Life Science	**78**	**77**
Engineering and Computing	**3175**	**3367**
Art and Design	**1**	**20**

Recent institutional audits provide evidence of some of the positive effects **sigma** has had across its host institutions. The Institutional Audit Report of LU commented that:

> "the ready accessibility of useful help was praised by both undergraduate and postgraduate students that met the audit team. Other students described the benefits of the support rooms and associated equipment. Postgraduate students were appreciative of the one-to-one help and individual study programmes provided for them by the Centre."

The submission to the audit process from Coventry University Students Union commented that the University "*should be commended for…the continued support of the Maths Support Centre (**sigma**).*" Similar praise was recorded from students within the Faculty of Engineering and Computing. Students believe that the staff, resources and technology are of a good standard, and they should be further publicised.

3. Statistics support

The establishment of the Statistics Advisory Service (SAS) at LU and CU was one of **sigma**'s first undertakings and has proved to be one of its most successful achievements. An appointment-based service, its aim is to offer support to final year undergraduates and taught masters degree students who are undertaking a project/dissertation, and to provide statistical advice to research students. Support and advice on a wide range of topics is provided, but the main focus of the SAS is the design of surveys or experiments, the development of questionnaires and the use of appropriate statistical techniques for analysis. **sigma**'s original target was to have supported 700 students by the end of the CETL programme in July 2010. Table 4 shows that this target has already been surpassed, 840 students having been supported over the past four years.

Table 4. Appointments for statistics advice by location and year.

Number of appointments	2004 baseline	2005/06 (target)	2006/07 (target)	2007/08 (target)	2008/09 (target)
Loughborough	**0**	**63 (50)**	**136 (60)**	**125 (70)**	**165 (75)**
Coventry	**0**	**66 (50)**	**98 (60)**	**92 (70)**	**95 (80)**
Total	**0**	**129 (100)**	**234 (120)**	**217 (140)**	**260 (155)**

The SAS service is much appreciated, regularly receiving positive comments:

> "The one-to-one approach was essential as was the non-judgemental attitude of the lecturer who took pains to explain what would no doubt would have been ordinarily, an elementary point, with patience and understanding."
>
> "As a PhD student I have a lot of stats to do and after being out of education for a couple of years this was very scary! It's been extremely helpful to have someone to chat through ideas with as this has helped clarify things for me whilst not having to ask my supervisors each time"

In addition to the face-to-face appointments, the SAS also offers drop-in sessions. These sessions are open to students looking for guidance on statistical problems or help with a statistical software package. Often these sessions enable the tutors to identify those students who would benefit from accessing the appointment-based service.

Another successful element of the SAS has been the development and delivery of a series of workshops on key statistics topics. These have been developed to give students the opportunity to revise, practice and develop basic statistical skills, which will be useful in their mainstream studies. Originally advertised to postgraduate students, there has been an increasing demand for places on these workshops from staff. Topics covered include Questionnaire Design, Introduction to Analysis of Variance, Simple Linear Regression and Correlation and Experimental Design and workshops on the SPSS and Minitab statistical software packages.

In 2008/09 around 75% of delegates surveyed after the Coventry workshops, agreed that the workshops "*covered topics that are potentially useful for their future research*". In addition, almost 90% agreed that they "*will use what they have learned in the future*", and almost 95% agreed that "*The teaching staff worked hard to make their topics interesting".* One delegate commented that: "*I really found the first* *session useful and liked your informal style of working with us. I find stats quite intimidating so your approach helped a lot to overcome my initial anxieties!"*

Collaboration has been vital in the development of these successful services. Statistics staff at both institutions work very closely with each other and also meet regularly with staff from the Royal Statistical Society Centre for Statistical Education and Lancaster University's Postgraduate Statistics Centre.

4. Investigating innovative uses of technology

sigma aims to implement and evaluate innovative teaching and learning approaches using state-of-the-art and emerging technologies, linked to a programme of pedagogic research. Capital funding has enabled **sigma** to purchase a wide range of equipment to improve the student learning experience. This included a full recording studio as well as mobile editing equipment, electronic voting systems, hand-held devices (mobile phones, iPods, etc.), Bluetooth technologies (e.g. writing tablets), classroom management software, Interactive Whiteboard technology and software, and robotic and technical LEGO®. Interactive classrooms have also been created at both LU and CU. This work has been co-ordinated through the Innovative Uses of Technology Group, a joint working group that enables cross-fertilisation of ideas and experiences.

The focus has now turned to evaluating the use of this equipment in different settings and to disseminating the findings to the wider teaching and learning community. To facilitate this, opportunities have been provided by **sigma** for academics in departments across LU and CU to bid for funding to purchase technologies and implement them into their teaching. Each funded project will report back with an evaluation of whether their chosen technology has influenced the way they teach. The chosen technologies included interactive panels and Bluetooth writing tablets, specialist acoustic software, Bluetooth server, Matlab Wavelet Toolbox and Interactive Whiteboards.

In Spring 2009, **sigma** agreed to focus continued efforts on four key technologies, namely: Tablet PCs, Elluminate® Online Classroom, Robotic LEGO® and Electronic Voting Systems. A recent call for project funding proposals (in May 2009) relating to these technologies has resulted in the funding of an additional five projects, all of which will provide evaluation reports before the end of July 2010.

5. Proactive teaching interventions

Whilst drop-in centres can deal with problems on an individual basis they are *reactive* rather than *proactive* – the system relies on students identifying that they have a problem and having the motivation and courage to come and get it sorted out. Anecdotal evidence suggested that many students who might benefit did not take advantage of the support on offer, and thus there was a need to target these students much more energetically. Consequently, a key driver of **sigma**'s work has been to move the balance from *reactive* to *proactive* support in order to address problems for the mass of students in need. Its strategy has been to develop a portfolio of activities, which have become known as Proactive Teaching Interventions (PTIs). Since its inception, **sigma** has undertaken 70 PTIs, as detailed in Table 5.

For example, in one PTI within the Department of Economics at LU, **sigma** staff have taught small groups of students requiring additional mathematics support

Table 5. Proactive teaching interventions by mode and year.

Mode\Year	2005-6	2006-7	2007-8	2008-9	Total
Alternative provision	**5**	**6**	**0**	**4**	**15**
Secondments into sigma				**5**	**5**
Supplementary Teaching (at risk)	**2**	**3**	**2**	**4**	**11**
Supplementary Teaching (whole cohort)	**5**	**3**	**2**	**5**	**15**
Secondments out of sigma			**1**	**1**	**2**
Enhanced teaching	**3**	**3**	**7**	**9**	**22**
Total	**15**	**15**	**12**	**28**	**70**

for their Economics programmes. All 1st year undergraduates take a diagnostic test on arrival and those identified as in need of extra support are provided with a personal development plan to address their particular weaknesses. The module leader commented:

> "The experience of the Department of Economics is that this activity has been wholly beneficial. ... The undergraduate support has been the subject of very favourable feedback … There is no doubt in my mind that the CETL project and the MLSC activity in general is of major benefit to students of Economics and consequently is a major support to the Department."

The whole PTI programme of activity is still being evaluated and a part-time researcher has been employed to assist in this. Even at this early stage lessons have been learned, some of which are reported below.

First, the PTIs involving secondments have provided excellent staff development opportunities. For example, a lecturer in Politics and International Relations tasked with designing a second year research methods module was awarded a **sigma** secondment to work with a **sigma** colleague to be trained in the general area of statistical methods and to develop resources for the module. He later commented "*I could never have carried out this type of teaching or research prior to my **sigma** secondment, so I can safely say it has been one of the most valuable experiences of my career*".

Second, several of the interventions have enabled staff to integrate new technologies into their teaching. For example, there have been opportunities for staff to use video tutorials and other on-line resources, and personal response systems. The reported feedback from students has been good.

And third, where students have been identified as at risk of failing and have been provided with alternative learning opportunities, in general outcomes have not been as good as we would have hoped. A common theme across many of these interventions is that the selected students fail to engage enthusiastically, and generally (but not exclusively) pass rates have not improved as a consequence of these interventions. There is some evidence that it is not a good idea to select weaker students and teach them separately. Where students self-select, engagement is much better, and staff have repeatedly reported this across several interventions. However, the very students that **sigma** needs to engage with most of all are the least likely to self-select.

Finally, where innovative opportunities for whole groups of students are provided, it is essential that the module leader is 'on-side' and promotes the intervention in lectures. It is essential to ensure that students believe that participation in the intervention will improve their chances of success and we have not always succeeded in this. Formally assessing students on elements of the intervention is likely to improve their engagement.

6. Support for students with additional needs

sigma is extending existing specialist support for students with additional needs or specific learning difficulties, and ensuring the transfer of expertise between LU, CU and beyond. This includes working with students with a range of support needs, such as dyslexia, dyscalculia, maths phobia and blindness as well as those students who lack confidence in their maths or statistics. In addition, support is provided to help build confidence with employer numeracy tests and workshops are offered focusing on the aims of numeracy tests, how to prepare for them, pre-requisite skills you should have and can practice, as well as looking at typical questions.

Work at LU has included the provision of dedicated learning space through the creation of the Eureka Centre for Mathematical Confidence (see http://eureka.lboro.ac.uk/). Students were consulted in the design and naming of the Eureka Centre, which aims to provide a non-threatening environment for those students who often feel very anxious and lack confidence with mathematics. As a by-product of **sigma** development, the UK's first Postgraduate Certificate course relating to dyslexia and dyscalculia in mathematics in FE/HE has been developed by the Eureka Centre; its first intake was in April 2007.

7. Pedagogic Research

A key objective of **sigma** is to underpin all developments with an extensive programme of pedagogic research. As access to HE continues to widen and the demand for quantitative and mathematical skills amongst non-specialists increases, this sector provides fertile ground for both basic and applied research, which can help shape policy and practice as well as help individual students. **sigma**'s current research focus includes:

- What factors influence student attitudes towards mathematics and statistics?
- What are the key motivators to study for students?
- What underlies the low attendance rates of some sections of the student body?
- What measures can be used to quantify the effectiveness of support mechanisms?
- Can the potential of emerging mobile technologies be effectively utilised for lasting benefit?
- What are effective technologies for the support of students with additional needs?

CETL funding has enabled considerable growth in research capacity through the appointment of a Senior Research Fellow and the funding of additional PhD studentships. All PhD students are offered support through the **sigma** PhD Research Group, which meets every couple of months and provides a forum for discussion as well as skills training and advice. As part of their skills development, the group organised the inaugural Higher Education Mathematics Education Conference (HEMEC), which was held at LU in September 2008. This successful event, attended by over 40 delegates, explored the themes of mathematical thinking, technology and socio-cultural influences at the tertiary level of mathematics education. A second HEMEC is planned for November 2009 and will be held at the University of Limerick in the Republic of Ireland.

Research has been published relevant to all of **sigma**'s core cluster groupings. In addition, a body of work has been collated regarding measuring the effectiveness of mathematics support centres
(see www.mathcentre.ac.uk/staff.php/all_subjects/measuring_effectivess/resources).

8. Extending our reach: sigma's dissemination strategy

Since in its inception, **sigma** has been keen to extend its reach (and the benefit of its funding) into its partner institutions and beyond. As such **sigma** actively encourages other institutions to visit its centres to find out more about the provision offered and the spaces provided. It has also implemented a wide range of dissemination and collaboration activities as well as developing projects and initiatives with a growing number of institutions. **sigma** continues to play a key role in mathematics support nationally, working closely with the Higher Education Academy MSOR Network (http://www.mathstore.ac.uk/) and acting as advisor to other UK and non-UK institutions interested in implementing a model of support or further developing their provision.

8.1 Secondments and Joint Projects

Over the past 4 years, **sigma** has funded nine secondments, with six of the secondees coming from outside LU and CU. Two types of secondments are available – short term (e.g. one week at one of the partner institutions) or long term (covering one semester or one academic year). Some of the secondments have fed into the Proactive Teaching Interventions strand of activities, whilst others are focussed on technical skills development, generic resource development or piloting new provision. All secondees have been encouraged to present their work at the annual CETL-MSOR Conference (see http://mathstore.ac.uk/index.php?pid=253) and publish a paper in MSOR Connections.

sigma also works closely with a number of academics through its Visiting Fellows scheme. This has enabled individuals to link with **sigma** members to work on research projects, including the development of a series of motivational sessions for the second year mathematics students and the learner identities of mathematics students and their use of social learning spaces.

8.2 Funded Centres Initiative

One of the initial objectives of the CETL was to provide "pump priming" funding for the establishment of new mathematics support provision at other institutions. As part of its original bid, the University of Leeds was identified as the initial institution that would receive funding for two years. This provision has continued to be funded from within the University of Leeds and is now embedded in terms of management and delivery into a comprehensive level of provision supporting students, researchers and staff to enhance their learning, teaching and research under the global banner of Skills@Library.

Following the success of the mathematics support initiative at Leeds, in summer 2007 **sigma** funded the establishment of two new mathematics and statistics support initiatives, at the Universities of Bath and Sheffield. In addition to the two-year funding from **sigma**, both of these universities have made significant additional funding available to support the respective initiatives, enabling the recruitment of support staff and the allocation of space. At Sheffield, this provision continues to receive strong support from the University via the Pro-Vice Chancellor for Teaching and Learning; this has resulted in the centre being adopted by the University and the Centre Manager's post being made permanent. At Bath, funding has been secured for an additional year and plans for longer-term funding have been submitted. The explicit inclusion of the provision within the University Learning and Teaching strategy augurs well for further funding.

8.3 Local Hub Network

Although not part of the original **sigma** proposals, one of the unexpected outcomes of its outreach programme has been the inception and development of the **sigma** Local Hub Network. The key purpose of these hubs is to provide more local access to **sigma** events and workshops. In addition, they aim to:

- Facilitate the sharing of information and materials across their constituency
- Build a local network, with representatives from HE, FE and secondary schools, who are interested in mathematics support as it applies to them
- Co-ordinate views from the local network to feed into an annual forum
- Help raise the profile of mathematics support locally

Two hubs were established as part of the initial pilot; the South West Hub is co-ordinated through the University of Bath and the Scottish Maths Support Network is co-ordinated through the University of St Andrews and Glasgow University. Both hub networks have held events to promote mathematics support. **sigma** has also been offering advice on establishing similar networks in Ireland and North East England.

8.4 Student Engagement

The key to the success of many of **sigma**'s activities is to engage successfully with its core clientele – the students. This is particularly true when trying to promote the drop-in centres and associated resources. **sigma** has been active in working with students, particularly through the use of student ambassadors and summer interns, as well as offering Year 11/12 school students placements through Nuffield's Science Bursary Scheme (see http://www.nuffieldfoundation.org/go/grants/nsb/page_390.html).

At LU, in 2007, the **sigma** Student Ambassador Scheme was introduced, where six students from a range of departments were employed (3 hours per week during term time) to assist with the promotion of the MLSC, the Eureka Centre for Mathematical Confidence and the Statistics Advisory Service to fellow students within their home departments. The ambassadors provided valuable insight in terms of the promotion of the support on offer and how these are perceived by fellow students. As well as providing "shout-outs" in lecturers and representations within their home departments, the ambassadors organised a series of information events just before the Easter Break. They also wrote and produced a short promotional video that has been very well received (see http://mlsc.lboro.ac.uk/services.php).

Given the success in its pilot year, the scheme was continued in Academic Year 2008/09 at LU and introduced at CU. In addition to raising awareness through events and representations, the ambassadors have produced more publicity material, including poster and beer mat designs, and have reviewed and redesigned the MSC website at CU. The LU scheme is now included as part of the University's Employability Award Scheme.

Over the past two summers, **sigma** employed students through its Summer Intern Programme on 4-6 week projects. These projects have included development of question banks for use with electronic voting systems, creation of surveys and other resources for an interactive kiosk, evaluation of interactive whiteboard technologies and software, and creation of learning resources using robotic LEGO®. The Summer Intern Programme provides valuable personal development opportunities for those involved, enabling them to enhance their skills, gain work experience and thereby increasing their employability.

9. Concluding remarks

As a collaborative CETL, **sigma** is impacting on the systems and practices at both partner institutions. As part of its ongoing evaluation strategy, the opinions of the Pro-Vice Chancellors (Teaching) at CU and LU have been sought and both view the dissemination of activities and lessons learned through the CETL process as essential to emerging policies on teaching and learning; outcomes of CETL activities will be used to inform and promote change within the wider teaching and learning environment. The Pro-Vice Chancellors underlined the importance of maintaining the widening of provision of mathematics support, made possible by CETL funding, long term. Both have spoken of the need to ensure that some of **sigma**'s activities are integrated into a centrally funded teaching and learning facility. This should allow significant CETL-initiated activity to continue post 2010. It also underlines the importance of maintaining and enhancing **sigma**'s mechanisms of communication, as a way to drive forward and influence learning and teaching strategy.

10. References

[1] G. Roberts, *The supply of people with science, technology, engineering and mathematical skills*, SET for Success: the Report of Sir Gareth Roberts' Review, HM Treasury, April 2002. Available at: http://www.employment-studies.co.uk/pubs/report.php?id=1440robert [accessed 7 August 2009].

[2] A. Smith, *Making mathematics count*, The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education, Stationary Office Limited, February 2004. Available at: http://www.tda.gov.uk/upload/resources/pdf/m/mathsinquiry_finalreport.pdf [accessed 7 August 2009].

[3] *Staying the course: The retention of students in higher education*, National Audit Office, July 2007. ISBN: 9780102947687, Available at: http://www.nao.org.uk/publications/0607/student_retention_in_higher_ed.aspx [accessed 24 August 2009], p32

AN ASSESSMENT COMPONENT TAXONOMY FOR ALTERNATIVE MATHEMATICS ASSESSMENT FORMATS

BELINDA HUNTLEY[§§§§§§§]
University of the Witwatersrand, South Africa

JOHANN ENGELBRECHT; ANSIE HARDING
University of Pretoria, South Africa

Abstract: In this study we propose an assessment taxonomy of mathematics, which we call the *assessment component taxonomy*, to be used to identify those components of mathematics that can be successfully assessed using alternative assessment formats, such as the multiple choice format. Based on the literature on assessment models and taxonomies in mathematics, this taxonomy consists of seven mathematics assessment components, hierarchically ordered by cognitive level, as well as by the nature of the mathematical tasks associated with each component.

Keywords: Mathematics assessment, mathematics assessment taxonomy, mathematics assessment components, multiple choice questions.

1. Introduction

In South Africa, as in the rest of the world, the changes in society and technology have imposed pressures on academics to review current assessment approaches. The issue of appropriate assessment of mathematics learning at tertiary level is one of rapidly growing importance. Changes in education assessment are currently being called for, both within the fields of measurement and evaluation as well as in specific academic disciplines such as mathematics. Geyser [1] summarises the paradigm shift that is currently under way in tertiary education as follows:

> The main shift in focus can be summarized as a shift away from assessment as an add-on experience at the end of learning, to assessment that encourages and supports deep learning. It is now important to distinguish between learning for assessment and learning from assessment as two complementary purposes of assessment….(p 90).

Mathematics at tertiary level remains, in general, conservative in its use of alternative formats of assessment. As goals for mathematics education change to broader and more ambitious objectives [2], such as developing mathematical thinkers who can apply their knowledge to solving real problems, a mismatch is revealed between traditional assessment and the desired student outcomes. In order for tertiary mathematics departments and instructors to consider new modes of assessment, it is inappropriate to assess student mathematical knowledge using general assessment taxonomies, because these taxonomies are not pertinent to mathematics and do not identify those levels of mathematics that can be assessed using alternative formats of assessment, such as the multiple choice format.

§§§§§§§ E-mail: belinda.huntley@wits.ac.za

Today's needs demand multiple methods of assessment, integrally connected to instruction, that diagnose, inform and empower both educators and students. One of the aims of tertiary education in mathematics should be to develop proficiency within all components of mathematics. A greater knowledge of taxonomies in mathematics can assist educators and assessors to improve their assessment programmes [3].

With this background, we propose a taxonomy of mathematics, which we call the *assessment component taxonomy*, to identify those components of mathematics that can be successfully assessed using alternative assessment formats. Using a model that we developed earlier [4] for measuring the quality of mathematical test items, we investigate which of the assessment components can be successfully assessed in the provided response question (PRQ) format, in particular multiple choice questions (MCQs), and which can be better assessed in the constructed response question (CRQ) format where students have to construct and supply their own responses [5].

In this paper we describe the taxonomy that was developed. The other parts of the study are reported on elsewhere, eg [4].

This paper then focuses on the first of three parts of a bigger research project, which are:

(1) Developing an assessment taxonomy for different mathematics assessment formats.

(2) Developing a model, called the Quality Index (QI) model, to measure a good mathematics question.

(3) Measuring the quality of different mathematics questions, PRQs and CRQs, in all seven assessment components of the mathematics assessment taxonomy developed in this study. Parts 2 and 3 will be addressed elsewhere.

2. Mathematics assessment models

An assessment model emerges from the different aspects of assessment: what we want to have happen to students in a mathematics course, different methods and purposes for assessment, along with some additional dimensions. According to Niss [6], the first dimension of a mathematics assessment model is WHAT to assess, which may be broken down into: concepts, skills, applications, attitudes and beliefs.

Niss [6] uses the term *assessment mode* to indicate a set of items in an assessment model that could be implemented in mathematics education.
These items include the following:

- The *subject* of assessment i.e. who is assessed
- The *objects* of assessment i.e. what is assessed
- The *items* of assessment i.e. what kinds of output are assessed
- The *occasions* of assessment i.e. when does assessment take place
- The *procedures* and *circumstances* of assessment i.e. what happens, and who is expected to do what
- The *judging* and *recording* in assessment i.e. what is emphasised and what is recorded
- The *reporting* of assessment outcomes i.e. what is reported, to whom.

For the purpose of this study, the focus will be on the *objects* of assessment in the Niss model outlined above i.e. types of mathematical *content* (including methods, internal and external relations) and which types of student *ability* to deal with that content. This varies greatly with the place, the teaching level and the curriculum, but the predominant content objects assessed seem to be the following:

[a] *Mathematical facts*, which include definitions, theorems, formulae, certain specific proofs and historical and biographical data.

[b] *Standard methods* and *techniques* for obtaining mathematical results. These include qualitative or quantitative conclusions, solutions to problems and display of results.

[c] *Standard applications* which include familiar, characteristic types of mathematical situations which can be treated by using well-defined mathematical tools.

[d] *Heuristic* and *methods of proof* as ways of generating mathematical results in non-routine contexts.

[e] *Problem solving* of non-familiar, open-ended, complex problems.

[f] *Modelling* of open-ended, real mathematical situations belonging to other subjects, using whatever mathematical tools at one's disposal.

[g] *Exploration* and *hypothesis generation*.

With regards to the students' ability to be assessed, the first three content objects require knowledge of facts, mastery of standard methods and techniques and performance of standard applications of mathematics, all in typical familiar situations.

As we proceed towards the content objects in the higher levels of Niss's assessment model, the level of the students' abilities to be assessed also increase in terms of cognitive difficulty. In the proof, problem-solving, modelling and hypothesis objects, students are assessed according to their abilities to activate or even create methods of proof; to solve open-ended, complex problems; to perform mathematical modelling of open-ended real situations and to explore situations and generate hypotheses.

In the Niss assessment model, objects [a] – [g] and the corresponding students' abilities are widely considered to be essential representations of what mathematics and mathematical activity are really about. The first three objects in the list emphasise routine, low-level features of mathematical work, whereas the remaining objects are cognitively more demanding. Objects [a], [b] and [c] are fundamental instances of mathematical knowledge, insight and capability. Current assessment models in mathematics education are often restricted to dealing only with these first three objects. One of the reasons for this is that methods of assessment for assessing objects [a], [b] and [c] are easier to devise. In addition, the traditional assessment methods meet the requirement of validity and reliability in that there is no room for different assessors to seriously disagree on the judgement of a product or process performed by a given student. It is far more difficult to devise tools for assessing objects [d] – [g]. Inclusion of these higher-level objects into assessment models would bring new dimensions of validity into the assessment of mathematics. Webb and Romberg [7] argue that if we assess only objects [a], [b] and [c] and continue to leave objects [d] – [g] outside the scope of assessment, we not only restrict ourselves to assessing a limited set of aspects of mathematics, but also contribute to actually creating a distorted and wrong impression of what mathematics really is [6]. Exploration and hypothesis generation, although rarely encountered as objects of assessment in mathematics, are crucial to the process of doing mathematics.

3. Assessment taxonomies

According to the World Book Dictionary [8], a *taxonomy* is any classification or arrangement. Taxonomies are used to ensure that examinations contain a mix of questions to test skills and concepts. A leader in the use of a taxonomy for test construction and standardization was Ralph W. Tyler, the “father of educational evaluation” [9] who in 1931 reported on his efforts to construct achievement tests for various university courses.

The next step was taken by Benjamin Bloom [10], who organised the objectives into a taxonomy (dedicated to Tyler) that attempted to reflect the distinctions teachers make and to fit all school subjects. In Bloom’s *Taxonomy of educational objectives,* objectives were separated by *domain* (cognitive, affective and psychomotor), related to *educational behaviours,* and arranged in hierarchical order from simple to complex.

Bloom’s taxonomy has often been seen as fitting mathematics especially poorly [11]. It is quite good for structuring assessment tasks, but Freeman and Lewis [12] suggest that Bloom’s taxonomy is not helpful in identifying which levels of learning are involved.

As Ormell [13] noted in a strong critique of the taxonomy, Bloom’s categories of behaviour “are extremely amorphous in relation to mathematics. They cut across the natural grain of the subject, and to try to implement them – at least at the level of the upper school – is a continuous exercise in arbitrary choice” (p7).

Since its publication, variants of Bloom’s taxonomy for the cognitive domain have helped provide frameworks for the construction and analysis of many mathematics achievement tests ([11]; [14]). Attacking behaviourism as the bane of school mathematics, Eisenberg [15] criticised the merit of a task-analysis approach to curricula, because it essentially equates training with education, missing the heart and essence of mathematics. Expressing concern over the validity of learning hierarchies, he argued for a re-evaluation of the objectives of school mathematics. The goal of mathematics, at whatever level, is to teach students to think, to make them comfortable with problem solving, to help them question and formulate hypotheses, investigate and simply tinker with mathematics. In other words, the focus is turned inward to cognitive mechanism.

Smith, Wood, Crawford, Coupland, Ball and Stephenson [16] propose a modification of Bloom’s taxonomy called the MATH taxonomy (Mathematical Assessment Task Hierarchy) for the structuring of assessment tasks. The categories in the taxonomy are summarised in Table 1.

Table 1. MATH Taxonomy.

Group A	**Group B**	**Group C**
Factual knowledge	Information transfer	Justifying and interpreting
Comprehension	Applications in new situations	Implication, conjectures and comparisons
Routine use of procedures		Evaluation

(Adapted from Smith *et al.*, 1996, [16]) In the MATH taxonomy, the categories of mathematics learning provide a schema through which the nature of examination questions in mathematics can be evaluated to ensure that there is a mix of questions that will enable students to show the quality of their learning at several levels. It is possible to use this taxonomy to classify a set of tasks ordered by the nature of the activity required to complete each task successfully, rather than in terms of difficulty. Activities that need only a surface approach to learning appear at one end, while those requiring a deeper approach appear at the other end. Previous studies have shown that many students enter tertiary institutions with a surface approach to learning mathematics [17] and that this affects their results at university. There are many ways to encourage a shift to deep learning, including assessment, learning experiences, teaching methods and attitudinal changes. The MATH taxonomy addresses the issue of assessment and was developed to encourage a deep approach to learning. It transforms the notion that learning is related to what we as educators do to students, to how students understand a specific learning domain, how they perceive their learning situation and how they respond to this perception within examination conditions.

Recently, work on how the development of knowledge and understanding in a subject area occurs has led to changes in our view of assessing knowledge and understanding. For example, in Biggs [18] SOLO Taxonomy (Structure of the Observed Learning Outcome), he proposed that as students work with unfamiliar material their understanding grows through five stages of ascending structural complexity.

In South Africa, the National Senior Certification (NSC) process includes a formal external assessment at the end of Grade 12, the final year of secondary schooling. The National Senior Certificate Mathematics examinations consist of two compulsory papers and one optional paper. The level of complexity of the mathematical questions in the examinations are in line with the taxonomical categories given in Table 2 [19].

Table 2. Taxonomical categories of questions on Grade 12 NSC Mathematics papers.

TAXONOMICAL CATEGORIES
Knowledge
Performing routine procedures
Performing complex procedures
Problem solving

(Adapted from Subject Assessment Guidelines: Mathematics – January 2007, [19])

The above taxonomical categories are based on the 1997 TIMSS Mathematics Survey, and are hierarchically ordered from lower to higher cognitive levels of mathematical demand.

In the interests of higher quality tertiary education, a deep approach to learning mathematics is to be valued over a surface approach [16]. Students entering university with a surface approach to learning should be encouraged to progress to a deep approach. Studies have shown [17], that students who are able to adopt a deep approach to study tended to achieve at a higher level after a year of university study.

4. Mathematics assessment components

Based on the literature on assessment models, and taxonomies in mathematics ([6]; [10]; [11]; [16]; [19]), we argued that for purposes of this study it was necessary to adapt the reviewed taxonomies in order to address the issue of assessing the cognitive level of difficulty of mathematical tasks, as well as the cognitive skills associated with each level.

Many of the reviewed taxonomies are more general assessment taxonomies, and can be used for mathematics as well as any other discipline. As these taxonomies are not fine enough for the purposes of this study, we felt the need to develop another assessment taxonomy which can be used for the types of questions which are specific to undergraduate mathematics mainstream courses.

With this background, we propose a taxonomy of mathematics, which we call the *assessment component taxonomy*, to identify those components of mathematics that can be successfully assessed using alternative assessment formats such as MCQs. This taxonomy consists of a set of seven items, hereafter referred to as the *mathematics assessment components*. This set of seven mathematics components was ordered by the cognitive level, as well as the nature of the mathematical tasks associated with each component. The seven mathematics components are hierarchical in cognitive levels, and were developed through a classification of questions specific to first year mathematics courses. This mathematics assessment component taxonomy is particularly useful for structuring assessment tasks in the mathematical context. The proposed set of seven mathematics assessment components are summarised below:

(1) Technical
(2) Disciplinary
(3) Conceptual
(4) Logical
(5) Modelling
(6) Problem solving
(7) Consolidation

In this proposed set of seven mathematics assessment components, questions involving manipulation and calculation would be regarded as *technical.* Those that rely on memory and recall of knowledge and facts would fall under the *disciplinary* component. Assessment components (1) and (2) include questions based on mathematical facts and standard methods and techniques. The *conceptual* component (3) involves comprehension skills with algebraic, verbal, numerical and visual (graphical) questions linked to standard applications. The assessment components (4), (5) and (6) correspond to the *logical* ordering of proofs, *modelling* with translating words into mathematical symbols and *problem solving* involving word problems and finding mathematical methods to come to the solution. Assessment component (7), *consolidation*, includes the processes of synthesis (bringing together of different topics in a single question), analysis (breaking up of a question into different topics) and evaluation requiring exploration and the generation of hypothesis.

Using Bloom's taxonomy [10], and the MATH taxonomy [16], the proposed mathematics assessment components can be classified according to the cognitive level of difficulty of the tasks as shown in Table 3.

Table 3. Mathematics assessment component taxonomy and cognitive level of difficulty.

Mathematics assessment components	Cognitive level of difficulty
1. Technical 2. Disciplinary	Lower order / Group A
3. Conceptual 4. Logical	Middle order / Group B
5. Modelling 6. Problem solving 7. Consolidation	Higher order / Group C

Table 4 summarises the proposed mathematics assessment components and the corresponding cognitive skills required within each component. Based on the literature on assessment, the necessary cognitive skills required by students to complete the mathematical tasks within each mathematics assessment component were identified [3].

Table 4. Mathematics assessment component taxonomy and cognitive skills.

Mathematics assessment Components	Cognitive skills
1. Technical	• Manipulation • Calculation
2. Disciplinary	• Recall (memory) • Knowledge (facts)
3. Conceptual	Comprehension: • algebraic • verbal • numerical • visual (graphical)
4. Logical	• Ordering • Proofs
5. Modelling	Translating words and real-world concepts into mathematical symbols
6. Problem solving	Identifying and applying a mathematical method to arrive at a solution
7. Consolidation	• Analysis • Synthesis • Evaluation

5. Question examples in assessment components

In the following discussion, question examples within three of the mathematics assessment components have been identified according to Table 4. These items, one MCQ (Multiple choice question) and one CRQ (Constructed response question), were selected from the tests and examinations in the first year Mathematics Major course (MATH109) at the University of the Witwatersrand, Johannesburg. The classification of the question according to one of the assessment components was validated by a team of lecturers (experts) involved in teaching the first year Mathematics Major course at the University of the Witwatersrand. In addition, the examiner of each test or examination was asked to analyse the question paper by indicating which assessment component best represented each question. In this way, the examiner could also verify that there was a sufficient spread of questions across assessment components, and in particular, that there was not an over-emphasis on questions in the technical and disciplinary components. This exercise of indicating the assessment component next to each question also assisted the moderator and external examiner to check that the range of questions included all seven mathematics assessment components, from those tasks requiring lower-order cognitive skills to those requiring higher-order cognitive skills.

Example 1: Technical mathematics assessment component

Assessment Component 1: Technical (MCQ)

If $z = 3 + 2i$ and $w = 1 - 4i$, then in real-imaginary form $\frac{z}{w}$ equals:

A. $\frac{-5}{17} + \frac{14i}{17}$

B. $\frac{5}{15} - \frac{14i}{\sqrt{15}}$

C. $3 - 4i$

D. $\frac{11}{17} + \frac{14i}{17}$

MATH109 August 2005, Tutorial Test: Question 5.

In this t*echnical* question, students are required to manipulate the quotient of complex numbers, z and w, by multiplying the numerator and denominator by the complex conjugate $\overline{w}$, and then to calculate and simplify the resulting quotient by rewriting it in the real-imaginary form, $\alpha + bi$.

Example 2: Conceptual mathematics assessment component

Assessment Component 3: Conceptual (MCQ)

State why the Mean Value Theorem does not apply to the function $f(x) = \dfrac{2}{(x+1)^2}$

on the interval $[-3, 0]$

A. $f(-3) \neq f(0)$
B. f is not continuous
C. f is not continuous at $x = -3$ and $x = 0$
D. Both A and B
E. None of the above

MATH109 June 2006, Section A: Question 7.

In the above *conceptual* question, the student is required to apply his/her knowledge of the Mean Value theorem to a new, unfamiliar situation which requires that the student selects the best *verbal* reason why the Mean Value theorem does not apply to the function $f(x)$ and the interval given in the question. This question requires a comprehension of all the hypotheses of the Mean Value theorem and tests the students' understanding of a situation where one of the hypotheses to the theorem fails.

Example 3: Problem solving assessment component

Assessment Component 6: Problem solving (CRQ)

This question deals with the statement

$$P(n): n^3 + (n+1)^3 + (n+2)^3 \text{ is divisible by } 9, \text{ for all } n \in \mathrm{N}, n \geq 2$$

(1.1) Show that the statement is true for $n = 2$.

(1.2) Use Pascal's triangle to expand and then simplify $(k+3)^3$.

(1.3) Hence, assuming that $P(k)$ is true for $k > 2$ with $k \in \mathrm{N}$, prove that $P(k+1)$ is true.

(1.4) Based on the above results, justify what you can conclude about the statement $P(n)$.

MATH109 June 2006, Section B: Algebra, Question 1.

In the *problem solving* CRQ, the students are required to use the principle of Mathematical Induction to prove that the statement $P(n)$ is true for all natural numbers $n \geq 2$. The CRQ has been subdivided into smaller subquestions involving different cognitive skills to assist the student with the method of solving using mathematical induction. In subquestion (1.1), the students need to establish truth for $n = 2$ by actually testing whether the statement $P(n)$ is true for $n = 2$. Hence (1.1) assesses within the technical mathematics assessment component. Subquestion (1.2) involves a numerical calculation, the result of which will be used in the proof by induction. Hence (1.2) also assesses within the technical assessment component. In subquestion (1.3), students are required to complete the proof by induction, by assuming the inductive hypothesis that $P(k)$ is true for $k > 2, k \in N$, and proving that $P(k+1)$ is true. Since subquestion (1.3) requires the cognitive skills of identifying and applying the principle of Mathematical Induction to arrive at a solution, (1.3) assesses within the problem solving mathematics assessment component. Subquestion (1.4) concludes the proof by requiring the students to justify that both of the conditions of the principle hold, and therefore by the principle of induction $P(n)$ is true for every $n \geq 2, n \in N$. Hence (1.4), requiring no more than a simple manipulation, assesses within the technical assessment component. This problem solving CRQ illustrates that often those questions involving higher order cognitive skills subsume the lower order cognitive skills.

6. Using the Mathematics Assessment Component Taxonomy

Response data from 14 different mathematics tests, both MCQ and CRQ format, written between August 2004 and June 2006 were collected. The study was set in the context of a first year level Mathematics Major course at the University of the Witwatersrand, Johannesburg. In total, 207 test items were analysed in this study.

The seven assessment component taxonomy levels were developed through a classification of the 207 test items, both MCQs and CRQs, specific to the first year Mathematics Major assessment programme. In the follow up part to this study, presented elsewhere, the Quality Index (QI) model, developed by Huntley, Engelbrecht and Harding [4], was used to identify those components of mathematics that can be successfully assessed using the provided response question format, in particular MCQs, and the constructed response question format. To assist with this process, we used the mathematics assessment component taxonomy proposed in this study.

7. Discussion and conclusion

The present paper is a summary of the first stage in a larger study of mathematics assessment conducted by the authors [4]. This paper has focused on developing a mathematics assessment taxonomy that can be used to test which of the mathematics assessment components can be successfully assessed using the MCQ assessment format and which of the mathematics assessment components can be successfully assessed using the CRQ assessment format.

In this study we used two types of questions, MCQs and CRQs. Such a variety of assessment formats is more easily addressed by the proposed mathematics assessment component taxonomy. The assessment component taxonomy can be used to identify those components of mathematics that can be successfully assessed using alternative assessment formats. Thus, as this taxonomy addresses two assessment issues simultaneously, that of assessing the cognitive level of difficulty of different types of mathematical tasks as well as the cognitive skills associated with each level, it is a good taxonomy to use for alternative assessment methods, such as those specific to undergraduate mathematics courses. The assessment component taxonomy, with its seven hierarchically ordered mathematics assessment components, is particularly useful for structuring alternative assessment tasks in the mathematical context.

8. References

[1] Geyser, H. (2004). Learning from assessment. In S. Gravett & H. Geyser. (Eds.), *Teaching and learning in higher education* (pp. 90-110). Pretoria, South Africa: Van Schaik.

[2] National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

[3] Huntley, B. (2008). *Comparing different assessment formats in undergraduate mathematics.* PhD Thesis submitted to University of Pretoria. http://upetd.up.ac.za/thesis/available/etd-01202009-163129/

[4] Huntley, B., Engelbrecht, J. & Harding, A. (2009). A model for measuring the quality of a mathematics question. Submitted to the Journal for Research in Mathematics Education (JRME), March 2009.

[5] Engelbrecht, J. & Harding, A. (2003). Online assessment in mathematics: multiple assessment formats. *New Zealand Journal of Mathematics, 32 (Supp.)*, 57-66.

[6] Niss, M. (1993). *Investigations into assessment in mathematics education. An ICMI Study*. Netherlands: Kluwer Academic Publishers.

[7] Webb, N. & Romberg, T.A. (1992) Implications of the NCTM standards for mathematics assessment. In T.A. Romberg (Ed.), *Mathematics Assessment and Evaluation: Imperatives for Mathematics Educators* (pp. 37-60). Albany: State University of New York Press.

[8] World Book Dictionary (1990). Chicago, London, Sydney Toronto: World Book. Inc.

[9] Romberg, T.A. (1992). *Mathematics assessment and evaluation. Imperatives for mathematics educators.* Albany: State University of New York Press.

[10] Bloom, B.S. (Ed.) (1956). *Taxonomy of educational objectives. The classification of educational goals. Handbook 1:* The cognitive domain. New York: David McKay.

[11] Romberg, T.A., Zarinnia, E.A. & Collis, K.F. (1990). A new world view of assessment in mathematics. In G.Kulm (Ed.), *Assessing higher order thinking in mathematics* (pp. 21-38). Washington, DC: American Association for the advancement of Science.

[12] Freeman, R. & Lewis, R. (1998). *Planning and implementing assessment.* London: Kogan Page.

[13] Ormell, C.P. (1974). Bloom's taxonomy and the objectives of education. *Educational esearch, 17,* 3-18.

[14] Begle, E.G. & Wilson, J.W. (1970). Evaluation of mathematics programs. In E.G. Begle (Ed.), *Mathematics Education* (69th Yearbook of the National Society for the study of Education, Part I, 376-404). Chicago: University of Chicago Press.

[15] Eisenberg, T. (1975). Behaviorism: The bane of school mathematics. *Journal of Mathematical Education, Science and Technology, 6(2),* 163-171.

[16] Smith, G.H., Wood, L.N., Crawford, K., Coupland, M., Ball, G. & Stephenson, B. (1996). Constructing mathematical examinations to assess a range of knowledge and skills. *Int. J. Math. Educ. Sci. Technol., 27(1),* 65-77.

[17] Ball, G., Stephenson, B., Smith, G.H., Wood, L.N., Coupland, M. & Crawford, K. (1998). Creating a diversity of experiences for tertiary students. *Int. J. Math. Educ. Sci. Technol., 29(6)*, 827-841.
[18] Biggs, J. (1991). Student learning in the context of school. In J. Biggs (Ed.), *Teaching for learning: the view from cognitive psychology* (pp. 7-20). Hawthorn, Victoria: Australian Council for Educational Research.
[19] National Curriculum Statement Grades 10-12 (General), (2007). *Subject Assessment Guidelines Mathematics,* South Africa: Department of Education.

A note on linear independence

STEPHAN V. JOUBERT[********] AND TEMPLE H. FAY

Department of Mathematics and Statistics, Tshwane University of Technology, South Africa

Abstract: Linear independence is an important notion appearing in many different contexts and in particular in beginning differential equations. Given the diverse student body taking such courses, the authors feel it is well worth a little extra time to develop the ideas around linear independence as it not only makes much of the theory of beginning differential equations better understood, but also permits the discussion of some fundamental notions of mathematical logic which non mathematics majors might otherwise never be aware of. These notions include proof by contradiction, counter examples, if and only if statements and their proofs, converse, contrapositive, and negation. In this way the student's analytical reasoning can be improved. In this article we develop the idea of linear independence for a finite set of real valued functions and discuss the Wronskian and its use as a test for linear independence and how solutions to a differential equation have a particularly nice Wronskian. We place emphasis on using functions from elementary calculus for examples and we suggest a number of observations to be used as student exercises which illustrate the theory as well as the logical notions being brought out. It is hoped that from this discussion an instructor could easily develop an enrichment lecture which could be used to augment the development in any of the popular beginning texts on differential equations. A new (to the best of our knowledge) easily understood proposition is introduced and an open question is posed.

Keywords: ode; Wronskian; linear; independence

1. Introduction

Most beginning texts in differential equations discuss linear independence and the Wronskian when developing the theory of linear differential equations. Linear independence is a fundamental notion in linear algebra and is used in many contexts - odes, pdes, numerical analysis, Fourier series - just to mention a few. Beginning students seem to find the idea difficult at first as the definition given is a bit abstract in comparison with much of their mathematical experience up to this point. Beginning differential equations is a popular course populated by a wide range of students majoring in fields other than mathematics, e.g. physics, computer science, engineering, chemistry, polymer science, biology. Many of these students take this course as their terminal course in mathematics and hence it makes sense to discuss a small amount of mathematical logic that arises in a natural way when discussing linear independence. Not only do the authors feel that the idea of linear independence becomes better understood, but that the students better appreciate the way mathematics texts are written when the mathematical logic being used is made explicit.

2. Linear independence

Throughout our discussion we shall be manipulating real valued functions defined over a real interval I as these are the types of functions the students have already encountered in the calculus and in the differential equations course. We like to

[********] E-mail: joubertsv@tut.ac.za

emphasize that often there is an abuse of notation in beginning texts that is generally subtle. Consider the differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

which one might ordinarily encounter in such a course; the coefficient functions may be constant or variable and x is understood to belong to the interval I. The observation we wish to make is that the left-hand side of the equation consists of functions defined over I, but the right-hand side is the number 0. Of course context takes care of any ambiguity and it is no big deal, except when it comes to discussing linear independence. Thus we first would like to emphasize that the right-hand side of this differential equation is the zero function on I; one can't have anequality between functions and numbers. So to be precise, we shall denote the zero function by $\mathrm{O}(x); x \in \mathrm{I}$ or simply O, and let the interval I be understood. And thus the differential equation should appear written as

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = \mathrm{O}..$$

Of course we are not advocating a rewriting of all texts, but we do wish to make clear our distinction between the number 0 and the function O. Let $S = \{f_1, \cdots, f_n\}$ be a set of real valued functions defined over the interval I. We say that the set S is **linearly independent over the interval** I provided for every set of constants $\{c_1, \cdots, c_n\}$ with the property that

$$c_1 f_1 + \cdots + c_n f_n = \mathrm{O} \tag{1}$$

it follows that $c_1 = c_2 = \cdots = c_n = 0$. What is hidden here, but emphasized by using the notation O, is that the equation

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0 \tag{2}$$

holds for all x belonging to I if and only if each of the constants is zero. Of course, the idea is that the functions are fundamentally different and any one of them can not be written as a linear combination of the others. When $S = \{f_1, f_2\}$ linear independence means that one is not a constant multiple of the other.

We shall call the set of functions S **linearly dependent over** I if they are not linearly independent over I. We shall see in a moment that it is important to carry the interval I along with us for if we change I, we may change from independence to dependence and *vice versa*.We like to point out to our students that if one of the constants, say without loss of generality c_1, is not zero and equation (1) holds, then we can solve for f_1 in terms of the others:

$$f_1 = \sum_{i=2}^{n} \frac{-c_i}{c_1} f_i$$

and this equation holds when evaluated at any x belonging to I and in this way make clear what a **linear combination** is.

Examples of sets of linearly independent functions abound, we like to use sets of polynomials of different degrees, sets of sines and cosines and of exponentials of differing exponents since these functions arise naturally from solving linear differential equations with constant coefficients. We also like the example $S = \{x^2, x|x|\}$ (see Boyce and DiPrima [1], page 136). This set is linearly independent over $I = (-1,1)$, over $\mathbb{R}$, but not over $(-\infty, 0)$.

Some elementary observations to discuss with students include:

Observation 1. If $S = \{f_1, \cdots, f_n\}$ is linearly independent over I, then S is linearly independent over any interval J containing I.

Observation 2. The example $S = \{x^2, x|x|\}$ shows that the converse to Observation 1 fails.

Observation 3. If $S = \{f_1, \cdots, f_n\}$ is linearly independent over I, then any proper nonempty subset is linearly independent over I.

Observation 4. The converse to Observation 3 fails. The set $S = \{1, x, 2x + 3\}$ is linearly dependent over $\mathbb{R}$, but every two element subset is linearly independent over $\mathbb{R}$.

Observation 5. If $S = \{f_1, \cdots, f_n\}$ is linearly independent over I and

$$a_1 f_1 + \cdots + a_n f_n = b_1 f_1 + \cdots + b_n f_n$$

for constants $a_1, \cdots, a_n, b_1, \cdots, b_n$, then $a_i = b_i$ for $i = 1, \cdots, n$.

Observation 6. If $S = \{\sin x, \cos x\}$, then S is linearly independent over $\mathbb{R}$ and hence the following equation is easy to solve for constants a and b

$$(a + b)\cos x + 6\sin x = \cos x + (a - b)\sin x.$$

Observation 7. If $S = \{f_1, \cdots, f_n\}$ is linearly independent over I and each f_i is differentiable, then $S' = \{f_1', \cdots, f_n'\}$ need not be linearly independent over I.

3. The Wronskian

The set $S = \{\sin x, \cos x, \sin 2x, \cos 2x\}$ is linearly independent over the entire real line, but this is not at all obvious. Thus it would be helpful to have a test for linear independence that is easier to apply than trying to verify all the c_i's are zero in equation (1). To that end and since students already know something about solving simultaneous equations, and since they are studying functions that are differentiable, it seems natural to develop the following.

Suppose we are given a set $S = \{f_1, \cdots, f_n\}$ and we wish to see if S is linearly independent over an interval I. We have to test equation (1):

$c_1 f_1 + \cdots + c_n f_n = \mathrm{O}.$

Since we are dealing with n constants and n functions, it might be helpful to have n equations. So we assume that we can differentiate this equation and obtain a second equation:

$c_1 f_1' + \cdots + c_n f_n' = \mathrm{O}.$

Now we have two equations which generally is not quite enough, so we suppose that we can continue to differentiate until we have exactly n equations

$$\begin{aligned} c_1 f_1 + \cdots + c_n f_n &= \mathrm{O} \\ c_1 f_1' + \cdots + c_n f_n' &= \mathrm{O} \\ \cdot \qquad & \;\cdot \quad \cdot \\ \cdot \qquad & \;\cdot \quad \cdot \\ \cdot \qquad & \;\cdot \quad \cdot \\ c_1 f_1^{(n-1)} + \cdots + c_n f_n^{(n-1)} &= \mathrm{O}. \end{aligned}$$

We view this as a system of linear equations with the unknowns being the constants c_i and coefficients being the various functions and their derivatives. Students should know that a simultaneous system of n equations in n unknowns has a unique solution when the determinant of the coefficients is nonzero. It is clear that there already is one evident solution $c_1 = c_2 = \cdots = c_n = 0$. So we want to examine the determinant of the coefficients

$$\begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \qquad (3)$$

and ask is this determinant (which is a function of x over the interval I) the zero function O or not. If not then S is linearly independent. This gives rise to naming this determinant and the test theorem for linear independence.

Definition. Let $S = \{f_1, \cdots, f_n\}$ be a set of functions defined over an interval I and suppose each function has $n-1$ derivatives so that the determinant (3) can be formed. Then the determinant (3) is called the **Wronskian of** S and is denoted $W[S]$. Note again that $W[S]$ is a function defined over the interval I .
We re-phrase (using O) the following familiar theorem (see [1] and [2]):

Theorem 1. *Let $S = \{f_1, \cdots, f_n\}$ be a set of functions defined over an interval I and suppose each function has $n - 1$ derivatives. If $W[S] \neq O$, then S is linearly independent over I. Phrased alternatively, if there exists (at least) one value x_0 belonging to I for which $W[S](x_0) \neq 0$, then S is linearly independent over I.*

This test is very valuable and easy to apply for small sets S. However, there are sets when the test is no help. Recall the set of functions in Observation 2.

Observation 8. The set $S = \{x^2, x|x|\}$ is linearly independent over the entire real line $\mathbb{R}$ but $W[S] = O$. This follows from the routine exercise in left-hand and right-hand limits and the definition of derivative to show that if $g(x) = x|x|$, then g is differentiable everywhere and $g'(x) = 2|x|$. This observation shows that the converse to Theorem 1 is false. The next proposition is a converse to Theorem 1 when the set S consists only of two functions.

Proposition 1. Let $S = \{f_1, f_2\}$ be a set of functions defined, differentiable and nonzero over the interval I. If S is linearly independent over I, then for some x_0 belonging to I, $W[S](x_0) \neq 0$.

Proof. Assume that S is linearly independent over I, but, by way of contradiction, that for every x belonging to I, $W[S](x) = 0$. Then for every such x,

$$f_1(x)f_2'(x) - f_1'(x)f_2(x) = 0.$$

The nonzero hypothesis permits division by $f_1(x)f_2(x)$, and consequently we can separate the variables and integrate, obtaining

$$\int \frac{f_1'(x)}{f_1(x)}dx = \int \frac{f_2'(x)}{f_2(x)}dx.$$

Hence

$$\ln|f_1(x)| = \ln c|f_2(x)|$$

where c is a nonzero constant. It follows that

$$f_1(x) = \pm c f_2(x)$$

for every x belonging to I. Since both f_1 and f_2 are nonzero over I and are continuous, they must be of constant sign over I and thus either

$$f_1(x) = c f_2(x)$$

or

$$f_1(x) = -c f_2(x)$$

for every x. Neither case is possible since S is linearly independent over I. This contradiction shows that our assumption: "for every x belonging to I, $W[S](x) = 0$" is incorrect and consequently the negation of this statement: "for some x_0 belonging to I, $W[S](x_0) \neq 0$" must be true and we are done.

Remark 1. We were unable to extend this proposition to sets S of cardinality larger than 2.

Observation 8 might seek to be a drawback to using the Wronskian as a test, but the next theorem shows that when dealing with solutions to a differential equation the Wronskian is exactly what we need. The following theorem's proof rests entirely on the Existence and Uniqueness Theorem which we assume has been discussed with students prior to encountering linear independence.

Theorem 2. (See [1] or [2]). Let $S = \{f_1, \cdots, f_n\}$ be a set of functions defined over an interval I and suppose each function f_i is a solution over the interval I to the differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = O$$

where each coefficient function is continuous over the interval I.

A. The set S is linearly independent over I if and only if $W[S](x) \neq 0$ for each x belonging to I.
B. The set S is linearly dependent over I if and only if $W[S] = O$ over I.

Proof. A $\Leftrightarrow$ B. This follows immediately from Theorem 1.
A $\Rightarrow$ B. Suppose that S is linearly independent over I and assume, by way of contradiction, that there is an x_0 belonging to I so that $W[S](x_0) = 0$. (Here one has the opportunity to mention the logical equivalence of the statements "$P \Rightarrow Q$" and "P and not Q" or symbolically "$P \wedge \neg Q$".) This means that the system of equations

$$\begin{aligned}
c_1 f_1(x_0) + \cdots + c_n f_n(x_0) &= 0\\
c_1 f_1'(x_0) + \cdots + c_n f_n'(x_0) &= 0\\
\vdots \qquad\qquad & \quad \vdots\\
c_1 f_1^{(n-1)}(x_0) + \cdots + c_n f_n^{(n-1)}(x_0) &= 0
\end{aligned}$$

has more than one solution and in particular there is a set $\{c_1, \cdots, c_n\}$ of constants, not all zero that is a solution to the system. Without loss of generality, assume that we have

$$c_1 \neq 0. \tag{4}$$

Define the function
$y_0(x) = c_1 f_1(x) + \cdots + c_n f_n(x), \quad x \in I.$

Since the differential equation is linear, it follows that y_0 is also a solution to the differential equation over the interval I. Observe also that O is a (trivial) solution to the differential equation. Consequently both y_0 and O are solutions to the initial value problem

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$
$$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$$

Hence, according to the Existence and Uniqueness Theorem, $y_0 = 0$, that is, $c_1 f_1(x) + \cdots + c_n f_n(x) = 0$ for all x belonging to I. The linear independence of S yields $c_1 = c_2 = \cdots = c_n = 0$, which contradicts Equation (4). Hence our assumption that "there is an x_0 belonging to I so that $W[S](x_0) = 0$" leads to a contradiction, and is consequently false. Therefore, its negation "for all x belonging to I, $W[S](x_0) \neq 0$" is valid. This completes the proof of (A).

Statement B is merely the negation of statement A.

4. An application

Aside from the theoretical interest for linear differential equations, linear independence is fundamental in a wide variety of contexts, not the least of which is partial differential equations. Beginning students might not appreciate the statement of a boundary value problem, say solving Laplace's equation over a disk, nor at this stage appreciate a Fourier series solution form. But they will understand a statement to the effect that a common way to solve multivariable pdes is to turn them into several odes and solve each ode separately.

In the case of doing this over a disk, the problem is usually converted into polar coordinates and the angular solution is required to be periodic of period 2π. To make this long story short (and suitable for discussion at this beginning level) the idea is to find solutions of period 2π to a function of the form

$$\Theta(\theta) = c\cos\lambda\theta + d\sin\lambda\theta \neq 0 \qquad (5)$$

where θ is the angular variable which is allowed to take on any real value and with c and d constants and $\lambda > 0$. Note that there are three parameters c, d, λ and that it is λ that controls the periodicity. The constants c and d are determined after an investigation of what values of λ are permissible. It is customary to cryptically state that "periodicity implies that λ is a non-negative integer" (see for instance, [1] Chapter 10.7, Equation 29), but this is not obvious. Here we deduce conditions on the positive parameter λ so that $\Theta(\theta)$ has period 2π.

Requiring such a period means

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

so we have

$$c\cos\lambda\theta + d\sin\lambda\theta = c\cos\lambda(\theta + 2\pi) + d\sin\lambda(\theta + 2\pi),$$

or equivalently,

$$\cos l\, q + d \sin l\, q = c \cos l\, q \cos 2l\, p -$$
$$c \sin l\, q \sin 2l\, p + d \sin l\, q \cos 2l\, p + d \cos l\, q \sin 2l\, p.$$

Because $\{\cos l\, q, \sin l\, q\}$ is a linearly independent set over R (apply the Wronskian), it follows that after simplifying, we can equate the coefficients from each side and obtain the pair of equations

$$(\cos 2l\, p - 1)c + (\sin 2l\, p)d = 0$$
$$(-\sin 2l\, p)c + (\cos 2l\, p - 1)d = 0.$$

The determinant of the coefficients must be zero otherwise Cramer's rule yields $c = d = 0$, contradicting (5). Hence

$$\begin{vmatrix} \cos 2l\, p - 1 & \sin 2l\, p \\ -\sin 2l\, p & \cos 2l\, p - 1 \end{vmatrix} = 0.$$

This means that

$$\cos 2l\, p = 1$$

and so l is a positive integer.

Arguments of this type, invoking linear independence to find permissible parameter values (called eigenvalues) pervades the technique of separation of variables.

5. Conclusion

In this article we developed the idea of linear independence for a finite set of real valued functions suitable for a discussion group in a general undergraduate mathematics classroom. We discussed the Wronskian and its use as a test for linear independence and how solutions to a differential equation have a particularly nice Wronskian. The emphasis was on using the functions from elementary calculus for examples and we suggested a number of observations to be used as student exercises which illustrated the theory as well as the logical notions being brought out. It is hoped that from this discussion an instructor could easily develop an enrichment lecture which could be used to augment the development in any of the popular beginning texts on differential equations.

6. References

[1] Boyce, W.E. & DiPrima, R.C. 1992. ***Elementary Differential Equations and Boundary Value Problems, Fifth Edition,*** John Wiley & Sons,New York.

[2] Edwards, C. H. and Penney, D. E. 1993. ***Elementary Differential Equations, Third Edition***, Prentice Hall, New Jersey.

Students' transfer of learning of eigenvalues and eigenvectors

GULDEN KARAKOK[††††††††] AND BARBARA EDWARDS

[*]*UMERC, Umea University, Umea, Sweden;*
Department of Mathematics, Oregon State University, Corvallis, United States

Abstract: One of the post-calculus mathematics courses required by mathematics and other disciplines is linear algebra. Most of the topics covered in a typical undergraduate linear algebra course are often the prerequisite topics for many client disciplines such as physics, economics, statistics, and engineering. Eigenvalues and eigenvectors are among topics covered in a linear algebra course and revisited in quantum mechanics courses in physics. Students are expected to apply their knowledge of linear algebra topics in quantum mechanics courses; in other words, students are expected to transfer their knowledge of linear algebra topics from mathematics to physics courses. Transfer of learning has been discussed by researchers from various fields, and yet there has not been an agreement among findings. Some researchers claim transfer is rare and others point out that people transfer successfully during their everyday life. To reconcile these differences some researchers proposed new methods to explore the issue of transfer. The purpose of this study was to investigate third year college students' transfer of learning of the concepts of eigenvalues and eigenvectors by implementing one of the new approaches to transfer, namely the Actor-Oriented Transfer (AOT) framework. The participants of this study were interviewed before and after being introduced to concepts of eigenvalues and eigenvectors in a series of quantum physics courses, and the participants' transfer of learning of these concepts from the courses to the interviews were investigated. The results of this study suggest the importance of the framework used in exploring the issue of transfer. The researcher found that the actor-oriented transfer analysis provided evidence of transfer from the physics courses to the interviews.

Keywords: transfer of learning, linear algebra, actor-oriented transfer, eigenvalues, eigenvectors, algebraic interpretation, geometric interpretation

1. Background

Transfer of learning has been discussed by researchers from various fields for over 100 years [1-5]. Early psychology researchers were interested in the issue of how information learned at one time influenced the learners and their performance at a later time [2]. Similar ideas have been observed in educational research studies, since transfer of learning is directly related to the generalization of learning experiences [6].

Early researchers developed studies to measure the learners' ability to transfer knowledge and skills. These studies were focused on the question of whether a particular piece of knowledge or a skill transfers from the initial learning situation to the target situation. The results of these studies were very similar to each other indicating that learners *did not* transfer their knowledge and skills [7]. However, this conclusion did not help the educators understand the reasons behind the problem of transfer. Some researchers suggested that the definition of transfer of learning was the source of the problem. Transfer of learning is defined as *the ability to apply knowledge learned in one context to new contexts* and according to a strict application of this definition learners mostly lacked the ability to transfer [7]. However, this definition of transfer suggested that knowledge could be separated from the situation in which it was constructed [5, 8, 9]. It was not just the definition of transfer but the

[††††††††] E-mail: gulden.karakok@math.umu.se

methodologies used were also problematic. The studies often used tasks (problems) developed by researchers and learners' answers were analyzed for evidence by researchers without further in-depth interviews with students [10,11].

To investigate the issues surrounding the research construct of transfer as well as to explore learner's transfer of learning, different approaches (definitions, theories and methodologies) have been suggested and are being developed by researchers [1, 5, 12, 13]. These researchers have questioned the methodology and the experimental evidence for lack of transfer provided by earlier studies. Several researchers have claimed that the previous findings contradict the everyday experiences in which most learners do perform successfully in new situations by finding similarities from previous situations [4, 12, 14, 15]. These researchers have observed that the methodology adopted in the earlier studies was problematic because it only focused on the researchers' and experts' perspectives, especially when developing the tasks (problems) and analyzing data.

Transfer studies often used tasks developed by researchers, and learners' answers were analyzed for evidence of transfer of learning of a predetermined piece of knowledge or a skill [10]. Researchers often tried to answer the question "Do students transfer?" and expected students to provide complete and correct answers. The target tasks in the earlier studies were conducted after students were introduced to the tested knowledge or skills [8]. The context of the task, or surface feature, was designed to be different from the context in which the initial "learning" occurred, but the methods needed to solve the task, structural features, were kept the same.

In the earlier paradigm the researcher would decide on the surface and structural features of the problems. These approaches to transfer assumed transfer and learning were passive processes; either they happened or did not happen. To complement these earlier perspectives on transfer of learning, new approaches were conceived to offer a dynamic process in which the learner continues to build knowledge even during the studies.

One of these contemporary approaches to transfer, the Actor-Oriented Transfer (AOT) framework was developed by Lobato [1]. AOT views transfer as the "personal construction of similarities between the two situations- the initial learning situation and the target (new) situation" [9, p.89]. The main focus of this framework is the learner (actor) and how he or she sees the target situation in relation to the initial learning situation. Obtaining evidence for actor-oriented transfer is done by *scrutinizing* [9, p.89] a given task together with data. Any indication of influence from previous tasks on the given task is considered to be evidence for actor-oriented transfer. Lobato and Siebert's (2002) case study demonstrated these main ideas of the AOT framework. In the case study, researchers focused on an eighth-grade student who participated in a 10-day (3 hr/day) teaching experiment. The goal of the teaching experiment was to improve quantitative reasoning of students on slope. Throughout the teaching experiments students were also interviewed individually three times; on the first, fifth, and last days of the teaching experiment. The researchers focused on the last interview with this particular eighth-grade student. They were looking for possible occurrences of transfer between the events before the last interview and the students' problem solving activities during the last interview. The student chosen for the case study was working on measuring the slope of a wheelchair ramp when the length and height were given. The student had demonstrated difficulties distinguishing between height and the slope during the first interview. The student thought that if one walked upwards on a hill, then the path would get steeper and steeper as one got higher on the hill. The same confusion was seen when the student was asked to create wheelchair ramps with the same slope but different heights, one being higher than the first one. However, during the last interview the student seemed to have a sudden insight on how to create a one feet high wheelchair ramp that had the same steepness as a 15-foot long and 2-foot high wheelchair ramp. Students learned about slope formula (height over length) prior to the 10-day-teaching experiment. However this particular student did not use the slope formula. He found a different way to approach the problem. He calculated how much length went with 1ft of height first, then created the ratio 15:2 = 7.5:1. Then he tried to explain what the ratio meant to him.

The researchers were mainly investigating the possible occurrence of transfer. They did not pose a question of transfer prior to the student's insight. In other words, they did not ask the question: Did the students transfer previously learned slope formula to the new situation? However, with this interview data Lobato and Siebert asked the question: "Was the student's sudden insight a spontaneous event without any connection to what had taken place during the teaching experiment or can we find anything that has possibly foreshadowed and laid ground for the sudden insight?" The authors were looking for possible links between the teaching experiment and the sudden insight of the student during the last interview.

Researchers claimed that one of the teaching experiment sessions prior to the last interview could be linked to the student's approach in the interview. In this session, students were working on an activity in which they were trying to make an animated clown walk with the same speed. Students were using some computer software to make the animated clown walk across the computer screen. The clown initially was walking at a constant speed, covering 10 cm in 4 seconds. The students' task was to make a frog walk with the same speed but covering different distances and times. The researchers noticed that the case study student had built his own reasoning to explain that the frog walked 20 cm in 8 seconds, 30 cm in 12 seconds that were respectively 2 and 3 units of the ratio 10 cm: 4 seconds.

Researchers claimed that the case study student probably used the same reasoning he created with the clown-frog task in the wheelchair problem. They postulated that the student demonstrated transfer between two situations by creating his own similarities between these two situations. The researchers reported that if they were investigating the student's transfer of learning of slope formula to new problems, then the results would be negative. The researchers also wanted to demonstrate that researchers should not initially identify the tasks of the experiments in terms of structural and surface similarities since students could have different ways of looking at these similarities.

Most recent studies [9, 11, 16, 17] implemented contemporary approaches, especially the AOT framework, as well as earlier approaches in the same study. The most common reason to use both approaches was that if a lack of transfer was concluded by the traditional approach, researchers then could do an in-depth analysis of the topic with the contemporary approaches, namely deciding what students did transfer between the tasks. These studies have investigated transfer of learning at different educational levels from kindergarten through early college education, and within different subject areas (within mathematics, between calculus and physics, etc)[9, 10, 11, 17]. There have been no studies of transfer of learning in post-calculus mathematics or physics courses implementing the AOT framework.

The main purpose of this study was to investigate students' transfer of learning of a post-calculus mathematics topic, namely eigenvalues and eigenvectors. The participants of this study were enrolled in a series of quantum physics courses in which they were introduced and applied the concepts of eigenvalues and eigenvectors. The participants were interviewed three times, before, during, and after these courses to investigate their transfer of learning of the concepts of eigenvalues eigenvectors from the courses to the interviews. Transfer of learning for each student was explored by implementing the AOT framework [1]. More specifically, the question, “What ideas of eigenvalues and eigenvectors do students transfer from the courses to the second and third interviews?” was answered.

2. Method

Undergraduate students who were enrolled in a junior-level physics course offered during the fall term of 2007 and who planned to take all the junior-level physics courses during the winter term of 2008 were invited to participate in the study. On the recruitment day students were told that the study was about physics students’ understanding of linear algebra topics. Seven students out of twenty volunteered to participate in the study that started during the fall term of 2007 and ended in the spring term of 2008. Students were interviewed three times. First interview was conducted towards the end of the fall term of 2007; the second interview was conducted after the first week of the winter term of 2008, and the third interview was conducted at the beginning of the spring term of 2008. The pseudonyms for the seven participants were Deniz, Crosby, Gus, Milo, Ozzy, Tom and Joey.

Only one participant (Gus) did not take any mathematics courses in which he was introduced to the concepts of eigenvalues and eigenvectors. Tom, Joey and Deniz took a course (not necessarily the same course) in which linear algebra topics including eigenvalues and eigenvectors were introduced prior to the fall term of 2007. Ozzy, Milo and Crosby were enrolled in a linear algebra course during the fall term of 2007. In other words, six of the participants were introduced to eigenvalues and eigenvectors prior to the junior-level physics courses of the winter term of 2008.

The junior-level physics courses during the winter term started with a week of review of linear algebra topics prerequisite for the courses. Participants were introduced to eigenvalues and eigenvectors during this first week and continued to use them throughout the term. The first week of this term was considered to be one of the experiences participants had with eigenvalues and eigenvectors. Students spend seven hours in class to review topics including, but not limited to, matrix manipulations, the determinant of a matrix, inverse matrices, symmetric matrices, linear transformations, vector spaces, eigenvalues and eigenvectors, and properties of Hermitian matrices. This review week (which will be called Linear Algebra Week (LAW) in this study) was not a course by itself, but junior-level undergraduate physics students were required to participate. During this week the professor provided some lectures to remind the basic matrix algebra manipulations and to introduce Dirac notation.

Students also worked on two activities during this week. The first activity students were engaged in was called Linear Transformations (LT). The goals of this activity were to introduce matrices as linear transformations, to introduce the concept of eigenvector and its geometrical interpretation, and to help students generate a hypothesis that links the determinant of a matrix to a linear transformation. Students worked in small groups. The LT worksheet had 10 matrices and 5 vectors. Each group was given two matrices and asked to operate on these 5 vectors with their matrices. They were asked to graph the initial 5 vectors (all group had the same 5 vectors) and the transformed vectors. Then they were asked to find the determinant of their matrices, and to make note of any differences (inversions, length, rotations, etc) between the initial and transformed vectors. They were also asked if there were any unchanged vectors by the given transformation. Once all the students were done with the activity, they shared the results with the rest of the class. At the end of this activity eigenvectors were defined as the vectors that left unchanged by the linear transformation. The eigenvalue equation, $A\vec{v} = \lambda\vec{v}$, was introduced and algebraic and geometric interpretations of eigenvalues and eigenvectors were discussed as a class. Students also made conjectures to classify matrices according to their determinant. They hypothesized that matrices with positive determinants represent rotation transformations that also scale matrices, negative determinants represent reflections that also scale matrices, and determinant zero matrices put all the initial vectors on a line. They also discussed the possible eigenvectors of each type of the matrix in the LT activity.

In the second activity, Eigenvalues and Eigenvectors (EE), students worked in groups to find eigenvalues and eigenvectors of matrices (there were six matrices on the EE worksheet) by first finding eigenvalues from the characteristic equation, and then finding eigenvectors from the eigenvalue equation. However, they were also asked to figure out, if possible, what each matrix represented as a linear transformation and find eigenvectors geometrically. Students once again discussed algebraic and geometric interpretations of eigenvalues and eigenvectors as a whole class at the end of the activity.

During the rest of the winter term students took three junior-level physics courses each of which met for seven hours a week for three weeks. Table 1 shows the timeline of the winter term physics courses.

Week 1	Weeks 2 - 4	Weeks 5-7	Weeks 8-10
Linear Algebra Week (LAW)-	Quantum Measurements and Spin	Waves	Central Forces

Table 1. Winter term junior-level physics course sequence.

During spin and quantum measurements course, postulates of quantum mechanics were discussed through demonstrations for the simple spin ½ Stern-Gerlach experiments. Operators, eigenvalues probability, state reduction and time evolution were also examined. Students were introduced to Schrödinger's Equation and then Schrödinger's time evolution is applied to the case of a spin ½ system. In waves course students were introduced to the terminology to describe waves as they focus on waves in electrical circuits, waves on ropes and the matter waves of quantum mechanics. They also discussed eigenfunctions of operators and superposition of eigenfunctions in the context of wave functions. During central forces course, students first discussed central forces in classical mechanics while the importance of the conservation of angular momentum was emphasized. The separation of variables in Schrödinger's equation and related equations were explored when there was a spherical symmetry. The course ended with an in-depth investigation of the quantum theory of the hydrogen atom

The professor who regularly teaches LAW and central forces course outlined four specific goals that were directly related to understanding of the concepts of eigenvalues and eigenvectors during these winter term courses. These goals were "Students are expected to view matrices as linear transformations", "Students are expected to learn algebraic and geometric interpretations of the eigenvalue equation $Av=\lambda v$", "Students are expected to understand the expansion of any vector with eigenvectors. (Superposition idea); and, “Students are expected to be fluent in finding eigenvalues and eigenvectors of a matrix or an operator.”

In this study data were used from several sources to address the research question and to capture the different aspects of students’ learning experiences. Seven students were interviewed three times and each interview lasted for ninety minutes. All interviews were audio and video taped. A pre-quiz was given on the first day of LAW and a post-quiz was given on the last day of LAW. The researcher obtained copies of student work on two assignments given during that week, and she observed every class that week and for the remaining nine weeks (all the class sessions were videotaped).

Data analysis was an ongoing process beginning with the analysis of the first interviews collected during the fall term of 2007 before conducting the second interviews and a similar process was followed for the second and third interviews. All the interviews were transcribed and transcriptions were checked for accuracy before the next interview. As the second and third interview videos were watched to check the accuracy of transcriptions, the parts in which students seemed to use an example or an idea from the winter term courses were marked. The purpose of marking these episodes was to alert the researcher to these examples and ideas when the videos of the winter term courses were watched.

Once all the data were collected checks for accuracy, and notes were prepared from classroom videos; a case folder for each participant was compiled consisting of the printouts of transcriptions of their three interviews with marks, copies of pre- and post- quizzes, final exam and two homework assignments from LAW and notes from the winter term courses related to the individual student’s participation in LT and EE activities and also in other three courses. The researcher did case analyses of each participant’s data presenting the results based upon each individual’s experience throughout the study. The data from the second and third interviews were analyzed by implementing the actor-oriented transfer (AOT) framework. The purpose of this analysis was to explore and address the research question: *What do students transfer about the concepts of eigenvalues and eigenvectors from these series of courses to the interview setting?*

3. Results

The seven participants’ episodes containing evidence of AOT were analyzed according to the aforementioned four goals of the linear algebra week (LAW) and the three physics courses in the nine weeks following LAW. In this paper we will describe three students’ interview episodes related to students’ algebraic and geometric interpretations of eigenvalues and eigenvectors.

Algebraic and geometric interpretations

The professor of LAW expected students to understand the concept of eigenvalues and eigenvectors, the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$, and its algebraic and geometric interpretations, and to use these interpretations when solving problems.

Participants were asked to describe eigenvalues and eigenvectors in all three interviews, on the pre and post quizzes and on the final exam. The wording of the questions in all these instances was the same; students were asked, “What is an eigenvalue?” and “What is an eigenvector?” In the interviews students were also asked to give examples. To investigate students' descriptions further, they were asked questions related to eigenvalues and eigenvectors.

In six participants' descriptions of eigenvalues and eigenvectors there were episodes that constituted evidence of actor-oriented transfer of the concepts of eigenvalues and eigenvectors, the eigenvalue equation and the algebraic or geometric interpretations during the second and third interviews. It seemed that these six participants attempted to reconstruct their experiences from LAW (especially from LT and EE activities) or from one of the quantum physics courses to describe eigenvalues and eigenvectors at the interviews. The seventh student Ozzy did not explicitly refer to the interpretations of the eigenvalue equation during the interviews and no evidence of actor-oriented transfer was found in his data.

The episodes that follow give the flavor of three of participants' actor-oriented transfer of concepts of eigenvalues and eigenvectors and interpretations of the eigenvalue equation from the second and third interviews.

Episode from Milo's second interview. Milo could not recall what eigenvalues and eigenvectors were at the beginning of the first interview, even though he wrote the eigenvalue equation $Ax=\lambda x$. He stated that the symbol λ was the eigenvalue but he could not recall what an eigenvector was. Initially he did not seem to know that the variable x represented a vector in his equation. However, he was later able to decide what each variable represented and to construct the algebraic interpretation of the eigenvalue equation at the first interview. Milo referred to the geometric interpretation of the eigenvectors during the second and third interviews. Both algebraic and geometric interpretations of the eigenvalue equation were observed during the third interview with Milo.

In the second interview Milo was given the matrix $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ and asked what he thought about it. The researcher intentionally did not use the words "matrix", "linear transformation" or "operator" so that the word itself would not make the students focus on any particular experience. Milo stated that this matrix reminded him of the things that were done in class with determinants, in other words he was referring to the LT activity. He found the determinant of the matrix and said it was a reflection matrix and that they had talked about this idea in LAW. He was not sure that he had recalled correctly if the matrix was a reflection so the interviewer suggested that he check his conjecture. He multiplied the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with the matrix and he could not decide if it was reflected or rotated. He operated on the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ too and decided that the matrix was reflecting because the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ did not seem to be rotated by the same degree. Then he stated that if the matrix was a reflection there should be a reflection axis lying between the initial vectors and the transformed ones. He proposed a line between the vectors, and when he was asked to find the reflection line he stated that he needs to find the eigenvectors of the matrix and provided the geometric interpretation of eigenvectors as his reason. When he found the first eigenvector, he was asked if he had any intuition about where the second eigenvector could be, he stated it would be orthogonal to the first one. Then he mentioned the "opposite direction" idea from LAW.

> ***Milo:*** *I think it will be orthogonal to this vector [pointing to the eigenvector he found]. Because if it is a reflection then a vector along the line of reflection will be an eigenvector and any line orthogonal to the line of reflection will be an eigenvector because its direction isn't changed, just the way its pointing. But it is said in physics course; well the north and south are facing the same direction.*

Milo referred to his experience related to the "opposite direction" of eigenvectors from LAW as he explained his thinking about the relationship between two eigenvectors. Since Milo recalled his experiences in LAW as he was trying to find the reflection axis, and referred to the "opposite direction" idea from LAW, the episode provides evidence of actor-oriented transfer of geometric interpretation of eigenvectors. It seems that Milo was making a person construction of similarities between his experience with the LT activity and this particular situation of finding the reflection axis through eigenvectors.

Episode from Gus' third interview. Gus was the only participant who did not take any linear algebra courses prior to LAW. He stated he did not know what eigenvalues and eigenvectors were but he heard about them in one of his engineering courses. In the first interview he said that the variable λ was the eigenvalue but he did not know what it meant. He described the eigenvalue algebraically and the eigenvector geometrically on the post quiz. He used the geometric interpretation of eigenvectors during both second and third interviews.

In the third interview Gus was given two eigenvalues (1 and -1) and two eigenvectors ($\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$) which were associated with the given eigenvalues of an unknown operator M, and he was asked to describe the operator M as much as he could. Gus wrote the eigenvalue equation first and then substituted the given eigenvalues and eigenvectors into the equation as seen in Figure 1. It was noticed that Gus was using the notation from his physics courses ($\hat{M}$ to represent an operator).

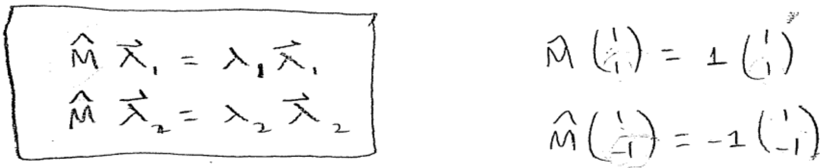


Figure 1.Gus' third interview question

Then he started to try different matrices with entries 0 and 1 as M. He said since the eigenvalues and eigenvectors involved only the number 1, he did not think the matrix M would have an entry greater than one. He guessed and checked a couple of different matrices. Then he said M could be one of the poly-spin matrices that they had in spin and quantum measurements course. Gus tried to recall the poly-spin matrices, but he could not successfully recall what they were exactly. He then said, "I am trying to think. How would I see, I can remember Professor Clayton[‡‡‡‡‡‡‡‡] was talking about this. I remember at some point she said we have the operator part. No, wait [he continued writing]." He rewrote M as a generic two by two matrix

[‡‡‡‡‡‡‡‡] Professor Clayton is a pseudonym for the professor who taught LAW and the central forces course.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and solved for the entries using the eigenvalue equation. Then he found the matrix M and said that it represented a reflection over the line $y=x$. It seemed that he recalled an experience in LAW or in one of the quantum physics courses. He mentioned the spin and quantum measurements course and then mentioned Professor Clayton's suggestion. After these ideas, he immediately wrote a generic matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (even though Professor Clayton did not do a similar problem in any of her classes). It seemed that he was trying to reconstruct ideas from one of these courses to solve the third interview problem. Once the LAW courses videotapes were watched again, the researcher observed that Professor Clayton suggested to students to start with a general matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when they don't know anything about the matrix. This event took place when students were discussing the properties of a generic rotation matrix during the third day of LAW. The same problem solving suggestion was addressed again in LAW, when students were asked to find a condition for a matrix A to be Hermitian. It seemed that Gus was making a personal construction of similarities between this third interview question and Professor Clayton's suggestion.

He also mentioned that given eigenvectors of the matrix made sense, because the first eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ was on the reflection axis and if the second eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ was reflected over the line $y=x$, it would become -1 "multiple" of itself, in other words the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ was not changing its direction. Notably, he implemented the geometric interpretation of the eigenvectors. Since Gus referred to his experience from LAW, this episode constitutes evidence of actor-oriented transfer.

<u>Episode from Deniz' third interview.</u> Deniz was also asked the same third interview question. He was given two eigenvalues and two eigenvectors that were associated with the eigenvalues of an unknown operator M and he was asked to tell everything he could about M with the given information. He tried to use the eigenvalue equation when he was working on this question. He said eigenvalues and eigenvectors probably came from a matrix like this one: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then he mentioned that he knew "an easy way" to do this problem and he had seen one of his classmates from the physics course do a similar problem and wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. He stated that $\begin{pmatrix} x \\ y \end{pmatrix}$ will be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ because of the eigenvalue equation and wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$. He continued working on the problem using the eigenvalue equation. Deniz did not mention the algebraic or geometric interpretation of the eigenvalue equation but he knew he could use the equation to solve the problem. He wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

claimed that the matrix was the identity matrix. He later said the other eigenvector would not satisfy the eigenvalue equation if M was an identity matrix and it would be "weird" to have two different matrices; so M could not be the identity matrix. He also wrote $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ but he had a sign error. He could not find what M was and gave up on the problem. However, it was noticed that Deniz found the third interview question similar to a previous question he worked on with a classmate during the winter term. The researcher checked the homework assignments from all the quantum physics courses and noticed in one of the assignments of spin and quantum measurements course students were given some properties of a spin matrix and asked to find the matrix using the given basis vectors. It may be possible that Deniz recalled this assignment. He seemed to use this experience to answer the third interview question and he explicitly stated that he had seen a classmate working on a similar problem. Since Deniz was trying to implement his previous experience of using the eigenvalue equation at the third interview, this episode seems to provide evidence of actor-oriented transfer of the eigenvalue equation and its appropriate use.

Overall, the actor-oriented transfer analysis of these participants' second and third interviews produced evidence that suggests the participants reconstructed their experiences from the LT and EE activities, problem solving ideas presented in LAW and one of the quantum physics courses. In other words, students seemed to implement algebraic and geometric interpretations of eigenvalues and eigenvectors and the eigenvalue equation.

4. Conclusion

The results of this study suggest the importance of exploring the issue of transfer by implementing the actor-oriented transfer framework. If the data from answers to all the interview questions from all participants were analyzed using the traditional transfer research paradigms, then the answer to the research question "Do students transfer the concept of eigenvalues and eigenvectors and other related ideas from winter term physics courses to the second and third interviews?" would indicate that only one participant (Milo) seemed to transfer because Milo provided complete and correct answers to almost all problems during the second and third interviews. However, the analysis with the actor- oriented transfer framework provides an in-depth exploration of how students make personal construction of similarities between experiences from the winter term courses and the second and third interview questions. Each student found their own similarities between the interview questions and their experiences in these courses. In other words, the actor-oriented transfer framework helps researchers to identify each student's perspectives on similarities between different experiences.

The results of this study also underline one of the most helpful features of the actor-oriented transfer framework. This framework can inform the researcher about the learning process rather than merely provide an observation of the end result of learning. The actor-oriented transfer framework focuses on how students connect their previous experiences (for example the experiences during teaching) to new ones (for example the experiences in the interviews) as they find explicit or implicit similarities between their experiences. In other words, this framework puts a lens on how students construct their understanding of a concept while they are constructing it.

The activity recalled by most of the participants was a small group activity at the end of which students had a whole group discussion. Further research is under way to examine the link between social learning aspects of small group activities and transfer of learning.

5. References

[1] J. Lobato, *Transfer reconceived: how "sameness" is produced in mathematical activity*, Ph.D. diss., University of California, Berkeley, 1996.

[2] T. Tuomo-Grohn, and Y. Engestrom, *Conceptualizing transfer: From standard notions to developmental perspectives,* in *Between School and Work: New Perspectives on Transfer and Boundary-crossing*, T. Tuomi-Grohn and Y. Engestrom, eds., Pergamon, New York, 2003, pp. 19-38.

[3] E. L. Thorndike, and R. S.Woodworth, *The influence of improvement in one mental function upon the efficiency of other functions,* Psychological Review, 8 (1901), 247-261.

[4] J.P. Mestre, *Transfer of Learning: from a Modern Multidisciplinary Perpective*. Information Age Publishing, Greenwich, Connecticut, 2005.

[5] J.G. Greeno, D.R. Smith, and J.L. Moore, *Transfer of situated learning*, in *Transfer on trial: intelligence, cognition, and instruction,* D. K. Detterman and R. J. Sternberg, eds., Ablex, Norwood, NJ, 1993, pp. 99–167.

[6] J.D. Bransford, A.L. Brown, and R.R. Cocking, *How people learn:brain,mind experience, and school,* National Academy Press, Washington,D.C, 1999.

[7] D.K. Detterman and R.J. Sternberg (Eds), *Transfer on trial: Intelligence, cognition, and instruction,* Ablex, Norwood, NJ, 1993.

[8] J. Lave, *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life*, Cambridge University Press, Cambridge, UK, 1988.

[9] J. Lobato and D. Siebert, *Quantitative reasoning in a reconceived view of transfer*, Journal of Mathematical Behavior, 21(2002), 87-116.

[10] M.L. Gick, and K. J. Holyoak, *Anagogical problem solving*, CognitivePsychology, 12(1980), 306-355.

[11] D.J. Ozimek, P. V. Engelhardt, A. G. Bennett, and S. N. Rebello, *Retention and Transfer from Trigonometry to Physics*, Proceedings of the Physics Education Research Conference, 2004, Sacramento, California.

[12] J. D. Bransford, and D.L. Schwartz, *Rethinking transfer: A simple proposal with multiple implications,* in *Review of research in education,* A. Iran-Nejad and P. D. Pearson eds., Washington, DC:AERA, 24 (1999), 61-100.

[13] K. Beach, *Consequential transitions: A sociocultural expedition beyond transfer education*. in *Review of research in education*, A. Iran-Nejad and P. D. Pearson eds., Washington, DC:AERA, 24 (1999), 61-100.

[14] H. Hatano, and J.G. Greeno, *Commentary: Alternative Perspectives on Transfer and Transfer Studies,* International Journal of Education Research, 31(1999), 645-654.

[15] J. Lobato, *Alternative Perspective on the Transfer of Learning: History, Issues, and Challenges for Future Research*, The Journal of the Learning Sciences, *15(4)* (2006), 431-449.

[16] D.J. Ozimek, *Student Learning, Retention and Transfer from Trigonometry to Physics,* M.S. thesis, Kansas State University, 2004.

[17] L. Cui, *Assessing College Students' Retention and Transfer from Calculus to Physics*, Ph.D. diss., Kansas State University, 2006.

Student Understanding and Misconceptions About Foundations of Sequences and Series

HARVEY B. KEYNES[§§§§§§§§], School of Mathematics
University of Minnesota, Minneapolis, USA
BRIAN J. LINDAMAN, Department of Mathematical Sciences
Montana State University, Bozeman, USA
REBECCA J. SCHMITZ, School of Mathematics
University of Minnesota, Minneapolis, USA

Abstract: This paper explores student understanding of foundational concepts from sequences and series. In particular, we examine student understanding of infinite decimals such as $.\bar{9}$. and their ability to connect them to alternate forms and notation such as .999… and $9/10 + 9/10^2 + 9/10^3…$ While the original focus was to measure pre-instructional and post-instructional student understanding and knowledge gained for a special university class of honors calculus for very talented students in grades 8-10, we expanded the study to include five different levels of university calculus and post-calculus courses, including general and engineering calculus, departmental and university honors calculus, and bridging courses for more advanced mathematics coursework. The overall preliminary results showed that the high school students and departmental honors students started from a high initial understanding and improved their knowledge significantly after instruction. Moreover, both groups retained their high level of understanding longitudinally. With the other groups, while pre-instructional knowledge varied, post-instructional gains were modest or non-existent. While all groups started with some difficulties understanding $.\bar{9}$, these misconceptions were significantly removed primarily in the two groups mentioned above. We will provide more details on these questions and other more fine-grained observations about less standard forms such as $.\overline{13}$.

While it is not unexpected that honors-level students would have greater understanding, it also appears that the instructional approach heavily influences improved foundational understanding. Both of the honors groups had instructional approaches which included emphases on foundational concepts, active student learning with group work and homework problems on these ideas. This contrasted to the typical instructional approach of quickly glossing over these concepts and concentrating on computational approaches in both recitation and homework. The data suggests that without explicit and appropriate instruction, students retain faulty understanding which then persists in later courses. We intend to further study this assumption in future papers.

Keywords: infinite decimals; sequences and limits; series representations; APOS Framework; Transitions from process-level to object-level understanding

1. Introduction

In this study, we explored student's understanding of foundational concepts of sequences and connections between limits and series. A central focus was the understanding of the meaning of infinite repeating decimals, and their alternate notations and forms in terms of sequences and series. Starting from the classical problem of determining the value of .999…, we explored several variations to determine depth of understanding. We investigated the effects of teaching approaches on understanding and longitudinal retention of these ideas. This paper will describe

[§§§§§§§§] E-mail: keyens@math.umn.edu

our initial approaches and findings, and suggest additional questions to verify our preliminary results. These results will also be examined in more advanced aspects of sequences and series.

The genesis for this study was an interest in measuring student understanding and knowledge gain in the honors calculus class for talented high school students in the University of Minnesota Talented Youth Mathematics Program (UMTYMP). UMTYMP was started 29 years ago in order to provide Minnesota's most mathematically talented students with an alternative educational experience. Each year approximately 400 students in grades 6-12 take their mathematics courses through UMTYMP instead of their own schools. During their first two years of the program, students cover four years of standard high school curriculum: algebra I and II, geometry, and precalculus. The final three years of the program are comprised of honors-level collegiate courses in calculus, linear algebra, differential equations and vector analysis. Along the way, students must develop a strong work ethic and excellent problem solving skills. Conceptual ideas are stressed with an emphasis on proof writing skills. Many continue on to upper-division and graduate level mathematics courses before finishing high school. For more details, go to *www.itcep.umn.edu*. Since these students typically encounter our rather rigorous approach to sequences and series typically while in grade 9, we wanted to gauge how effective our teaching approaches were for this important topic.

The major source of information was an instrument we designed to assess student understanding at several levels, and to compare the results internally in various ways to more precisely gauge levels of understanding. We compared UMTYMP students' pre- and post-instructional results to measure the effects of our instructional approaches. Because of our interest in gauging UMTYMP students' knowledge compared to regular college students, we also compared these results to pre- and post-instructional results in two of the large freshman calculus courses and to pre- instructional results in 2 sophomore post-calculus bridging courses whose major content focus was sequences and series. Finally, we gave the same assessment to the departmental freshman mathematics honors course, where students entered with advanced standing in calculus. Results in general suggest that the UMTYMP students were comparable for the departmental honors students, and both of these groups performed significantly better than either the calculus or bridging course students.

In this paper, we will discuss the theoretical framework used in this study, and how we developed our instrument to assess several aspects. We will provide details about the assessment results and look at data that indicates longitudinal retention of these concepts. We will discuss our preliminary findings on instructional impact on improving students' understanding, and indicate some of the questions that they raise. Since our overall population involved over 750 students and included engineering and science majors, honors students and talented high school students, these preliminary findings should be useful indicators of further directions for the study.

2. Theoretical Background and Related Research

In the past 25 years there has been sustained interest on the part of educational researchers in student understanding of the limit concept. Tall and Vinner's (1981) seminal study on concept image and definition explored students' difficulties with .999… and this study contributed to the development of other learning theories. For this study, we will use two more recent models of student thinking to frame our discussion. The first model, the Action-Process-Object-Schema (APOS) framework, has been used to explore students' thinking about many calculus concepts, including functions, sequences, and derivatives. A second model by Tzur and Simon (2004) is built on the analysis of participatory and anticipatory understanding. This provides a more fine-grained model for us to analyze student responses.

The APOS model proposes that the maturation of a concept, within a student's mind, can be categorized into four distinct levels of understanding (Asiala et al., 1996). The concept is viewed first as an action level, second as a process level, third as an object level third, and fourth at a schema level. A student with an action concept is only able to gather information about a concept by performing each step as a specific action (Cottrill et al., 1996). Students use the process concept when they view a concept as a process. The individual is able to describe, reverse, and reflect upon the steps of a process without actually carrying them out. When an individual reflects on operations applied to a process, sees the process as applicable to other processes, and understands the process in its totality, then he/she is at an object level of conception. A schema conception is the entire collection of consciously or unconsciously linked actions, processes, objects, or other schema in a student's mind. The APOS theory has been used to analyze students' understanding of several mathematical topics, including functions (Breidenbach, Dubinsky, Hawks, & Nicholas, 1992; Dubinsky & Harel, 1992), sequences (McDonald, Mathews, & Strobel, 2000), and derivatives (Zandieh, 2000).

The APOS framework seems well-suited for our use here. There is a wealth of findings on student understanding of concepts mathematically closely related to infinite decimals. Hence, APOS-based research suggests that students' understanding of limits and sequences may also be categorized into the action, process, and object stages of understanding. Since infinite decimals are mathematically limits of sequences, we hoped to find similar action-process-object level distinctions among students in our study. We do observe that, especially in the course of our interviews, students' responses indicated an APOS-type understanding at work in their thinking processes. Roughly stated, for the purpose of our study, .999… is a process and 1 is an object and more generally, limits are processes and the limit value is an object.

Another learning theory that sheds light on our findings is the theory of Stage Distinction and Activity Effect Relationships (SAER theory) described in Tzur and Simon (2004). This theory builds upon the APOS model, the reification theory proposed by Sfard (1991), and Von Glasersfeld's tripartite model (1995). In a sense, Tzur and Simon's (2004) theory both consolidates and extends these three models. It illuminates the subtle transition, in two stages, for students as they move from a process-level understanding to an object-level understanding. In the SAER theory, Tzur and Simon (2004) posited that the transition from a process-level understanding involves two stages of conceptual transformation: participatory and anticipatory. The mechanism by which students are able to make the transition is reflection on an activity-effect relationship (AE relationship).

In the participatory stage, learners can anticipate the effect of an activity, and explain why they work but only in the context of the activity (Tzur & Simon, 2004). Their understanding of the AE relationship is dependent upon the particular activity in which they first formed their understanding. In the anticipatory stage learners' use of the new activity-effect relationship is no longer context-dependent. The learner is able to independently abstract the situation and use a newly-formed AE relationship appropriate to the situation at hand. One reason for students' inability to call upon prior knowledge in a new context may be that students are still operating at a participatory stage, and thus are not at an object-level understanding yet. The failure of interventions to achieve this transition may be explained by the lack of attention in the second (anticipatory) stage (Tzur & Simon, 291, 2004). As an example, our assessments indicated that some students cannot call up newly developed geometric series AE relationships when presented with the goal of determining convergence of different representations of infinite decimals. It appears that the "…" used in, say, .777… does not imply a limit process to them.

3. Development of the Survey Instrument

For this study, we developed two versions (pre- and post-assessments) in order to assess in some detail student understanding of the infinite decimal $.\overline{9}$. The initial draft of this survey came from items on the Series Understanding Instrument (SUI) developed and used in a previous study on calculus learning (Lindaman, 2007). However, instead of focusing on students' understanding of infinite series in general, for this study we decided to target their specific understanding of infinite decimals. Did students view $.\overline{9}$ the same as the symbolic representation .999…? In particular, we were interested in learning more about what the notation "…" meant to them in the latter symbolic representation. In the language of sequences and limits, was it a particular n-term $\underbrace{.999...9}_{n}$ for some unspecified but fixed value of n, a representation for any n-term $\underbrace{.999...9}_{n}$, or the actual limit $\lim_{n\to\infty}\underbrace{.999...9}_{n}$? How is it related to the sum of the infinite series $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\dots$? Finally, of course, what were the numerical values of these symbols, and how did they compare?

The central method to address these issues was question (3), in which parts (a) and (b) addressed the first 2 questions above, and parts (d)-(f) addressed the last question. Part (c) was related to one method of computing .999… as equal to 9x(.111…) and then showing that .111… = $\frac{1}{9}$. To help determine student thinking in terms of SAER theory, question (1) was a close parallel to question (3) with 7 replacing 9. There are some indications that student thinking for integers other than 9 becomes more focused and anticipatory because of the lack of a priori knowledge about these types of questions. Also, success with question (1) could indicate a growing object-level understanding in SAER theory. To help discourage rote copying of answers, the orders of some parts of question (1) and (3) were scrambled.

For question (2), the major goal was to specifically probe student understanding of the symbol .mmm… for any integer m. In the pre-assessment, we chose m=5 and in the post-assessment m=6. Our question (4) was an attempt to be even more general than question (2) in gauging student's depth of object–level understanding by replacing single digit numbers with n=13, as well as asking for the actual value. Since the notation .131313… might be less familiar than the infinite decimal notation $.\overline{13}$, we used the latter notation.

To assess student understanding after instruction, the post-assessment was designed to be a virtual replication of the pre-assessment, with very slight variation. While question (1) and (3) were exactly duplicated, question (2) as previously noted exchanged digits, and similarly question (4) replaced n=13 with n=41. With these modifications, the study assumed that the ability to answer all parts of question (4) correctly would indicate a complete object-level understanding of infinite decimals.

4. Analysis of the Data

The questionnaire was given to over 750 students in several different calculus and post-calculus sequences as well as UMTYMP students at the University of Minnesota (UMN) during 2008 and 2009. The college students were in 5 different levels of calculus and post calculus-courses (general calculus, engineering calculus, university-wide honors, departmental honors, and bridging courses) and several different instructional formats. One major focus was to compare the UMTYMP students to the calculus students at UMN. The pre-assessment was given to UMTYMP calculus 1 students during Spring 2008 and Spring 2009. Due to an error in the questionnaire we are excluding Spring 2008 from the pre-assessment data. The various courses are denoted as: mainstream calculus classes (MC) and in the university-wide honors calculus class (HC), where sequences and series are first taught; the sequences and series and foundations bridge course (SSF) and the departmental honors (DHC). Post-assessment results are not currently available for the university-wide honors calculus class. The post-assessment is included as Figure 1 at the end of this paper.

Table 1: Pre-Assessment

Class	**Mean (out of 20)**	**Median (out of 20)**
UMTYMP Calc 1	16.25	16.50
MC	12.61	12.73
HC	15.21	15.00

Table 2: Post-Assessment

Class	**Mean (out of 20)**	**Median (out of 20)**
UMTYMP Calc 1	18.39	20.00
MC	14.37	13.99

Table 3: Longitudinal

Class	**Mean (out of 20)**	**Median (out of 20)**
UMTYMP Calc 2	19.21	20.00
UMTYMP Calc 3	18.05	20.00
SSF	13.29	13.00
DHC	18.72	20.00

There are several conclusions that can readily be drawn from the pre-assessment data. First, the UMTYMP students appear to have a stronger basic background for sequences and series than students in the standard calculus sequences. In fact, their pre-assessment knowledge is better than the post-assessment results for MC students. Also, UMTYMP students are better than students in the university-wide honors calculus sequence. However, even with this strong initial background, UMTYMP students still could increase their understanding with instruction.

The post-assessment data shows that both UMTYMP students and MC students made solid gains in understanding. However, UMTYMP students seem to have made more significant gains with 2.14 and 3.50 increases in mean and median scores, compared to 1.76 and 1.26 for MC students. Long-term retention is also shown by the longitudinal data above. The comparable population to UMTYMP calculus 2 and 3 for retention among the UMN students is the SSF bridge class for upper-level mathematics. In contrast to UMTYMP where we can track the same students through all three years of the calculus program, the SSF students come from several different classes but do form a representative sample. While the SSF students do show reasonable retention, the UMTYMP students' retention is notable since it is as strong as the post-assessment results. Finally, we have the departmental honors students, DHC, who we use as the longitudinal group for the HC students. Not surprisingly, these students show roughly the same retention as UMTYMP students.

As an overall measure, it appears that UMTYMP students in calculus 2 and 3 students are performing mathematically at about the same level as the DHC students. Also, when looking at this data, one should note that there is often a large age gap between UMYTMP and regular UMN students. Students in UMTYMP calculus 1 range in age from 14 to 16, where the typical UMN calculus student is 18 or 19. Thus UMTYMP students are mathematically performing at a level far beyond their maturity age-wise. We plan to pursue this observation in a future paper.

As previously noted, question (3) addresses student understanding of $.\overline{9}$. The body of the table above consists of the percentage of students who selected that response. For instance, (3a) is an incorrect response marked in bold and 15% percent of the UMTYMP calculus 1 students (pre-assessment) selected this response as true. Response (3b) is the other marked in bold incorrect response. The remaining table in this section will also follow the same convention.

Table 4: Longitudinal Question 3

	UMTYMP Calc 1: Pre	UMTYMP Calc 1: Post	UMTYMP Calc 2/3	MC-1: Pre	MC-1: Post	MC-2: Pre/Post	SSF
a: $.\overline{9}$ is a number less than $9/10 + 9/10^2 + 9/10^3 + \ldots$	.15	.07	.05	.20	.12	.17	.10
b: .999… is a number less than $9/10 + 9/10^2 + 9/10^3 + \ldots$	.31	.05	.03	.23	.12	.25	.22
c: .999… = 9 • 1/9	.65	.84	.87	.31	.71	.26	.33
d: $.\overline{9} = 1$	.69	.93	.92	.42	.90	.36	.44
e: $9/10 + 9/10^2 + 9/10^3 + \ldots = 1$	.77	.92	.93	.41	.87	.43	.38
f: .999… = 1	.63	.92	.91	.34	.85	.31	.38

All students, including UMTYMP students, started with initial confusion. UMTYMP calculus 1 students show a marked increase in understanding upon instruction and strong retention through calculus 2 and 3. By contrast, the UMN students did not show similar retention as SSF students, but one group of MC students, MC-1, did show substantial gains upon instruction. Our results indicated that among the MC population, another group, MC-2, remained very consistent and unchanged in their understanding of .999…, pre- and post-instruction. Thus we separated the two populations and have averaged the pre- and post-assessment results for MC-2. We noted the SSF students, the longitudinal population for all the MC students, show similar understanding at the level of the MC-2 students. Thus, even though the MC-1 students had improved understanding, those gains seem to have been lost as SSF students. We also noted that while (3d) – (3f) were initially more troublesome for UMTYMP students and MC-1 students, both populations showed large gains upon instruction, while the MC-2 students showed no significant improvement after instruction in understanding of these questions.

Finally, we compare student understanding on the matched set of questions: (1) and (3). For the sake of comparison, we match up the relevant parts. Surprisingly, despite the fact that both questions are asking the same things, there is initially significantly more pre-instructional confusion for the UMTYMP calculus 1 students when working with .999… as opposed to .777… This trend is true even among the strongest students. However, the gap narrows significantly post-instruction. These results are similar among MC students, with the rather large pre-instructional gaps between .999… and .777… narrowing due to large understanding gains of .999… Longitudinally, the understanding of SSF students slips back to pre-instruction levels.

Table 5: Question (1)[on .777...] and Question (3)[on .999...]

	Question	**a**	**b**	c	d	e	F
UMTYMP Calc 1: Pre	1	.02	.17	.79	.92	.88	.85
	3	.15	.31	.65	.69	.77	.63
UMTYMP Calc 1: Post	1	.04	.04	.91	.92	.89	.85
	3	.07	.05	.84	.93	.92	.92
MC-1: Pre	1	.08	.09	.76	.86	.70	.91
	3	.20	.23	.31	.42	.41	.34
MC-1: Post	1	.12	.09	.84	.83	.86	.95
	3	.12	.12	.71	.90	.87	.85
MC-2: Pre	1	.10	.19	.46	.52	.40	.78
	3	.18	.26	.21	.38	.40	.31
MC-2: Post	1	.13	.16	.55	.62	.51	.85
	3	.17	.25	.30	.34	.46	.31
SSF	1	.06	.13	.66	.71	.59	.78
	3	.10	.22	.33	.44	.38	.38
UMTYMP Calc 2/3	1	.01	.01	.95	.96	.93	1.00
	3	.05	.03	.87	.92	.93	.91
DHC	1	.06	.11	.94	.89	.89	1.00
	3	.00	.06	.83	.84	.93	.89

Questions (2) and (4) likewise probe a matched set of concepts, with the variations being the number of repeated digits in the decimal (.555… and .1313…) and the ordering of parts. Two striking observations can be made. First, even with instruction, the notation $\underbrace{.555\ldots5}_{n}$ remains confusing to students. This is true across the board in UMTYMP calculus 1, MC, and HC. The longitudinal data suggests little improvement in understanding of this notation in UMTYMP students while also demonstrating modest improvement in MC students.

Secondly, even though the number of digits being repeated changes from one in question (2) to two in question (4), student understanding did not seem to change, with the exception of confusion with the notation $\lim_{n\to\infty} \frac{13}{10^2} + \frac{13}{10^4} + \ldots + \frac{13}{10^{2n}}$. Note that this notation is no longer a standard decimal representation. This observation is emphasized by the MC pre-assessment data where a significantly larger number, 77%, were able to correctly answer the parallel question $\lim_{n\to\infty} \frac{5}{10} + \frac{5}{10^2} + \ldots + \frac{5}{10^n}$, while only 58% were able to answer the above more difficult question involving 13. In UMTYMP students, the longitudinal data suggests increased understanding of the general form of this notation over time, especially with respect to the non-traditional notations. Among the SSF students, we again see some loss in understanding from the MC post-assessment levels.

We also have collected information on the students' backgrounds, including AP Calculus tests taken, and have examined the influence of AP background on the current data. Evidence suggests that any level of AP Calculus has little influence on the post-assessment scores of MC students. This will be pursued further in a future paper.

To better understand instructional issues, we observed instructional practices for most of the non-UMTYMP groups in this study. In general, the undergraduate calculus courses were taught in a large lecture format, with most students sitting passively and sometimes taking notes. These lectures also had traditional recitation classes where students primarily asked homework questions.

On the other hand, the UMTYMP classes were taught in two distinct phases: a somewhat large (50-60 students) interactive lecture-based format for the first hour, including examples which pursued conceptual understanding, and smaller discussion groups for the second hour where students worked in groups of 3 or 4 on challenging problems based on the content presented that day in lecture.

The above observations about instruction suggest some approaches which will lead to improved foundational understanding. In both the UMTYMP and the DHC classes, foundational issues were included centrally as part of the lectures, including examples involving these concepts. Students subsequently did class group works and homework which involved these two issues. This contrasts to the more typical approach of quickly glossing over these concepts in lecture and concentrating on computational approaches in both homework and recitation. Our data suggests that without this explicit instructional approach, students retain a faulty understanding which continues to persist in later courses. The level of understanding demonstrated by SSF students does not bode well for having a sufficient background for upper-division analysis and topology courses. We will again address this issue in the following discussion section.

5. Discussion of Results

As noted in the previous section, one initial finding is that students' pre-instructional understanding of single-digit repeating decimals varies considerably. It appears that many students see the equality of .777… to 7/9 as fundamentally different than the equality of .999…. and 1. This confusion can be due to a number of causes. Some students may see "1" as significantly distinct from any fraction, being the first natural number, whole, and so forth. So for it to equal a decimal, and not be a fraction, per se, is confusing for them. Secondly, 7/9 indicates an operation (division) which results in the decimal representation. However, 1 indicates no such operation. So students have no way of "travelling" from 1 to .999…, as they can with 7/9. Finally, there may be entirely distinct stages of understanding at work for students as they consider 1 versus .999… Students may view 1 with an object-level understanding, while .999… is viewed with a process-level understanding, in the Action-Process-Object-Schema model described by Asiala et al. (1996). The SAER framework (Tzur & Simon, 2004) provides more detail on this, and is described more fully below as we note the differences in retention between the groups of students.

We also noted a significant notational confusion for some UMTYMP students and a much larger group of MC and SSF students, as seen in question (2). These students failed to recognize that .555… is the same as $\lim_{n\to\infty} \underbrace{.555...5}_{n}$. Similar proportions of students from these groups also believed that .555… is the same as $\underbrace{.555...5}_{n}$, where n is a positive integer. Some reasons for this behavior may include a failure to fully understand the "braced n" notation. One UMTYMP student indicated a different reason behind her incorrect response, namely that the notation "…" from .555… was confusing and ambiguous. She understood that "…" meant that a pattern was to continue, but that it could possibly terminate after some time, or, in the case of .555…, that the repetend could change from a 5 at some point. In contrast, the repeating bar $.\overline{5}$ and the "braced n" notation were unambiguous to her.

We also observed in the previous section that in comparing groups, the retention levels are much better for the UMTYMP students. Their ability to consistently anticipate the effect of a limit process, especially when working with a limit process which is out of their initial comfort zone (like .999…), indicates that these students are operating at an anticipatory stage of understanding. So, in the APOS sense, we can tentatively conclude that their understanding of infinite decimals is at an object level of understanding.

On the other hand, many of the college students seemed stuck at a process level of understanding. In the SAER framework, they were not able to call up their Geometric Series AE relationship understanding when confronted with infinite decimals. They were applying knowledge from a different source, most likely their knowledge of rational numbers and decimals. The college students' knowledge and understanding about geometric series seems to be inaccessible when these ideas are presented in a different setting. This finding corroborates with Tzur and Simon's (2004) finding; that is, that students who are at a participatory stage of understanding of a concept have knowledge that is context-dependent. Certainly, we would like to substantiate this claim further by asking more typical series-type questions, such as applying convergence tests, to students and see if their responses on infinite decimal questions are more consistent with their knowledge level about geometric series.

Finally, we have already noted that the impact of instruction on infinite decimal understanding was greater for the UMTYMP and DHC classes than the regular college classes. This suggests that the type of conceptually oriented and active student learning modes of calculus instruction present in the UMTYMP curriculum (i.e., focus on problem-solving and group work) provided students with a greater conceptual understanding than the traditional practices present in the regular college classrooms (lecture-based and computational homework questions). This finding is substantiated elsewhere in calculus research literature (Bookman & Friedman, 1994; Lindaman, 2007; Park & Travers, 1996). We intend to further explore this line of investigation into the influence of instructional practices in future work.

Figure 1. Post Assessment – Page 1

Calc 1 Post-Questionnaire 05/07/09 Name:

1. Which of the following are true? Circle your answer(s).
   (a) $\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\ldots=\frac{7}{9}$
   (b) $.777\ldots$ is a number less than $\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\ldots$
   (c) $.\overline{7}$ is a number less than $\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\ldots$
   (d) $.777\ldots$ is the same number as $.\overline{7}$
   (e) $.777\ldots=\frac{7}{9}$
   (f) $.\overline{7}=\frac{7}{9}$
2. Which of the following are the same as $.666\ldots$? Circle your answer(s).
   (a) $.\underbrace{666\ldots6}_{n}$ where n is a positive integer
   (b) $\lim_{n\to\infty} .\underbrace{666\ldots6}_{n}$
   (c) $\lim_{n\to\infty} \frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\ldots+\frac{6}{10^n}$
   (d) $\sum_{n=1}^{\infty}\frac{6}{10^n}$
3. Which of the following are true? Circle your answer(s).
   (a) $.\overline{9}$ is a number less than $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots$
   (b) $.999\ldots$ is a number less than $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots$
   (c) $.999\ldots=9\cdot\frac{1}{9}$
   (d) $.\overline{9}=1$
   (e) $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots=1$
   (f) $.999\ldots=1$
4. Which of the following are the same as $.\overline{41}$? Circle your answer(s).
   (a) $\lim_{n\to\infty} \frac{41}{10^2}+\frac{41}{10^4}+\frac{41}{10^6}\ldots+\frac{41}{10^{2n}}$
   (b) $.\underbrace{414141\ldots41}_{n}$ where n is an even positive integer
   (c) $\lim_{n\to\infty} .\underbrace{414141\ldots41}_{n}$
   (d) $\frac{41}{99}$

6. References

Asiala, M., Brown, A., Devries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A Framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 1-32). Providence, RI: American Mathematical Society.

Bookman, J., & Friedman, C. (1994). A comparison of lab based and traditional calculus. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education I* (pp. 101-116). Providence, RI: American Mathematical Society.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of functions. *Educational Studies in Mathematics,* 34, 247-285.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior,* 15, 167-192.

Dubinsky, E., & Harel, G. (1992). *The concept of function, aspects of epistemology and pedagogy*. Washington, DC: Mathematical Association of America.

Lindaman, B. (2007). Making sense of the infinite: A study comparing the effectiveness of two approaches to teaching infinite series in calculus. *Dissertation Abstracts International, 68*, 03A. (UMI No. AAI3255081)

McDonald, M., Mathews, D., & Strobel, K. (2000). Understanding sequences: A tale of two objects. In E. Dubinsky, A. Schoenfeld, & J. Kaput, (Eds.), *Research in collegiate mathematics education IV* (pp. 77-102). Providence, RI: American Mathematical Society.

Park, K., & Travers, K. (1996). A comparative study of a computer-based and a standard college first-year calculus course. In J. Kaput, A. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 155-176). Providence, RI: American Mathematical Society.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 26*, 114–145.

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics,* 12, 151-169.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education,* 2, 287–304.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput, (Eds.), *Research in collegiate mathematics education IV* (pp. 103-127). Providence, RI: American Mathematical Society.

Modelling, applications and inverse modelling: innovations in differential equations courses

VICTOR MARTINEZ-LUACES[*********]

I.I.Q., Faculty of Engineering, University of the Republic of Uruguay

Abstract: Differential equations and Laplace transform are widely used to solve problems concerned with mathematical modelling and applications. These analytical tools appear regularly in chemical engineering courses, due to the usefulness in modelling chemical kinetics, mixing problems and reactors design, among other problems. The discussion in this paper will deal with non typical inverse problems and how they can be used in engineering courses. Recommendations will be drawn for undergraduate courses in differential equations for chemical engineering and other related careers.

Keywords*:* Modelling and applications; Differential Equations; Engineering courses; Inverse problems

2000 Mathematics Subject Classifications: 97D50; 34A30; 34A55

1. Introduction

Modelling and application is nowadays an important and fast growing area in mathematics education at the tertiary level. On the other hand, differential equations (both ordinary and partial) and Laplace transform (abbreviated as O.D.E., P.D.E. and L.T. in what follows) are mathematical tools that since their beginning are strictly related with applications in other sciences and engineering.

These mathematical tools (i.e., O.D.E. and L.T.) are very useful when mixing problems are considered [1] and [2]. Moreover, as it was commented in other papers and books [2-3], a stirred tank system can, under certain conditions, be considered as a chemical reactor. For instance, under ideal conditions, a stirred tank is a "Continuous Stirred Ideal Reactor" (CSIR) [4]. This fact was mentioned, and analysed in [2-3-5], among other publications.

As a consequence, mixing and/or chemical reactor design problems are excellent sources for applications and modelling examples where O.D.E. and L.T. are widely used.

When mixing problems are included in differential equations texts, volume, geometry and flux data of the tanks are given and so, a trained student can solve such problems just as an exercise. Once they have solved a "typical set" of these problems, they become a routine task, which is interesting from the applications view point, but is no longer a problem to challenge students.

The inverse problem corresponding to the typical mixing situation, consisting in asking which tank system, if any, corresponds to a given function or a given O.D.E. linear system. As it is usual in inverse problems [6-7], the desired conditions of existence, uniqueness and stability are not present in many cases. This kind of

[*********] E-mail: victorml@fing.edu.uy

problem –called inverse modelling problems in previous papers [2-4-5]– is more interesting due to its mathematical richness and unpredictability. As was shown in other papers [2-4-5], inverse modelling situations related to stirred tanks and reactors, cannot always be solved, or there may not be a unique solution but even in this case, slight modifications in functions or O.D.E. coefficients produce big changes in the final results.

In the next section some new problems (not analysed in other papers), will be considered, taking into account their potential richness from a mathematical education view point.

2. Several inverse modelling problems

a) Previous concepts

The following ideas and concepts are included near the beginning of any O.D.E. and L.T. course.

i) An ideal stirred tank:

A stirred tank (CSIR reactor) is showed in the following figure:

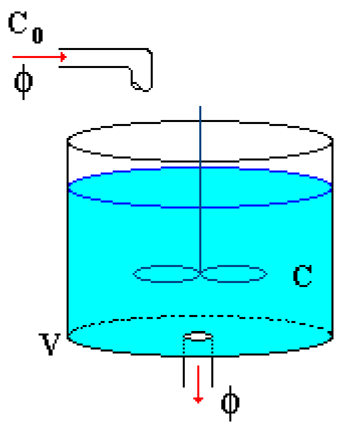


Figure 1 Ideal stirred tank

In this problem, the input is a water solution of salt with a volumetric flux **ϕ** (L/s) and a concentration $\mathbf{C_0}$ (g/L). The volume of the tank is **V** (L).

If **ϕ** and **V** are constants, then: $V\dfrac{dC}{dt} = \Phi C_0 - \Phi C$ **C**(0)=0

Applying L.T. we have: $V.(s.L\{C\} - C(0)) = \Phi L\{C_0\} - \Phi L\{C\}$

And rearranging we get: $L\{C\}\cdot(V.s + \Phi) = \Phi \cdot L\{C_0\}$

The transference function is defined as: $G(s) = \dfrac{L\ \{C(t)\}}{L\ \{C_0(t)\}}$, i.e., the quotient between L.T. applied to the output and input and it is a characteristic of the reactor.

In this case we have: $G(s) = \dfrac{\Phi}{V.s + \Phi}$

Rearranging again and putting $\dfrac{V}{\Phi} = \tau$, the result is: $G(s) = \dfrac{1}{1 + \tau.s}$, where τ is the average time of a water molecule in the reactor.

ii) Reactors in series and parallels:

Using the definition given for the transference function, then for a series of two chemical reactors (see figure 2), it is easy to obtain the following formula:

$G(s)=G_1(s).G_2(s)$

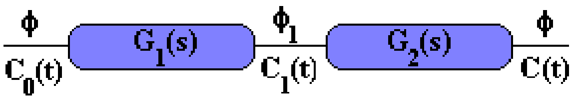


Figure 2. Series of two chemical reactors.

It is possible to obtain a general version $G(s) = \prod_{i=1}^{n} G_i(s)$ for a series of *n* chemical reactors.

A little more difficult is to derive the formula for a parallel system consisting of a pair of chemical reactors as in the following figure:

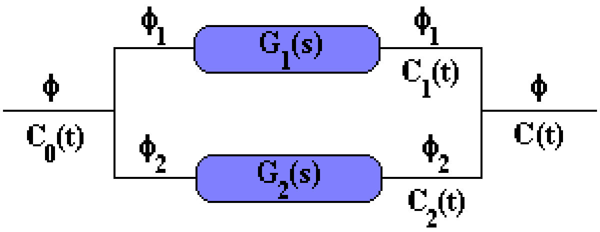


Figure 3. Parallel of two chemical reactors.

In this case, a mass balance at the second bifurcation point, and several algebraic manipulations give: $G(s) = \dfrac{\Phi_1}{\Phi}.G_1(s) + \dfrac{\Phi_2}{\Phi}.G_2(s) = f_1.G_1(s) + f_2.G_2(s)$,

where the coefficients f_i are $f_i = \dfrac{\Phi_i}{\Phi}$, so they can be interpreted as fractions of total flux. This formula can be easily generalised to give $G(s) = \sum_{i=1}^{n} f_i . G_i(s)$, where $\sum_{i=1}^{n} f_i = 1$ for obvious reasons.

b) Two little theorems:

In this second part, two theoretical results will be stated:

i) A series of n ideal stirred reactors:

A series of n CSIR reactors is considered, as in the following figure:

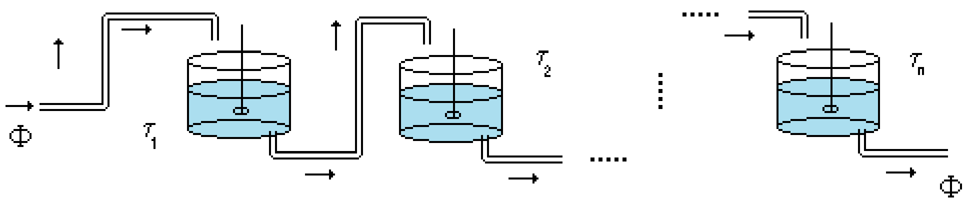


Figure 4. Series of n CSIR.

In this case, it is easy to get the following formula:

$$G(s) = \frac{1}{1+\tau_1 s} \cdot \ldots \cdot \frac{1}{1+\tau_n s} = \frac{1}{(\tau_1 \ldots \tau_n)s^n + \ldots + 1}$$

Moreover, it can be seen that $G(s) = \dfrac{1}{a_n s^n + \ldots + a_1 s + 1}$, where $a_n = \prod_{i=1}^{n} \tau_i$ and $a_i > 0 \quad \forall i$.

As a theoretical exercise, students can be asked to demonstrate this result using complete induction.

ii) A parallel of n CSIR:

As usually happens, the parallel equivalent is not so easy as the series example. For instance, if only two CSIR are considered, it follows that:

$G(s) = f_1 \dfrac{1}{1+\tau_1 s} + f_2 \dfrac{1}{1+\tau_2 s}$ being $f_1 = \dfrac{\Phi_1}{\Phi_1 + \Phi_2}$ and $f_2 = \dfrac{\Phi_2}{\Phi_1 + \Phi_2}$ Then, in this case, the transference function is: $G(s) = \dfrac{(f_1\tau_2 + f_2\tau_1)s + (f_1 + f_2)}{\tau_1\tau_2 s^2 + (\tau_1 + \tau_2)s + 1}$

The expression $f_1\tau_1 + f_2\tau_2$ is positive and, at same time, it is a convex linear combination of residence times τ_1 and τ_2, while the other coefficient $f_1 + f_2$ is

$$f_1 + f_2 = \frac{\Phi_1 + \Phi_2}{\Phi} = \frac{\Phi}{\Phi} = 1$$

Once more these results can be generalised for a parallel of f n CSIR reactors, as in the following figure.

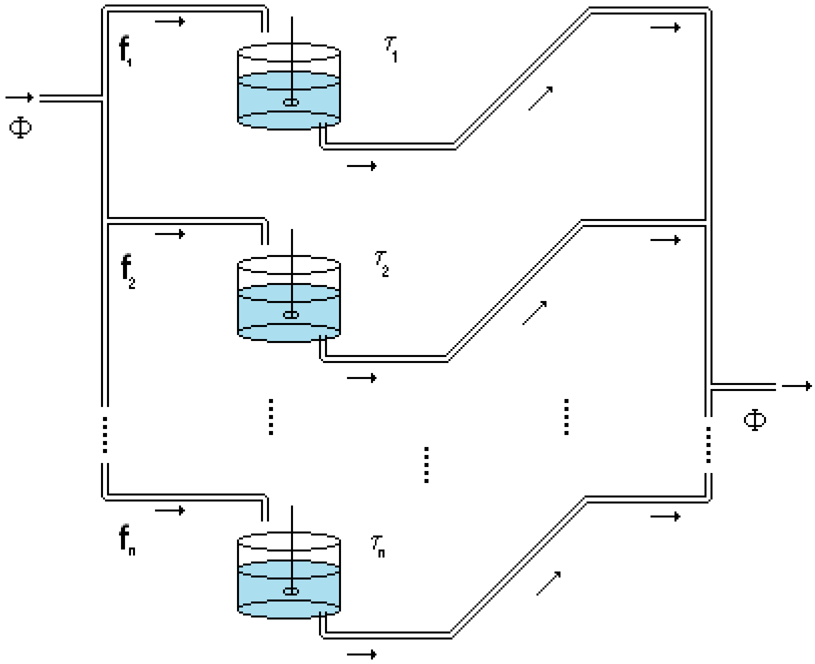


Figure 5. Parallel of n CSIR.

In this case it is possible to get $G(s)=\dfrac{b_{n-1}s^{n-1}+\ldots+b_1s+1}{a_ns^n+\ldots+a_1s+1}$ where $a_ns^n+...+a_1s+1$ is the same as in the series case, $b_j>0\ \forall j$ and the first coefficient b_{n-1} is a convex linear combination of $\tau_1\cdots\tau_{n-1}$,…, $\tau_2\cdots\tau_n$ (i.e., all the $n-1$ factors product of residence times).

Another possibility consists in writing $b_{n-1}=\left(\prod_{i=1}^{n}\tau_i\right)\cdot\left(\sum_{i=1}^{n}\dfrac{f_i}{\tau_i}\right)$.

As in the series case, all these results can be demonstrated by induction and this can be a theoretical exercise for the students.

c) Consequences of the previous results:

Interesting inverse modelling problems can be put to the students. For example, this function can be considered: $G(s)=\dfrac{s^2+s+1}{(1+s)(1+2s)(1+3s)}$. Does it correspond to a parallel of CSIR? In a heuristic approach the answer may be "yes", but analysing the previous results shows it to be impossible. This fact can be observed by taking into account that $G(s)=\dfrac{P(s)}{Q(s)}$, where $Q(s)=(1+s)(1+2s)(1+3s)$ suggest to consider three CSIR with residence times $\tau_1=1$, $\tau_2=2$ and $\tau_3=3$. Nevertheless, in this case $\tau_1\tau_2=2$, $\tau_1\tau_3=3$ and $\tau_2\tau_3=6$, so the first coefficient of $P(s)$ must be between 2 and 6, because this coefficient must be a convex linear combination of the two-factors products. This is not the case, because $P(s) = s^2 + s + 1$ and so, the function considered $G(s) = \dfrac{s^2 + s + 1}{(1+s)(1+2s)(1+3s)}$ does not correspond to a parallel of CSIR.

Even more, not all the convex linear combinations are allowed. For example $G(s) = \dfrac{3s^2 + 4s + 1}{(1+s)(1+2s)(1+3s)}$, seems to be a parallel of three CSIR, but it is not the case. After simplifying this quotient, it can be reduced to: $G(s) = \dfrac{1}{1+2s}$ (just one reactor with $\tau_2 = 2$).

Slight changes in the first coefficient can give: $G(s) = \dfrac{3.02s^2 + 4s + 1}{(1+s)(1+2s)(1+3s)}$, which corresponds to a parallel system with $f_1 = 0.01$, $\tau_1 = 1$, $f_2 = 0.98$, $\tau_2 = 2$ and $f_3 = 0.01$, $\tau_3 = 3$, or modifying again the first coefficient in the other sense $G(s) = \dfrac{2.98s^2 + 4s + 1}{(1+s)(1+2s)(1+3s)}$ which does not correspond to a parallel of CSIR (the last statement can be proved by a decomposition in simple fractions).

3. The mathematical education viewpoint

The examples already analysed (and/or others of the same kind) can be proposed as an interesting complement to traditional mixing problems. The typical mixing problem includes tanks, chemical solutions, fluxes, etc., and the main purpose is to write an O.D.E. linear system and to solve it. Taking into account the results and ideas of section 2, there are other possibilities, such as the following ones:
1.- Apply L.T. and consider the tank system as a chemical reactor (this idea was already exploited in [1]).
2.- Try to describe the reactor as a series and/or parallel of ideal reactors and ask the students if a solution exists, if it is unique and if it is stable under slight modifications (these questions were analysed in [2-4-5]).
3.- Ask the students to prove (using complete induction, simple fractions, etc.) theoretical results about series, parallels and combinations of both (examples of such theoretical results were shown in the previous section, but there are others that can be used).
4.- Finally, ask the students if given formulas (rational fractions for example) can be considered as the transference function of an ideal reactor or not. In the case of a positive answer, it can be asked if the reactor is the unique one and/or what happens if slight modifications in the coefficients are made.

These modifications, among others (for example, consider other kinds of reactor, recirculation, mix of series and parallels, etc.) can convert the typical mixing problem –which works more as an exercise for experienced students– in a real problem whose solution is not obvious.

The main idea is to substitute –or at least complement– routine exercises with real problems which imply an interesting challenge for engineering students.

4. Modelling and inverse problems in our courses

Modelling was introduced in U.D.E.L.A.R. differential equations courses for chemical engineering and related careers courses in 1996, and since then, mixing problems have appeared in the final examinations [1-5]. Inverse problems appeared in the assessment of this course, two years later in 1998. The questions had two different settings: firstly, tank dimensions and geometry were given, and students were asked to obtain an input for a desired output; and secondly, both input and output were given and the question was about what to put in the middle (i.e., how many tanks, which volumes and fluxes, what connections occurred between them, etc.). Finally, inverse-modelling issues were considered specifically since year 2005 [2-8-9], although inverse-modelling students' questions, appeared since the beginning of all this experience, in 1996 [5].

As it was mentioned before, all this teaching experience incorporated modelling, problem-solving and inverse-modelling. All of them were not just discussed in the classes, but played an important part of the assessment. This is a very important issue, for example Smith and Wood said that "…appropriate assessment methods are of major importance in encouraging students to adopt successful approaches to their learning. Changing teaching without due attention to assessment is not sufficient" [10].

5. Results

In several faculties (Engineering, Chemistry, Sciences, etc.) at U.D.E.L.A.R., the Education Department evaluates teachers, courses, assessment procedures (exams, etc.) by asking for students' opinion through questionnaires. The most typical questionnaire is composed of 25 questions using a Likert scale as follows:

Total agreement	10
Partial agreement	7.5
Indifference	5
Partial disagreement	2.5
Total disagreement	0

Table 1. *Likert scale used in the questionnaire.*

From this set of twenty-five questions, seven are particularly relevant and important to evaluate the inclusion of modelling and inverse modelling issues in differential equations courses. These questions are:

1) The pitch and pace of class can be followed by the students
2) The examples presented in the classroom illustrate the courses' main concepts.
3) Relationship with other subjects is established.
4) An applied approach is developed, giving examples and applications connected with real-life problems and professional practice.
5) Students are motivated in this course.

6) Students feel comfortable and enjoy classes.

7) Final exams and assessment problems and exercises can be solved using knowledge obtained in class.

Table 2 compares the average scores for the seven questions in both the innovative experience (group A, i.e., lecturers of the differential equations courses), with the other two traditional groups (group B and group C, corresponding to other mathematical courses of the same department).

	1	2	3	4	5	6	7
A	9.27	9.52	8.87	9.03	8.39	8.47	8.75
B	7.88	8.00	7.50	6.44	6.50	7.31	7.56
C	6.94	6.48	4.22	4.19	5.16	5.58	4.90

Table 2. *Comparison between "innovative" and "traditional" groups.*

Group A teachers' assessment can be compared in two different situations: before 1996, when the innovative experience started, and after 1996, i.e., after including modelling and inverse modelling problems. This comparison can be observed in the following average scores:

	1	2	3	4	5	6	7
Before 1996	8.45	8.87	8.20	7.61	8.63	8.57	-
After 1996	9.27	9.52	8.87	9.03	8.39	8.47	8.75

Table 3. *Group "A" teachers' assessment before and after including modelling and inverse modelling problems. Average scores.*

A comparison of the teaching style is possible before and after the innovative experience respectively, since teachers of the group “A” (i.e., four professors involved in the differential equations course) taught equivalent students both times (engineering courses, first O.D.E. course, etc.).

There are other issues that must be commented:

a) Question 7 was not included in the questionnaire before 1996, so, there is no result in the corresponding row in table 3.
b) The mean scores in questions 5 and 6 diminished slightly, this may be due to the dramatic increase in the number of students between 1993 and 1996.
c) It follows from Table 3, that the first four questions showed an important mean score increase. Moreover, if questions 1 to 6 are considered, there exists an average increase of 6.65 %, which is very important, because this improvement was obtained teaching the same syllabus plus an extra subject (chemical kinetics and mixing problems among other applications problems).
d) Teachers involved in this innovative experience achieved the best results in motivation (question 5).

All these results were reinforced by several students' comments about different topics as applications, motivation, etc. Several examples are the following sentences:

Student I: "I never thought that Maths had so many applications related with my career"
Student II: "Teachers help us in solving real-life problems, but they don't do all the task...we must work hard...this is the best way to learn"
Student III: "All Maths courses should be like this"

Another interesting consequence of this innovative approach was the improvement in comprehension when students had their second experience with this kind of problems in other subjects (Physical Chemistry, Design of Reactors, etc.) of their careers. We have remained in contact with many former students through other activities, and they continue to report that they are using skills and knowledge that they acquired trough our courses. This fact is in concordance with the experiences of other teachers in other parts of the world, when courses are based on problem solving activities [11].

In a previous paper, an expert group was consulted, and almost all the experts remarked the importance of teaching significant concepts and procedures in service courses. From a different point of view, engineering students showed an important preference for teachers who make the effort of presenting real-life problems, related with their own careers [12].

Finally, cluster analysis and other multivariate statistical methods showed a very similar situation [13]. More precisely, in our group of mathematical teachers (twelve teachers in the Mathematics Department), the cluster analysis of “applications” separate a group of them as the better ones. It is important to mention that this variable “applications” consists of a 12-components vector with the average results of two questions: the one related with real-life problems (question 4) and the other about the connection with other disciplines (question 3). The group with the best results (a group of five teachers) was integrated almost exclusively with differential equations’ teachers.

It is important to note that almost all of these teachers (the ones with better results in “applications”), participated in interdisciplinary work with researchers of other departments and laboratories. Moreover, two teachers of this group are researchers in Applied Mathematics.

From these comments and results, it is obvious that real applications produce positive reactions in engineering students, in concordance with experts’ opinions [12-14].

6. Conclusions

As it was mentioned in the previous section, an expert group was consulted about mathematics teaching and learning at the undergraduate level, focusing in the specific case of engineering careers [14]. Most remarked the importance of teaching significant concepts and procedures in service courses. On the other hand, chemistry and engineering students showed an important preference for teachers who make the effort of presenting real-life problems, related with their own careers [12-14].

When students of differential equations were consulted about these courses, there were positive reactions, when motivating examples are used to promote mathematical modelling and applications. Moreover, they enjoyed working together in project-work, trying to propose mathematical models and/or applying the different concepts, tools and techniques to solve them analytically or numerically [15].

The need for relevance was highlighted by many writers as being important in assisting students with learning mathematics. For example, Bajpai *et al.* [16] suggested a range of improvements including a modelling approach and providing more relevant examples. According to Wood *et al.* [17] 'To make a mathematics course seem relevant to engineering students – and hence worth an investment of time – the subject has to be made to seem valuable for their own specialization and future cases'.

Finally, Mc Alevey and Sullivan [18], asserted that there is a need for using real-life problems since, 'Students are best motivated by exposure to real applications, problems, cases and projects'.

Motivation is not the unique reason for introducing modelling and problem solving activities in engineering courses. For instance, it is possible to present a sophisticated mathematical tool or concept in a preliminary version, immersed in a motivating context, through real-life problems. To achieve this goal, these concepts and tools must be in the Zone of Proximal Development of the students [19], so, if the problem has a high degree of difficulty, it usually needs a Didactic Transposition [20], in order to convert the original problem in a suitable version for second year university students. This is exactly the situation in several chemical kinetics problems, while mixing problems can be introduced almost in their original versions, at least in differential equations courses, usually placed in the fourth semester (i.e., at the end of the second year). For this reason, mixing problems are used to illustrate several procedures, which are very common when operating with O.D.E. linear systems, such as diagonalization, changes of variables, etc.

Another important reason for including modelling, inverse modelling and problem solving is to expose the engineering students to other processes which are almost absent in other courses without these modelling and problem solving activities. In fact, as Cavallaro *et al.* mentioned "it is also important for an engineer's professional performance to make decisions and to make correct estimations. These estimations do not always follow the procedural characteristics that students have met in traditional courses" [21]. Chemical kinetics and mixing problems (among others) had been successfully to introduce this kind of procedural characteristics. For example, students are asked to estimate a priori the qualitative behaviour of a salt solution in a tank system and/or they are asked to predict which can be the internal geometry to get a desired result at the output. These estimations are compared with the final results and if there are differences between both (their *a priori* estimations and the *a posteriori* results), they are discussed in class. The following sequence: 1.- *A priori* estimation of the salt solution behaviour; 2.- Modelling or inverse modelling; 3.- Solve the O.D.E. linear system and 4.- Comparison with the *a posteriori* results; cannot be usually find in traditional courses.

Another aspect that must not be ignored is the type of assessment must reflect the teaching method of the topic. The evaluation process must not be dissociated from the style of teaching. So, if courses are have been instructed through problem-solving, modelling, etc., then assessment must be carried out to reflect this. This purpose can be put into practice through project-work, where students –with orientation of an interdisciplinary team of teachers and lecturers– try to solve real problems of their careers, in order to approve their mathematical courses [15]. Mixing and chemical kinetics problems are excellent sources for this purpose. Moreover, there exists an important set of real-life problems from these areas, which remain almost unexplored from the point of view of their mathematical education richness.

According to Blum and Niss in their classic paper about applications, modelling and applied problem solving [22], there are six different types of basic approaches to including relations to applicational areas in mathematics programmes. In our course, at the beginning (1966 to 2000), the "islands approach" was the selected one. In this approach, the mathematics programme is divided into several segments, each organized according to a two-compartment approach: a first part of a usual course in "pure" mathematics whereas the second one deals with one or more "applied" items, utilizing mathematics established in the first part or earlier. Gradually, the course changed to a "mixing approach", where elements of applications and modelling are invoked to assist the introduction of mathematical concepts and conversely, newly developed mathematical concepts, methods and results are activated towards applications and modelling situations whenever possible.

Finally, as it was suggested in the foreword of iJMEST special issue, devoted to Calafate Delta '07 Proceedings, 'Current students are presented with an enormous range of choices of courses and career pathways. In this competitive environment, the biggest challenge for mathematics departments around the world is to attract students to mathematics' [23]. In order to attract these students, it is important to note that for most engineering students, intrinsic motivation appears when it is felt that they are solving problems in which they have an interest. For example, Smith and Wood [24], said 'we feel that this is an effective way of introducing mathematical concepts to engineering students, whose main interest in mathematics is often limited to its usefulness in their future profession'. In the same direction, Brown *et al.* [25], argued that meaningful learning will only take place if it is embedded in the social and physical context within which it will be used, and if it involves what they call 'authentic activity' or 'ordinary practices of the culture'.

Searching for new real-life problems to be used in mathematical courses for engineering students, represents an interesting challenge for engineers, mathematicians and mathematical education researchers and, at the same time, it provides a good opportunity for interdisciplinary work in both research and teaching.

7. References

[1] Martinez Luaces, V., 2005, Engaging Secondary School and University Teachers in Modelling: Some Experiences in South American Countries, *International Journal of Mathematical Education in Science and Technology*, **36(2-3)**, 193-205.

[2] Martinez Luaces, V., 2007; Inverse-modelling problems in chemical engineering courses, *Proceedings of Calafate Delta '07, Sixth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning*. Pp 111-118. Montevideo: Ed. ISC-Delta, Uruguay.

[3] Westerterp, K.R., 1984; *Chemical Reactor Design and Operations*. New York: Ed. Wiley. U.S.A.

[4] Martinez Luaces, V., 2009; *Aplicaciones y Modelado: E.D.O., Transformación de Laplace y E.D.P.*, Montevideo: MATSER, Uruguay.

[5] Martinez Luaces, V., 2009; Modelling and Inverse-Modelling: Experiences with O.D.E. linear systems in Engineering Courses, *International Journal of Mathematical Education in Science and Technology*, **40 (2)**, 259-268.

[6] Groetsch, C.W., 2001; Inverse problems: the other two-thirds of the story, *Quaestiones Mathematicae,* Suppl. **1,** Supplement, 89-93, South Africa.

[7] Groetsch, C.W., 1999; *Inverse problems: activities for undergraduates*, Washington, D.C.: Mathematical Association of America**,** U.S.A.

[8] Martinez Luaces, V., 2008, Modelado Inverso y Transformada de Laplace en problemas de Diseño de Reactores Químicos, *C.D. Memorias de EMCI-14*, ISBN 13:978-950-42-0091-8. Mendoza: Ed. F.R.M.-U.T.N., Argentina.

[9] Martínez-Luaces, V., 2006, "E.D.O y sistemas de E.D.O.: Una experiencia de Molelado-Inverso" *C.D. Memorias de EMCI-13*. Misiones: Ed. Universidad de Obera, Argentina.

[10] Smith, G. and Wood, L., 2000, Assessment of learning in university mathematics. *International Journal of Mathematical Education in Science and Technology,* **31**, 125-132.

[11] McLeod, K. & Huinker, D., 2007, University of Winconsin-Milwaukee mathematics focus courses: mathematics content for elementary and middle grades teachers. *International Journal of Mathematical Education in Science and Technology*, **38**, 949-962.

[12] Martinez Luaces, V., 1998. Matemática como asignatura de servicio: algunas conclusiones basadas en una evaluación docente. *Números. Revista Española de Didáctica de Matemáticas*. **36**, 65-74.

[13] Gomez, A. & Martinez Luaces, V.; 2002, Evaluación docente utilizando análisis multivariado. *Acta Latinoamericana de Matemática Educativa*, **15**, 1016-1021.

[14] Martinez Luaces, V., 1998. Considerations about teachers for Mathematics as a service subject at the university. *Pre-proceedings of the ICMI Study Conference*. Singapore: NTU, pages 196-199.

[15] Martínez Luaces, V., 2006, Analytic solving of real P.D.E. problems applied to Mass Transfer: Experiences in project-work with Chemical Engineering and Food Technology Engineering students. *C.D. Proceedings of the 3rd International Conference on the Teaching of Mathematics at the Undergraduate Level*, ICTM-3, Turkey, ISBN 0471072709. Istanbul: ICTM-3 International Committee.

[16] Bajpai, A.C., Mustoe, L.R. and Walker, D., 1975, Mathematical education of engineers: Part I. A critical appraisal. *International Journal of Mathematical Education in Science and Technology,* **6**, 361-380.

[17] Wood, L., Mather, G., Kirkup, L. and Logan, P., 2003, Cross-disciplinary teaching of mathematics. In: D. Holton and I. Reilly (Eds).*Communications of the Fourth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning*, (Dunedin: International Delta Steering Committee), Pages 282-288.

[18] McAlevey, L.G. and Sullivan, J.C., 2001; Making statistics more effective for bussiness? *International Journal of Mathematical Education in Science and Technology*, **32**, 425-438.

[19] Wells, G.; 1999, *Dialogic inquiry: Towards a sociocultural practice and theory of Education.* New York: Cambridge Press.

[20] Chevallard, Y.; 1985, *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage.

[21] Cavallaro, M.I., Anaya, M., Garcia Argiz, E. and Aurucis, P., 2007, Estimative abilities in probabilistic judgments of engineering students. *International Journal of Mathematical Education in Science and Technology,* **38**, 917-925.

[22] Blum, W. and Niss, M., 1991, Applied mathematical problem solving, modelling, applications and links to other subjects: state, trades and issues in mathematics instruction. *Educational Studies in Mathematics,* **22.1**, 37-68.

[23] Martinez Luaces, V. & Varsavsky, C.; 2007, Foreword. Vision and change for a new century, *International Journal of Mathematical Education in Science and Technology*, **38-7**.

[24] Smith, G & Wood, L., 2001, Motivating Engineering students to study Mathematics, *Quaestiones Mathematicae,* Suppl. **1,** Supplement, 47-55.

[25] Brown, J.S., Collins, A. and Duguid, P., 1989, Situated cognition and the culture of learning, *Educational Researcher*, **18**, 32-42.

Modelling and inverse modelling with second order P.D.E. in Engineering courses

VICTOR MARTINEZ-LUACES
I.I.Q., Faculty of Engineering, University of the Republic of Uruguay.
Corresponding author: victorml@fing.edu.uy

Abstract: Partial Differential Equations (P.D.E.) are usually taught and learnt in South American countries engineering courses as a recipe where mechanical procedures and algebraic manipulations seem to be the only skills to be considered. In this paper, modelling and particularly inverse modelling, are utilised to give context to typical exercises trying to convert them in real-life problems. As a result, typical routine tasks and manipulations are changed for challenging problems that give place to deep understanding of concepts and procedures. Several examples are analysed here and changes in motivation and comprehension, among other outcomes, are commented from the Mathematical Education view point.

Keywords: Modelling and inverse modelling; Partial Differential Equations; Engineering courses; real-life problems

2000 Mathematics Subject Classifications: 97D50; 80A20; 34B05; 34B40

1. Introduction

In South American countries, P.D.E. are usually placed at the end of the Differential Equations courses for engineering careers. For this reason, among others, in most cases the part of the course devoted to P.D.E. reduces to just a couple of examples. This first one is the one-dimensional wave equation (a vibrant infinite string classical problem) where propagation method is exemplified and used to solve the problem. The second problem is the well known diffusion problem (usually named “heat equation problem”) also considered in the one dimension version, with “typical” boundary and initial conditions [1].

Due to this particular situation among the students there exists the recipe (personal communication) that if propagation method is not accurate (i.e., the problem to solve is not the typical vibrant string one), then, separation of variables, Fourier series, etc., must be applied. In this way, interesting problems are transformed into routine exercises where the only skills involved are the ability in performing algebraic manipulations and the expertise in following a routine recipe.

There exist other more sophisticated recipes well known among the students. For instance, when they solve the typical heat equation with null boundary conditions (B.C.) they get Fourier Sine Series [2]. On the other hand, if they had null derivative B.C., they know that the formal solution will be a Fourier Cosine Series [2].

Taking into account these recipes and mechanical techniques even the most interesting second order P.D.E. problem becomes an exercise, and so, there is nothing to assess except algebraic manipulations and routine tasks.

There are other disadvantages that must be analysed. For instance, the role of B.C. is not well understood if these conditions are just a small part of a routine. Algebraic manipulations do not illustrate how null B.C. or null derivatives B.C. can be applied when modelling real-world problems and/or how these conditions can modify the graphic of the final solution.

In this paper an alternative, and at same time, complementary approach, based on inverse problems [3-4] will be discussed and the results will be commented.

2. Inverse-problems in P.D.E. courses

Inverse problems and mathematical modelling can be combined and utilised in different parts of a Differential Equations course. For instance, when Ordinary Differential Equations (O.D.E.) linear systems are studied, chemical kinetics and/or mixing problems can be used for this purpose. Moreover, in other papers [5-6] it was studied which exercises can be converted in chemical kinetics and/or mixing problems and which ones cannot be used for this purpose. In a similar way, when Laplace Transform (L.T.) is the studied topic, mixing problems and/or chemical reactors design, under certain conditions [7-8] become a useful source of unusual and interesting problems.

In this paper we will analyse several examples, where simple exercises are substituted by inverse modelling problems to illustrate ideas and concepts related with second order P.D.E.

i) <u>One dimensional wave equation</u>:

A typical P.D.E. exercise is the following one:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & \text{P.D.E.} \\ u(x,0) = f(x) & \text{I.C.} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \text{I.C.} \end{cases} \quad \text{for an infinite string or}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & \text{P.D.E.} \\ u(0,t) = 0 & \text{B.C.} \\ u(L,t) = 0 & \text{B.C.} \\ u(x,0) = f(x) & \text{I.C.} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \text{I.C.} \end{cases} \quad \text{for a finite string which length is L}$$

The infinite string problem is usually solved using propagation method and the finite one is solved by separation of variables and Fourier series. Students usually learn how to apply these methods but they don't know which problems, if any, are they solving in the real world and they do not understand why they use one technique or the other.

The finite string exercise can be represented, in the real-world, by a string of a musical instrument (a guitar or a piano), and in these cases, the B.C. can be easily illustrated. Moreover, both B.C. ($u(0,t) = 0$ and $u(L,t) = 0$) are a consequence of the attached string (in the case of the piano) or fixed by the instrument player (in the case of the guitar). In a similar way, functions f(x) and g(x) can be easily interpreted as position and velocity profiles due to the initial perturbation in the string produced by the instrument player.

Students realise that if the sound corresponds to an A or a B in music, it depends only of the B.C. not of the I.C. The problem's I.C. will show, for example, the differences between a professional musician and an amateur guitar player.

In the infinite string exercise the string is not attached, so the initial perturbation is propagated both to right and left sides, with the same speed a, so the change of variables $\xi = x + at$, $\eta = x - at$ becomes obvious.

ii) One dimensional heat equation:

A typical P.D.E. exercise is the following one:
$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} & \text{P.D.E.} \\ u(-L,t) = 0 & \text{B.C.} \\ u(L,t) = 0 & \text{B.C.} \\ u(x,0) = f(x) & \text{I.C.} \end{cases}$$

Students are trained in Separation of Variables and Fourier series and they are able to arrive to the final solution, but almost none of them know a priori how to draw approximately the graphic of this solution.

It is possible to give a context to this exercise. This situation was analysed in a previous paper [9], so it will not be commented again, but just as a summary, the mass transfer problem of drying vegetables cut in slices can be adapted to this exercise and the final solution becomes intuitive, even for no-trained students.

iii) "Tendential behaviour" of solutions in bounded and unbounded conditions:

Tendential behaviour was studied by Cordero [10], among other Mexican researchers, when O.D.E. exercises are considered. The same ideas can be explored for P.D.E. problems and exercises. For instance, let's consider first a couple of parabolic P.D.E. exercises: in the first one there are no B.C. and in the second one, null derivatives B.C. are studied. In the first example, the solution tends to zero when $t \to +\infty$, while in the second the solution tends to a positive constant as can be observed in figures 1 and 2, respectively.

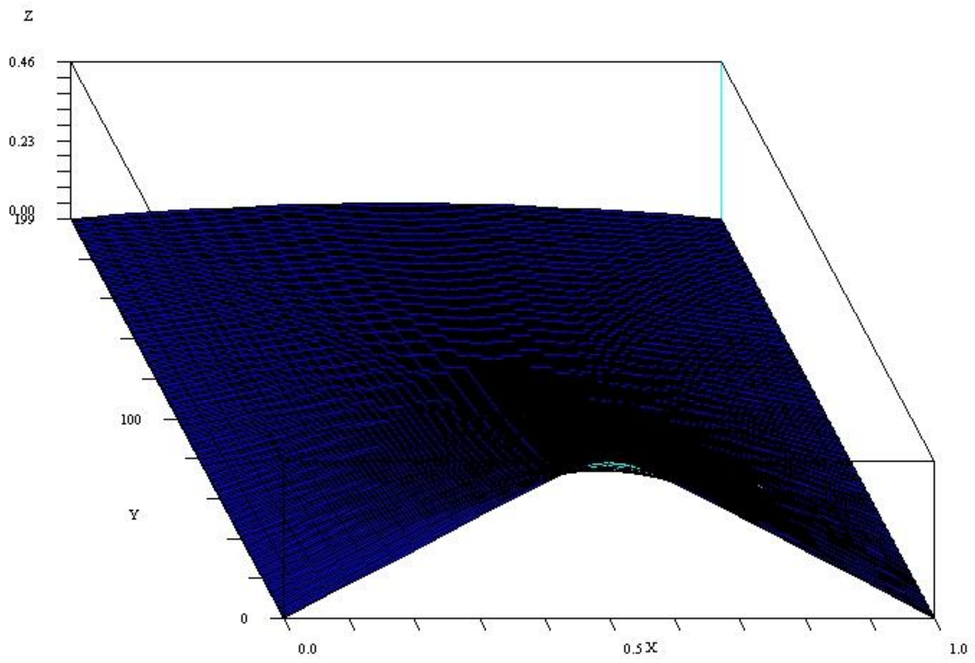


Figure 1 Solution of a parabolic P.D.E. problem without B.C.

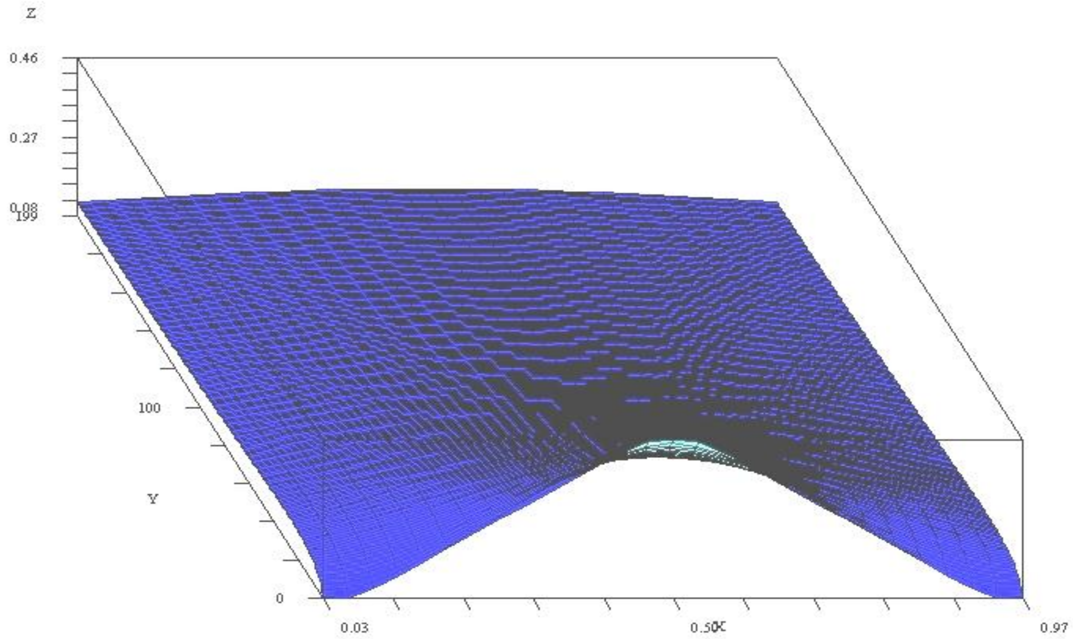


Figure 2. Solution of a parabolic P.D.E. problem with null derivatives B.C.

In both graphics, the independent variables x (position) and t (time) correspond to x-axis and y-axis respectively, while the function $u(x,t)$ was plotted in the z-axis. These figures show the horizontal and vertical diffusion of contaminants after Chernobyl nuclear accident [11]. If the exercises are substituted by these horizontal and vertical problems:

$$(P_H)\ \begin{cases} \dfrac{\partial u}{\partial t} = D_H \dfrac{\partial^2 u}{\partial x^2} & \text{P.D.E.} \\ \mathrm{u(x,0)} = Q_H.\delta(\mathrm{x}) & \text{I.C.} \end{cases} \quad \text{and}$$

$$(P_V)\ \begin{cases} \dfrac{\partial u}{\partial t} = D_V \dfrac{\partial^2 u}{\partial x^2} & \text{P.D.E.} \\ \dfrac{\partial \mathrm{u}}{\partial \mathrm{z}}(0,\mathrm{t}) = 0 & \text{B.C.} \\ \dfrac{\partial \mathrm{u}}{\partial \mathrm{z}}(\mathrm{H},\mathrm{t}) = 0 & \text{B.C.} \\ \mathrm{u(x,0)} = \mathrm{Q_V}.\delta(\mathrm{x}) & \text{I.C.} \end{cases}$$

then, their behaviour when $t \to +\infty$ can be easily understood, because in the horizontal problem the contaminants diffusion is unbounded, and so, it is logical to think that the concentration diminishes and tends to zero. In the vertical problem, contaminants diffusion is bounded between $z = 0$ and $z = H$ ($z = 0$ corresponds to ground level and $z = H$ corresponds to approximately 1000 ft where the thermal profile of atmosphere changes), and so, final concentration remains positive.

iv) <u>Transitory and steady-state solutions</u>:
It is well known and usually mentioned in P.D.E. courses that transitory and time-dependent solutions for diffusion problems (i.e., solutions of parabolic P.D.E.

problems) when $t \to +\infty$ tend to the steady-state solutions (i.e., the corresponding elliptic P.D.E. problem).

For the students, this idea can be more or less intuitive, but not easy to visualise. Real-world P.D.E. problems, carefully chosen can show clearly this behaviour of the solutions. An example is the electrode of diluted oxygen problem, already analysed in [11], where curves of the surface corresponding to the graphic of the solution, tend to a straight line as time tends to infinite as it can be observed in figure 3.

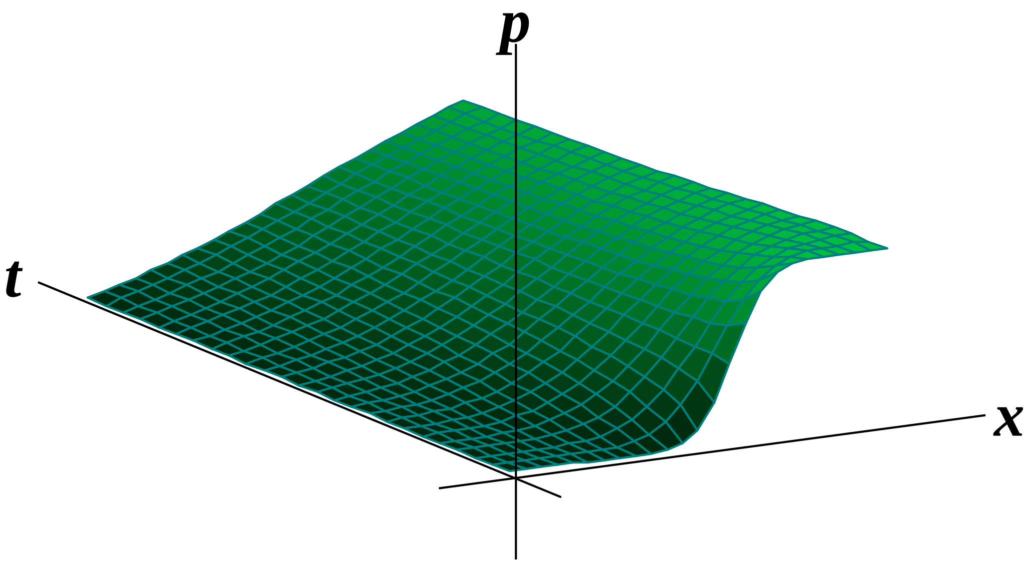


Figure 3. Graphic of the solution for the diluted oxygen electrode problem.

Other examples of how this "tendential behaviour" can be visualised were included and partially commented in [2]. The intention is to study these examples, among others, from the Mathematical Education view point in a subsequent paper.

v) <u>The role of the different parameters</u>:
In order to finish this section devoted to real-life P.D.E. problems, an interesting example corresponding to another mass transfer P.D.E. problem will be discussed in detail. In this case, the main purpose consists in studying the sugar concentration in a cherry immersed in a glucose/water solution and, as in the other examples, two independent variables are involved. The positional variable is r (radial position), because the problem is posed in spherical coordinates, due to the geometry of the fruit, and the other independent variable is t (time) as in all the other cases. If C represents sugar concentration, then Fick's law for this case takes this form:
$\frac{\partial C}{\partial t} = D_e \left(\frac{\partial C^2}{\partial r^2} + \frac{2}{r}\frac{\partial C}{\partial r} \right)$ where the right side of the equation is the product of D_e (diffusivity of glucose in the fruit) and $\frac{\partial^2 C}{\partial t^2} + \frac{2}{r}\frac{\partial C}{\partial r}$ which corresponds to the spherical-coordinates Laplacian.

It can be assumed that there is no diffusion at $r = 0$ (the centre of the sphere, i.e., the point at the middle of the cherry), for symmetry reasons. On the other side, at

$r = a$ (radius of the sphere), the equilibrium concentration C_e is reached instantaneously. These two logical assumptions give the boundary conditions for the P.D.E. problem.

Finally $C = C_0$ is the initial sugar concentration in the fruit before starting the mass transfer process.

Then, the P.D.E. problem takes this form:

$$\begin{cases} \dfrac{\partial C}{\partial t} = D_e \left(\dfrac{\partial^2 C}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial C}{\partial r} \right) & \forall r, 0 \le r \le a, \forall t > 0 \\ \dfrac{\partial C}{\partial r}(0,t) = 0 & \forall t > 0 \\ C(a,t) = C_e & \forall t > 0 \\ C(r,0) = C_0 & \forall r, 0 \le r \le a \end{cases}$$

The analytical solution is interesting because it uses a couple of non-trivial changes of variables. In the first one, C^* is defined as $C^* = C - C_e$ and so, C^* represents the difference between the concentration at time t and the "final" concentration when $t \to +\infty$. The second one is even more original, because a new variable $U = C^* r$ is defined in order to get null boundary conditions.

Using these changes of variables the analytical solution can be obtained by Separation of Variables method.

Nevertheless, the most interesting part for non-mathematical students consists in analysing the influence of the model parameters (a, D_e, C_0 and C_e) in the final solution. Intuitively, sugar concentration can be expected to grow from the initial value (C_0) to an asymptotic concentration C_e, theoretically reached at an infinite value of time (t). Then, C_0 will be at the bottom of the graphic and C_e will be at the upper limit of the surface. As a consequence $C_e - C_0$ will give the range in the vertical axis.

In the cherry, the radial position varies between $r = 0$ (at the middle of the fruit) and $r = a$ (in the fruit's surface), so, the radius a will give the width of the graphic.

Finally, the role of the diffusivity (D_e) is by far, the most interesting. In fact, this parameter works as a conductivity for the diffusion process and so, a big diffusivity means a low resistance to mass transfer and conversely, a small value of this parameter D_e will produce an important resistance, and so, sugar concentration takes much more time to grow. In the following figures (fig 4, 5 and 6) this behaviour can be observed easily.

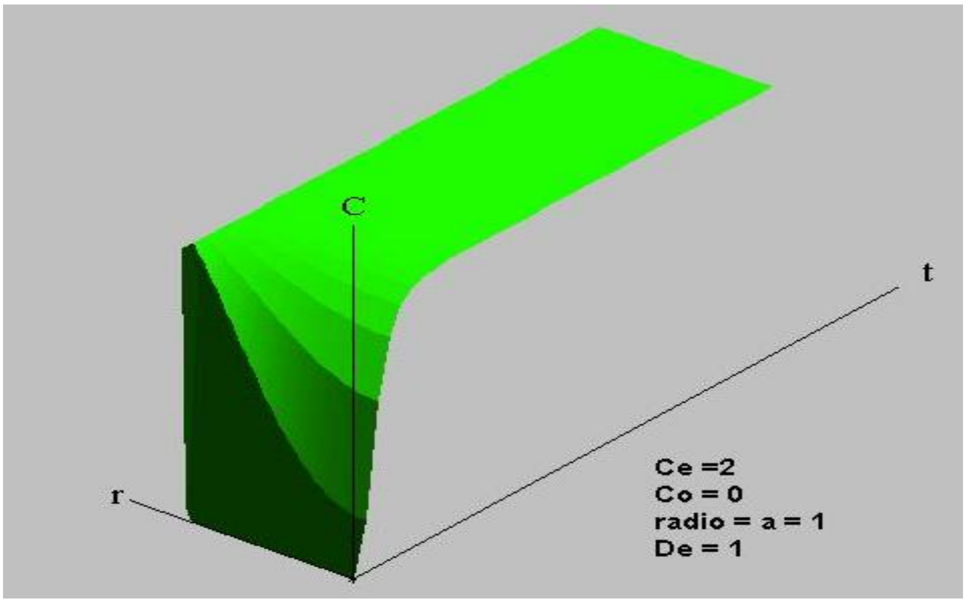


Figure 4. First graphic of the solution for the sugar diffusion problem.

In this first graphic corresponding to $C_e = 2$, $C_0 = 0$, $a = 1$ and $D_e = 1$, an almost linear behaviour can be observed at $t \approx 0^+$ and a fast growing curve is obtained at the right side of the surface (corresponding to $r = 0$). This is exactly the expected situation, because $D_e = 1$ is the greatest value used in these graphics.

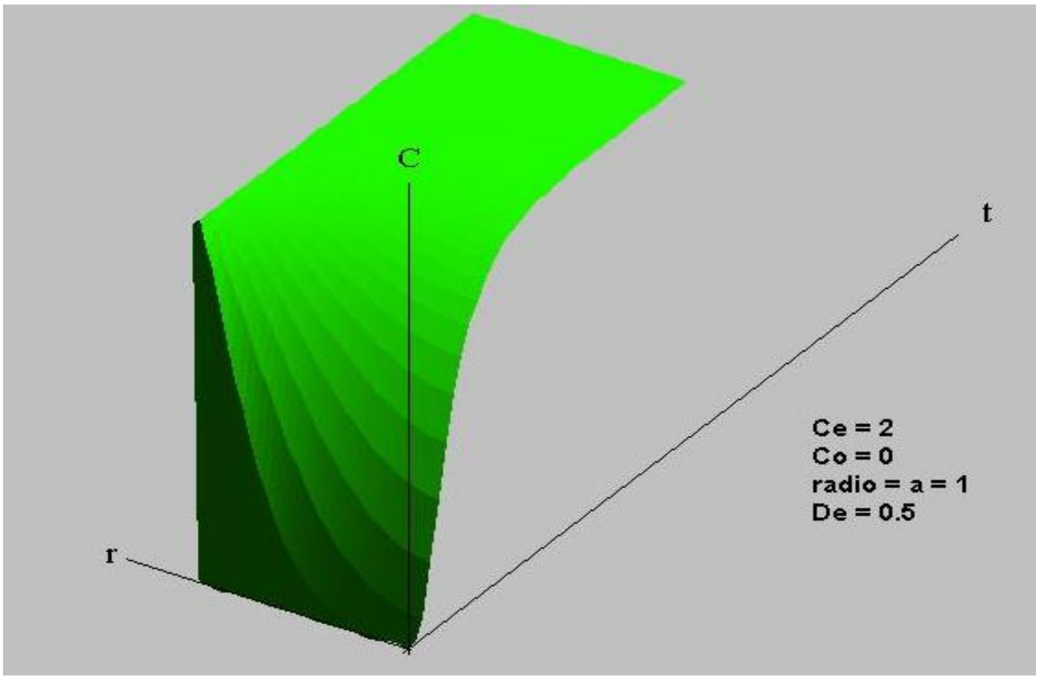


Figure 5. Second graphic of the solution for the diluted oxygen electrode problem.

In the second graphic, the diffusion coefficient was modified to a lower value ($D_e = 0.5$), while the others remained constant (i.e., $C_e = 2$, $C_0 = 0$ and $a = 1$). The growth of the curve corresponding to the middle of the fruit ($r = 0$) is not so fast as in the previous example and even more dramatic are the changes observed in the "almost initial curves" (corresponding to $t \approx 0^+$) where C remains C_0 for the lowest values of the radius r.

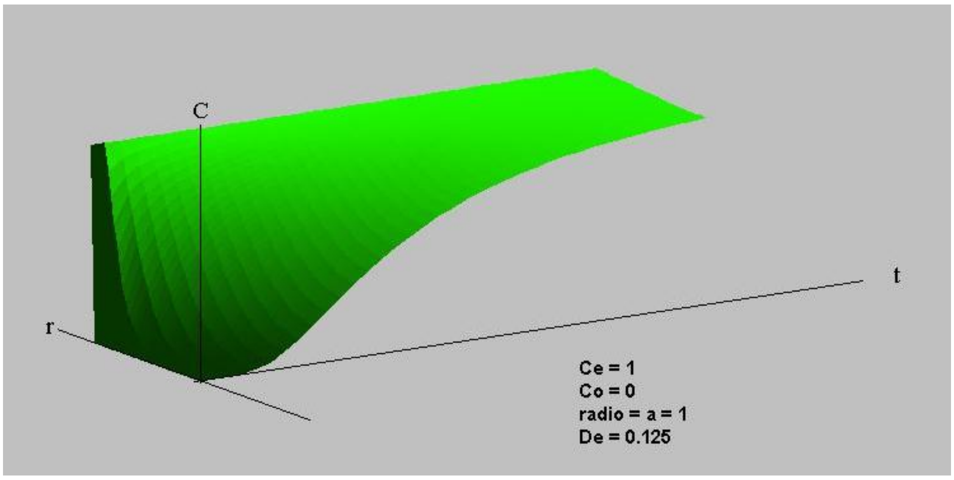


Figure 6. Third graphic of the solution for the diluted oxygen electrode problem.

In this third, last figure, diffusivity was changed to the lowest value considered here (i.e., $D_e = 0.125$). Both the "almost initial curves" ($t \approx 0^+$) and the "almost central" ones (i.e., $r \approx 0^+$) show very low concentrations. Moreover, the curve corresponding to $r = 0$ (middle of the fruit), grows very slowly towards the equilibrium concentration C_e.

After obtaining the analytical solution, students can get easily these graphics and many others, just using a standard computer and commercial or free software, in order to visualize the graphic of the solution from different angles. The following figure is an interesting example where time and radial position axes were changed.

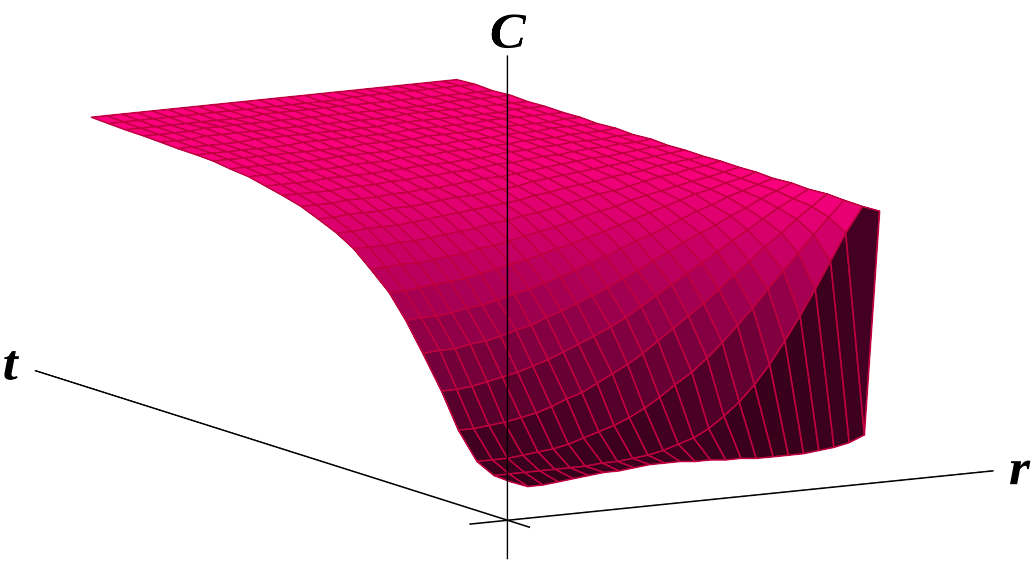


Figure 7. The graphic for the sugar diffusion problem from a different angle.

3. Results

This approach was put in practise in undergraduate and postgraduate courses in Chemistry and Engineering faculties at U.D.E.L.A.R., since 1996 and it showed interesting results in comprehension and motivation.

Students commented that they were able to understand deeply the role of B.C. and I.C. in P.D.E. problems and their influence in the final solutions, not just as a recipe (e.g., null B.C. implies Cosine Series, etc.) like in the "traditional" approach. Moreover, changing parameter values and visualizing solutions from different angles, allowed them tho get a deeper comprehension of these parameters and physical constants. For instance, sometimes the role of diffusivity is not well understood (or remembered), until they are able to change its value and see the results on the plotted solutions. In fact, graphics showed in figures 4, 5, 6 and 7, among others, were obtained by Enrique, a second year engineering student.

They also found useful the typical classification in parabolic, elliptic and hyperbolic second order P.D.E., but not for taxonomic reasons. In fact, they established a relationship between this classification and real-life problems, like these of Table 1.

P.D.E	Real-world problems
Parabolic	Heat transfer, mass transfer, other diffusion problems
Hyperbolic	Waves problems, vibrant strings, signal transmission
Elliptic	Steady-state problems corresponding to previous rows when $t \to +\infty$

Table 1. P.D.E. and real-life problems

Students found very useful this classification and the relationship with the real-life problems, because they know –and understand– which changes of variables can be used in a given problem and/or which kind of solution can be expected in each case.

Once again, they said that solving real-life problems instead of simple second order P.D.E. exercises, helped them to get a deep understanding. Moreover, if the main ideas, concepts, methods, etc. can be explained in a context given by real-world situations, they feel that their physical intuition works and really makes the difference with a purely mathematical exposition.

Considering the big amount of data produced since 1996, it is impossible to exhibit them with all the details, including the corresponding statistical methods used. For this reason, the most important results will be described here, and references will be given for the statistical analysis reports [12-13].

The first source of statistical data corresponds to the Mathematics Department teachers' assessment, carried out through anonymous questionnaires.

In the particular case of the Differential Equations courses, questions about illustrative examples, connections with other subjects, real life applications, etc., shown better results than other service courses. Moreover, differences between this course and others can be easyly visualized using Multivariate Analysis methods. In order to confirm the semi – quantitative impressions given by the Clusters Analysis, several statistical tests were carried out, including Contingency Tables, Kruskal-Wallis and Wilcoxon tests and Comparison of paired samples, among others. The statistical tests confirmed clearly the best results of the Differential Equations course, although it was considered by students as the most exigent course taught by the Mathematics Department. This opinion can be explained because final and partial examinations were based on real problems instead of routine exercises [12-13].

Another consequence of this success, was the invitations to teach other courses, like the following ones:

- The Chemical Engineering Institute proposed since 1997 a new postgraduate course to be taught by the Differential Equations teachers team. This new course was also positively evaluated by students.
- More than ten Permanent Education courses, with very good evaluations, were taught by the same teachers team since 1996.
- Several Differential Equations courses were taught in other countries, by the same teachers and lecturers; among them: "Differential Equations with Applications" (RELME XV, Buenos Aires, 2001), "Laplace Transform with applications" (EMCI XII, San Juan, 2005) and "Direct and inverse problems with Differential Equations and Laplace Transform" (EMCI XV, Tucuman, 2009).

Coming back to the students of the Differential Equations course, it is important to mention that they had an active participation during classes and even after the course itself. As an example several students and former students contributed with new problems to be included in following courses. Due to their potential richness, these problems were modified and after that, they were proposed in examinations. Moreover, several of these problems, in their original or modified versions, were included in the posters that won two first, and three second prizes in the Latin American Meeting on Mathematical Education (RELME in Spanish). Concretely, they won the best poster award in Cuba 2002 and Montevideo 2005 and the second one in Republica Dominicana 1999, Panama 2000 and Argentina 2001 [14].

Then, as a resume of these results, the Differential Equations course was the starting point for postgraduate courses, Permanent Education courses, research papers, international awards, etc. [14].

It is important to analyse what happened after putting in practice the new syllabus for chemical careers since year 2000. In order to get a deeper analysis of this transition, two different groups of results will be considered here: the first one is related to the "transition course" (in year 2000) and the second one involves other courses since 2001, after consolidation of the new syllabus.

In the "transition course", one hour and a half per week was devoted to applications and half an hour more was offered optionally, for practical classes and tutorials. At the end of this transition course, students gave their opinions in a semi-open questionnaire. The following sentences are just a selection of these opinions:

"Now, I found the utility of mathematics…"

"…the course was really good…it was a great contribution for my studies…"

"…an interesting course, with enough applications to real life…"

Particularly, applications were explicitly mentioned in other answers:

"…without applications, it would be just another maths course…it would be hard, just working with methods, calculations, numbers…"

These opinions agree with those expressed in year 1999, during the old syllabus course period. At that time, there was no questionnaire prepared to know about students' reactions to this course and so, they were asked to put their opinions in a sheet of paper, without any structure. The following sentences are just a few examples of their opinions:

"The course has a lot of applications to my career and it gives a new taste for maths…it is well taught…they (the teachers) guide the solution of exercises, without doing the whole task…"

" This course seems to me very useful and dynamic and I think it will be applied often in the following years."

Once again, about the "applications" issue, they said:

"…they encourage students since the utility of mathematics is observed in real life and

the course shows clearly the relationship with other subjects."

"…I had just done several courses in my career, where the same ideas considered here were applied…"

"…they are very useful for going on with my career…"

Finally, since year 2001, once established the new syllabus in chemical careers, a new questionnaire was designed for courses assessment. This new questionnaire used to have four questions about lectures (about the lecturer itself, the technical level, comprehension of classes, etc.), other four questions about tutorials and other classes for practise, laboratory, and solving exercises (with the same content as the lectures questions), and finally, four more questions about organisation and evaluation issues (didactic material, differences between classes and examinations contents, etc.).

It is important to mention that this new questionnaire was put in practise in five different courses simultaneously: Algebra, Analysis, Differential Equations, Physics II (Electromagnetism) and Probability and Statistics. In four of these courses the results were very good, and not so good in the other one. In the first four questions (those about lectures), the best scores were those corresponding to Differential Equations courses and in three of these four questions, showing important differences when comparing with the other courses. For administrative reasons, the "applications" classes were assessed in this four questions, together with lectures, and both, theoretical and applications classes, represented 60 % of the whole time (in weekly hours) assigned to this course. This fact gives more significance to the successful results corresponding to the first four questions. On the other hand, practical classes (laboratory classes, tutorials, etc.) also had very good scores as it was observed when processing the second group of four questions. Finally, results were not so successful in the last four questions (about organisation items), at least in years 2001 and 2002. This fact can be explained taking into account the inconveniences experienced as a result of changes due to new syllaburs in chemical careers, as it was already mentioned.

4. Conclusions

Mathematical textbooks for other educative levels, particularly in secondary school, really do an effort in order to give context to routine exercises and try hard to convert these exercises into real problems. In several courses in South American universities, it seems that the opposite way was chosen. In fact, rich and potentially motivating real-life problems where second order P.D.E. are widely used, seem to be substituted by routine exercises with poor educative results.

In this experience, the main idea was just substitute –or at least complement– routine exercises with real-life problems. These problems showed, for example, the reasons for a change of variables, or explained the "tendential behavior" of the solutions, among other results observed in our courses in the last twelve years. At same time, these problems resulted in an interesting challenge for students, with the consequent changes in their interest in the main topics of the P.D.E. part of the course.

Giving context to routine exercises it is not the final solution for the poor educative results observed in several Differential Equations courses, but may be the beginning of changes for many courses that fail in the purpose of motivate Engineering students.

5. References

[1] I.ME.R.L., 1992, *Análisis Matemático II*, Montevideo: Ed. CEI, Uruguay.

[2] Martínez Luaces, V., 2009; *Aplicaciones y Modelado: E.D.O., Transformación de Laplace y E.D.P.*, Montevideo: MATSER, Uruguay.

[3] Groetsch, C.W., 2001; Inverse problems: the other two-thirds of the story, *Quaestiones Mathematicae*, Suppl. **1,** Supplement, 89-93, South Africa.

[4] Groetsch, C.W., 1999; *Inverse problems: activities for undergraduates*, Washington, D.C.: Mathematical Association of America**,** U.S.A.

[5] Martinez Luaces, V., 2007; Inverse-modelling problems in chemical engineering courses, *Proceedings of Calafate Delta '07, Sixth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning*. Ed. ISC-Delta, Uruguay.

[6] Martínez Luaces, V., 2009; Modelling and Inverse-Modelling: Experiences with O.D.E. linear systems in Engineering Courses, *International Journal of Mathematical Education in Science and Technology*, **40 (2)**, 259-268.

[7] Martínez Luaces, V., 2005, Engaging Secondary School and University Teachers in Modelling: Some Experiences in South American Countries, *International Journal of Mathematical Education in Science and Technology*, **36(2-3)**, 193-205.

[8] Martínez-Luaces, V., 2008, "Modelado inverso y Transformada de Laplace en problemas de diseño de reactores químicos" *C.D. Memorias de EMCI-14*. Ed. UTN-FRM. Mendoza. Argentina.

[9] Martínez-Luaces, V. & Guineo-Cobs, G., 2005, P.D.E. real-life problems: visualization and intuitive approach, *Proceedings of the Fifth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning (Delta 05).* ISBN 1-86499-840-7. Australia. Pp. 238-245.

[10] Cordero, F., 1998, El entendimiento de algunas categorías del conocimiento del Cálculo y el Análisis: el caso del comportamiento tendencial de las funciones, *Revista Latinoamericana de Investigación en Matemática Educativa*, **1(1)**, 56-74.

[11] Martínez Luaces, V., 2003, Mass transfer: the other half of parabolic P.D.E., *New Zealand Journal of Mathematics*, **Vol. 32**, Supplementary issue, 125-133.

[12] Gómez, A., Guineo Cobs, G., Martínez Luaces, V., 2002. "Análisis Multivariado: una metodología para procesar encuestas sobre calidad de la enseñanza en carreras de Ingeniería", página web de *Memorias de X EMCI*, *http://ing.unne.edu.ar/emci*.

[13] Gómez, A. y Martínez Luaces, V., 2002. "Evaluación Docente utilizando Análisis Multivariado". *Acta Latinoamericana de Matemática Educativa* **XV.II** pp1016 - 1021

[14] Martínez Luaces, V., Guineo Cobs, G., 2002. "Un trabajo de Investigación-Acción en un curso de Ecuaciones Diferenciales para Ingeniería de Alimentos", página web de *Memorias de X EMCI, http://ing.unne.edu.ar/emci*.

Using History to Enrich Undergraduate Mathematics: Beyond the Anecdote

CLEMENCY MONTELLE[††††††††††] AND KATHLEEN M CLARK

University of Canterbury, New Zealand

Abstract: The value of using excerpts from the history of mathematics to enrich and consolidate undergraduate calculus classes as well as motivate and enthuse students has long been recognized. From the casual anecdote to dedicated modules which let students grapple with actual primary-source mathematical documents, there are many ways in which the history of mathematics can enrich existing curricula. This paper summarizes the particular practice of using mathematical anecdotes in classrooms and lectures, as well as present some practical ideas for ensuring they have both educational impact as well as their entertainment value.

Keywords: History of Mathematics, Undergraduate Mathematics, Enrichment, Mathematical Anecdote.

1. Introduction

The pedagogical advantages of using aspects of the history of mathematics in the teaching of mathematics is well-known. There are extensive resources both documenting and validating the benefits of this practice, as well as those which offer ideas, strategies, methods, modules, and lesson plans for its implementation. (See, for example, the many studies in Fauvel [1], Fauvel and van Maanen [2], Jankvist [3], and Katz [4].) However, even with an abundance of resources and the best intentions on the part of teachers, time is usually limited, and general unfamiliarity can lead to a certain hesitation for fully embracing this opportunity. More realistically, perhaps the most common way that teachers actually include history of mathematics into their classroom is through the simple anecdote. About to cover arithmetic sequences and series? Recount the amazing insight of the seven-year old Gauss. Ready to teach calculus? Detail the feuding Newton and Leibniz. Covering noncommutative algebras? Recall the story of Hamilton and the quarternions inscribed on the Broome Bridge. Who hasn't added some spice into their lectures by relishing in the sibling rivalry of the Bernoulli brothers, the unrequited love and swashbuckling foolhardiness of the 21-year old Galois, the tragic end of the talented Hypatia of Alexandria murdered by religious fanatics, or else sparked some interest between theorems by means of the mathematician of the day—a birthday, a death, a breakthrough, a publication, a fracas. Scholars and students alike love anecdotes—they are the equivalent of intellectual gossip.

[††††††††††] E-mail: c.montelle@math.canterbury.ac.nz

2. Pedagogical implications: the role of anecdotes

However the use of anecdotes is somewhat problematic on a number of levels. While these tales may be momentarily entertaining students, there are some deeper implications arising from their presence. As Siu [5] observed, one of the issues with anecdotes is that when recounting them we usually disregard our concerns with authenticity and genuineness. This is somewhat strange for mathematicians who are by profession committed to precision, accuracy, proof, and integrity. Anecdotes are frequently partially fabricated, typically embellished, more often than not apocryphal, tales at best, rumors at worst. Their authenticity is tenuous. Could you be being subliminally counter-productive or unfaithful to your discipline when you promulgate these apocrypha?

As historians of mathematics we do have some serious reservations about the anecdote as a resource, as well as concern over their historicity and relevance. But we do recognize their value in the lecture theatre. Every collective has its common tales. Recounting stories about our heroes and champions is a time-honored tradition in any community. Furthermore, anecdotes have the potential to be inspiring as well as instructional; entertaining as well as educational. Mathematics is no exception. Modern mathematical mythologies are cherished, repeated, and actively preserved in the oral tradition of academics. Their importance has necessitated their compilation into large compendia, such as Eves [6], Krantz [7], Littlewood [8] to name but a few. Some academics have even contributed to collecting and recounting their own personal anecdotes; the colorful and ever curious Richard Feynman epitomizes this [9].

Not all anecdotes are equal. Some are shamelessly risqué. Some are marginalizing. Some are racist, sexist, or stereotyping. Some can be demoralizing. Some seem too good to be true—and there is probably a very good reason for that. Others though are exclusive insights into the profession, are inspiring, are educational, illuminating, and instructive. They portray mathematicians as endearing, resourceful, witty, brave, even exotic. They can bear testimony to a field populated with scholars endowed with intelligence of a singular kind.

Beyond their immediate colorful narrative, though, almost always lie further important insights. The best anecdotes often contain subtleties beyond their face value. Some have a metonymous significance over and above their primary effect. With this in mind, we offer strategies to turn the most simple, well-told anecdote into a message about the richness and diversity of mathematics. We argue that even the driest pithy aphorism can be spun into a pointed and directed reflection about the mathematical craft, with a bit of extra perspective. Mathematical gossip can be turned into mathematical gifts. With this in mind, we also offer some direction on how to weave anecdotes which are edifying, and entertaining as well as mathematically relevant. We seek to instill an awareness in you as educators about the details and features of this genre, so you can use and apply it more mindfully.

3. The anecdote as a literary genre

The anecdote as a literary genre falls in domain of folklore. Taylor [10, p.223] notes that the literary community, despite their obvious importance, numerousness, and presence in everyday oral culture, has largely neglected anecdotes. Defining an anecdote is a somewhat difficult task, however for the most part they are short concise narratives which are circulated predominantly orally to convey a single incident, usually quirky, out of the ordinary, or unusual, about a real person, an actual event, or an item. They are realistic in their intentions—they allege to reveal what actually occurred. It is this very aspect which gives them their impact. They are typically anonymous and rarely contain a reference to the actual original source. They are seen as common property to a community. Taylor [10, p. 224] relays: "Anecdotes are no one's property…They are an aspect of folklore". Generally, they contain an element of humor or wit.

Further formal features of this genre can be identified. Commonly, there are two protagonists, one is typically the star, the other, the foil. Because of this, Bauman [11, p. 55] noticed that "anecdotes also tend to be heavily dialogic in construction, often culminating in a kind of punch line, a striking, especially reportable statement rendered in direct discourse…quoted stylistic speech is a significant stylistic feature of the genre". This reported speech component is foundational as well as fascinating. It forms the very focus of the narrative. But more broadly, this quoted speech component is contextualizing and humanizing. It opens up sociolinguistic dimensions as well as caricatures the main protagonist. As a result, all the elements of the anecdote are arranged with this reported speech element. Typically, those anecdotes which have quoted speech as their dénouement stay more stable over time, than those that do not. In this way the reported speech provides a tag for the remembering and transmission.

Beyond their entertainment value though, anecdotes are in some way intended to be instructive. They embody deeper functional and generic considerations. Many convey a metaphorical force or a metonymic meaning [11, p. 76; 12, pp. 34-5]. That is they exemplify an insidious problem, a conflict, a vulnerability, a weakness, a common failing or oversight. They may often suggest a strategy (successful or not) for dealing with the situations they portray. They may function as a cautionary tale.

Mathematicians love anecdotes. Their prominence and pervasiveness in this discipline has the potential to contribute to the study of the anecdote in broader literary fields, but this has not been done so far. Significantly, mathematical anecdotes are of a special quality. Not only do they pick out the human quality of an incident and its significance, but they make telling remarks or insights into the field of mathematics. They rely on a basic (sometimes expert) knowledge of mathematics. They are directed largely at a specific audience—a generally mathematically literate audience—and they typically have little effect on those not acquainted with the field. In this way, they are designed with a target subpopulation in mind. Generally the more mathematically specific an anecdote is, the more satisfaction it confers upon the receivers.

For example, the following anecdote is lost on those who don't have the relevant mathematical background or an appreciation for the pressures of academic oral assessments:

> Herman Amandus Schwarz (1843 – 1921)…who was noted for his preciseness, would start an oral examination at the University of Berlin as follows:
> Schwarz: Tell me the general equation of fifth degree.
> Student: $x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$
> Schwarz: Wrong!
> Student: …where *e* is not the base of the natural logarithms.
> Schwarz: Wrong!
> Student: …where *e* is not *necessarily* the base of the natural logarithms. [5, p. 4]

In a similar sentiment is this anecdote, which also gives a nice example of the importance of the final reported speech component as essential to the impact:

> Shizuo Kakutani is one of the great experts on Brownian motion. He understands better than most that Brownian motion in two dimensions is generically recurrent: motion beginning at a point *P* has positive probability of returning to *P*; but in three dimensions this is false. He likes to describe the situation by saying, "A drunken man will usually find his way home. A drunken bird has no hope." (This story is also attributed to George Pólya.) [7, p. 76]

Another aspect of the special nature of mathematical anecdote is that they can anchor salient mathematical observations in human-centered events. Constance Reid in Krantz [7, p. x] comments about the particular nature of mathematical anecdotes: "I have always thought of mathematical anecdotes as pretty much unique to mathematicians. The best ones, for me, are those that encapsulate a mathematician's character or personality with all the economy of a formula". For example,

> Luitzen Egbertus Jan Brouwer's (1881 – 1966) life was one punctuated with conflicts. As well as doing fundamental work in logic and intuitionistic mathematics, he was a committed solipsist. The tenacity with which he held this position was somewhat ironic as his ultimate conflict was that with a car, which tragically killed him. So much for that philosophical stance, where anything beyond you is an illusion. [Personal Collection]

This example neatly combines three factors—solipsism, a logician, and a car accident—to convey a deep irony about the tension between an intellectual idealism and the reality of everyday life.

Some anecdotes work through the introduction by reference to the problematic, mathematically loaded attribute that will make for the focal conflict of the story. For example, a comment on the difficulty and perhaps futility of the many attempts to prove a (then) outstanding intractable mathematical proposition—Fermat's Last Theorem—is neatly captured in the following anecdote:

> The analyst Edmund Landau kept a printed form in his office for handling "proofs" of Fermat's last theorem that came in over the transom. It read "On page ____, lines ____ to _____, you will find

> that there is a mistake." Actually *finding* the error was a task that fell to a Privat Dozent. [7, p. 64]

The spirit of the anecdote is to delight. Inherent in its form is that it can be repeated. It is in this way self sustaining: short enough to remember—amusing enough to be repeatable. More importantly the recounting of these narratives educates emerging mathematicians with the particularities of mathematical culture, the preferences of the community, and their values. It directly conveys how mathematicians practice. Where a large portion of mathematical practice is related by means of imitation, anecdotes form an important part of collective identity. Anecdotes can be a sort of pat-on-the-back for mathematicians and how they operate. They can also reveal the limitations of a mathematician, as well as their brilliance. The humility expressed in the following is humbling and reveals a charming affection between teacher and student!

> The speed with which John von Neumann could think was unnerving. G. Polya once admitted that, "Johnny was the only student I was ever afraid of". [Personal Collection]

For the mathematical community, anecdotes can relay the sorts of social hazards common to the profession. By sharing experiences which may reveal a more widespread issue, a sense of belonging is encouraged. For example, many mathematicians will attest to feeling similar to the sentiments in this anecdote at some time in their career:

> Richard Bellman (1920 – 1984) was quite a distinguished mathematician in his day, the founder of *The Journal of Mathematical Analysis and Its Applications*. He liked to say that he got tired of responding to the social question, "And what do you do for a living?" by saying he was a mathematician. This usually got a tiresome response, or no response at all. So he often told people that he was a tennis coach. At parties, this proved to be a much more salubrious social touch. [7, p. 7]

Anecdotes can be particularly useful for mellowing relations between professionals and students:

> G H Hardy (1877 – 1947) and J E Littlewood (1885 – 1977) discussed the concept of stage fright. They agreed that, for a lecture in front of the Royal Society, or a lecture at a foreign university, stage fright was not a problem. You knew what you were talking about, you were a ranking expert, you were among equals, and you could get up and strut your stuff. But in front of a calculus class, first lecture of the Fall term, there was definitely stage fright. [7, p. 3]

They can also poke fun while validating the particular nature of teaching mathematics and acquiring mathematical knowledge. The following anecdote gives a funny account of assessment, but also reaffirms the various challenges in the subject for students:

> My friend Ed Dunne (b. 1958), now at the AMS, was a graduate student at Harvard. One semester he was a TA, and part of his duties was to write certain exams. One of the exams was in a subject with which Ed was not entirely conversant. So he went to the math library, where a large book is kept that contains old exams. Ed passed through it looking for ideas. Along the way he came across an old algebraic geometry exam written by expert David Mumford (1937 –). Dunne was suitably awed. Mumford is a great genius, holder of the Fields Medal and the MacArthur Prize, and admired by all. Ed paused to see what sort of algebraic geometry exam Mumford might write. It had just two questions. They read:
> 1. Write an exam for this course
> 2. Take it
>
> It is rumored that the Harvard philosophy department has a similar exam. It has a third question:
> 3. Grade it [7, p. 79]

Mathematical anecdotes can reveal certain preconceptions of the field. The following reveals what mathematicians consider 'modern':

> One day two Cambridge mathematics professors were discussing an impending policy change in the math department. Finally, out of some frustration, the first savant said, "But we have been doing it this way for the past 400 years!" "Quite so," said the second professor, "but don't you think that the last 400 years have been rather exceptional?" [7, p. 46]

Another form of mathematical anecdote that is special to the discipline of mathematics is what we will call the biographical anecdote. In any other context, this form of anecdote would fail in its power. There would seemingly be little point for telling it. These types of anecdotes reveal a tragedy, a troubling episode, a bizarre behavior, an affliction—bordering on cruel to recount. But under the scrutiny of mathematicians, it becomes a reason for admiration. It demarcates that individual as being singular, and thus confers an ever greater respect on them. It recognizes that the brilliance and depth of what the protagonist suffers is a result of pushing the very limits of human intelligence which guarantees certain idiosyncrasies. For example, what sort of collective celebrates mental imbalance?

> It cannot be complete coincidence that several outstanding logicians of the twentieth century found shelter in asylums at some time in their lives. Cantor, Zermelo, Gödel, Peano, and Post are some. Alonzo Church was one of the saner among them, though in some ways his behavior must be classified as strange, even by mathematicians' standards. [Personal Collection]

Or what of this depressing story about Gödel:

> Gödel was always quite phobic, and something of a hypochondriac. He was sometimes committed to institutions for depression and exhaustion. He avoided human contact—handshakes and other shows

> of affection. He could be seen, at tea and other social occasions, weaving though the crowd in a strange dance designed to avoid touching other people. Gödel died in 1978; he was convinced that people were trying to poison him, and he starved himself to death. [7, p. 20]

Or Heaviside's particular proclivities to furnishings and appearances:

> It would be a mistake to think that the honours that Heaviside received gave him happiness in the last part of his life. On the contrary he seemed to become more and more bitter as the years went by. In 1909 Heaviside moved to Torquay where he showed increasing evidence of a persecution complex. His neighbours related stories of Heaviside as a strange and embittered hermit who replaced his furniture with: ... *granite blocks which stood about in the bare rooms like the furnishings of some Neolithic giant. Through those fantastic rooms he wandered, growing dirtier and dirtier, and more and more unkempt - with one exception. His nails were always exquisitely manicured, and painted a glistening cherry pink.* [13]

These stories could be seen as rather nasty to recount, but for scholars they have the opposite effect: they in fact convey something of the qualities of genius and the destabilising effects of such gifts. It is a validation of peculiarities, rather than a criticism of them.

4. Anecdote examples

Given these various illustrating qualities mathematical anecdotes possess, they can be powerful in the learning environment. However, they can be made all the more powerful by a few simple additional observations. Their attention-grabbing qualities can be further exploited as an opportunity to give more background to students. With the audience captured, you can deliver more profound insights and make more evident the subtext of the messages the anecdotes contain.

As Meaghan Morris [14, p. 5] observes, "Anecdotes work to make contact and catch peoples attention, although they can fail in their nudging, insinuating mission". Within any anecdote there is a setting, a context, and a deeper principle being alluded to. Anecdotes can have a powerful import if these are made explicit. Sometimes their point may be lost if these aren't emphasized, as Morris suggests. These additional points are generally best addressed in the forms of questions. The following is a selection of time-loved anecdotes with some exemplary additional elements that can be pointed out to make these excerpts both entertaining as well as instructive.

Opportunities to relay the details behind the discovery of the so-called Pythagorean theorem will be plentiful:

> The Pythagorean theorem is …attributed to Hippasus of Tarentum, a renegade Pythagorean whom, according to one account, Pythagoras pushed off a boat for revealing to outsiders the tragic secret of the

> Pythagorean Theorem, which was irrationality or incommensurability. [15, p. 85]

The veracity of this anecdote is highly contentious, however there are many relevant points to be added to this tale of unlucky Hippasus. Themes to emphasize include the following: the seriousness for these individuals by this breach between mathematics and geometry—that lines can be constructed geometrically but not described arithmetically was an ideological challenge for the Pythagoreans, for whom mathematics was fundamental to their basic world view; the ways in which mathematicians respond to challenges to their established theories; how mathematics progresses and the effect that this discovery had on mathematics.

Another clever anecdote which brings to light some historical as well as philosophical mathematical themes is the following:

> Charles Babbage (1791 – 1871)…and the Analytical Society wished to remedy the severe situation into which the English mathematicians had worked themselves following the earlier bitter controversy between Newton and Leibniz over priority of the discovery of the calculus….While on the continent the mathematicians were using Leibniz's much more fluent differential notation *dy/dx* for the derivative, the English mathematicians were clinging to Newton's far less fortunate fluxional notation $\dot{y}$ for the derivative. Accordingly, in Babbage's humorous words, the Analytical Society advocated "the principles of pure *d*-ism as opposed to the *dot*-age of the university". [6, p. 276]

Here of course, the anecdote's effect relies on the connection between Newton's dot-notation for the derivative and the term dotage (a state of senility) and Leibniz's differential notation and 'd-ism', or more properly deism, (the state of godliness), and as a result Babbage's obvious preference for one notation over the other. The richness in the interactions between Newton and Leibniz is well known—anecdotes related to this episode show that mathematics is rarely linear in its development; the reception of an idea is not just because of its mathematical merits, people play a huge role in which ideas get accepted over others. Importantly, in this quote is the question of mathematical notation: what role does mathematical notation play in facilitating insights? Are some notations better than others? Why have some notations persisted and others ceased to be used? What of seeming parallel insights? Should one mathematician be accorded preeminence over another?

There has been much recent attention for the so-called Fermat's last theorem:

> In his personal copy of Diophantus' *Arithmetica,* Fermat (in about 1637) inscribed his so-called 'Last Theorem' in the margin next to the sum-of-squares problem but did not give details of the proof, claiming a lack of space!: *Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.* [Personal Collection]

Fermat's last theorem is a great example of the sort of international attention mathematics can generate (albeit rarely!). Other points worth emphasizing include the way in which it gripped the mathematical community for an extended period of time, the mathematical phenomenon of great problems which remain intractable for many, many years and the drawing power that they have on mathematicians to tackle them, and the inspiration mathematicians find in reading mathematical texts of times past. Another interesting contrast with contemporary mathematical practice is the rhetorical nature of Fermat's statement of this theorem. This is mathematics before its practitioners were fully conversant with exponential and other sorts of notation. Mathematical statements, which we would capture with highly refined symbolic notation, were expressed using long descriptive sentences.

Pi has always captivated mathematicians, largely because of its irrationality:

> Daniel Tammet (b. 1979), author of "Born on a blue day", has Asperger's syndrome. He recited the first 22,514 digits of pi in 5 hours and 9 minutes by memorizing visual numerical landscapes. [Personal Collection]

Is mathematics somehow hard-wired into our brain? How do our brains conceive of mathematical objects and processes? What can these prodigious achievements reveal about the human capacity to calculate and conceptualize numbers?

One frequently told anecdote concerns the great Greek philosopher Plato (427 – 347 BCE) and the phrase that was inscribed above the main entrance door to his famous school, the Academy. It read:

> *ageómetrétos mēdeis eisitō*

This phrase is perhaps the mathematical equivalent of Chinese whispers, given the number of versions that are circulated! It has been variously translated as:

> Let no one enter who is ignorant of geometry
> Let no one who is ungeometrical enter
> Let no one destitute of geometry enter my doors
> Let no one who is not a geometer enter
> Let no one un-versed in geometry enter
> Let no man ignorant of geometry enter here
> Let no one enter who is lacking in geometry

As well as being useful to consider the status and importance of mathematics for ancient philosophers, and the high ambitions Plato held for mathematics as being the paradigm for wisdom, this example raises some technical issues about the medium of mathematical expression. Are these statements all equivalent? Is translation interpretation, even in mathematical contexts? How well do we know the early mathematical cultures when we rely so much on the interpretation of a translator?

The following anecdote about Descartes hints at some important points about the generation of mathematical insight:

> …the initial flash of analytical geometry came to Descartes when he was watching a fly crawl about on the ceiling near the corner of his room. It struck him that the path of the fly on the ceiling could be described if only one knew the relation connecting the fly's distances from the two adjacent walls. [6, p. 174]

Mathematics is usually seen as abstract or disconnected from reality, but there are some powerful examples of mathematical inspiration from experiences in the real world. What is the role of everyday experiences in capturing the imagination of mathematicians? Are mathematical ideas independent of experience, or are they contingent upon it? Is mathematics *a priori* or does it require a reference to reality to be valid?

These are but a few of the many additional observations that can enrich the power and impact of an anecdote. The anecdote becomes no longer the focus of discussion, but rather an inspiration to cover some deeper relevant themes that are illuminating and instructive of our mathematical culture and its history.

5. Conclusion

Academics love to tell stories. However anecdotes are seemingly counter to many of the principles we are trying to instill in our students—such as authenticity, precision, and exactness. For the historian, they pose even more of a dilemma, they are unverifiable and often spurious references. Many of the favorite anecdotes about great individuals would never survive scrutiny from the probing of historians. However, they will continue to be used by the scholarly community. Therefore, with a bit of additional attention and contextualizing as we suggest in this paper, anecdotes become opportunities to be instructive as well as entertaining, providing vital glimpses into the wider context of the mathematical community.

6. References

[1] Fauvel, J. (1991) Using history in mathematics education. *For the Learning of Mathematics* **11**: 2, pp. 3 – 6.

[2] Fauvel, J., and van Maanen, J. (eds.) (2000) *History in Mathematics Education: The ICMI Study,* Kluwer, Dordrecht, The Netherlands.

[3] Jankvist, U. T. (2009) A categorization of the "whys" and "hows" of using history in mathematics education. *Educ. Stud. Math.* **71**: 3, pp. 236 – 261.

[4] Katz, V. (ed.) (2000) *Using History to Teach Mathematics An International Perspective, MAA Note #51,* Mathematical Association of America, Washington, DC.

[5] Siu, M.-K. (2000) The ABCD of using history of mathematics in the (undergraduate) classroom. In V. Katz, (ed.) *Using History to Teach*

Mathematics An International Perspective, *MAA Note #51* (pp. 3 – 9), Mathematical Association of America, Washington, DC.
[6] Eves, H. W. (1969) *In Mathematical Circles: A Selection of Mathematical Stories and Anecdotes*, Prindle Weber and Schmidt, Boston, MA.
[7] Krantz, S. G. (2002) *Mathematical Apocrypha: Stories and Anecdotes of Mathematicians and the Mathematical*, Mathematical Association of America, Washington, DC.
[8] Littlewood, J. E. (1953) *A Mathematician's Miscellany*, Methuen, London.
[9] Feynman, R. P., Leighton, R., and Hutchings, E. (1989) *Surely You're jJoking, Mr Feynman!" Adventures of a Curious Character*, Bantam Books, Toronto/New York.
[10] Taylor, A. (1970) The anecdote: A neglected genre. In J. Mandel and B. A. Rosenberg (eds.) *Medieval Literature and Folklore Studies* (pp. 223 – 228), Rutgers University Press, New Brunswick, NJ.
[11] Bauman, R. (1986) *Story, Performance, and Event: Contextual Studies in Oral Narrative,* Cambridge University Press, Cambridge, UK.
[12] Stewart, S. (1982) The epistemology of the horror story, *J Am Folk.* **95**, pp. 33 – 50.
[13] O'Connor, J. J., and Robertson, E. F. (2003) Oliver Heaviside. *The MacTutor History of Mathematics archive*, http://www-history.mcs.st-andrews.ac.uk/ Mathematicians/Heaviside.html.
[14] Morris, M. (2006) *Identity Anecdotes: Tradition and Media Culture*, Sage Publications, London.
[15] McEvilley, T. (2002) *The Shape of Ancient Thought: Comparative Studies in Greek and Indian Philosophies,* Allworth Press, New York.

Teachers' learning during mathematics content based professional development: Results and a model of processes and outcomes

JUDY PATERSON‡‡‡‡‡‡‡‡‡

Department of Mathematics, The University of Auckland,

Auckland, New Zealand

Abstract: This paper reports results from a study of secondary teachers' engagement in a two-year professional development intervention. A model of processes and outcomes derived from evidence in data obtained during the intervention is presented. Four key behaviours were identified: engagement in mathematics; realisations about teaching; re-energisation for teaching; and openness to change. Implications of the model for professional development are discussed. It is suggested that empathy and energy play key roles in the consideration and enactment of change of practice. The study was part of a larger initiative involving senior mathematics teachers from low socio-economic schools in Auckland, New Zealand.

Keywords: change, growth, professional development, mathematics content

1. Introduction

This paper reports on findings from a group of secondary mathematics teachers' engagement in a series of professional development learning opportunities. In each encounter the teachers were placed in the position of being learners of mathematics. It was hypothesised that these opportunities to learn might encourage the teachers to re-view their own teaching and to consider changes in their pedagogy. Fullan and Hargreaves [1] contend that change is an integral part of professional growth and professional teachers are the key to effective teaching. In addition, most educational reforms have a poor record of success when measured against the criterion of classroom impact and teacher change or growth [2,3]. Since it is acknowledged that teachers are often resistant to change [4-7] and have few opportunities to discuss their pedagogy [8] finding means to encourage consideration of change and to promote discussion of pedagogy are crucial in professional development programmes.

The structure of the professional development opportunities drew on three strands of research: firstly the provision of some kind of rationale or catalyst to engage in professional learning [9]; secondly, the creation of a forum within which teachers can engage with new ideas and challenge existing assumptions and beliefs [10-12]; and thirdly, that the teachers be challenged in an academic field in which they had previously experienced success [13,14].

The catalyst for the perturbation was exposure to unfamiliar tertiary level mathematics content [15-17]. The encounters aimed to perturb teachers' existing views of teaching and

‡‡‡‡‡‡‡‡‡ E-mail: j.paterson@auckland.ac.nz

learning and to provide opportunities for discussion of pedagogy. The forum for discussion was developed in a series of workshops within a larger professional development initiative. The teachers were in a non- threatening domain and had multiple opportunities to learn new ideas in mathematics. This combination of circumstances was thought likely to encourage the teachers to begin to consider the possibility of change, to erode the "hardened husk of habituated practice" [17, p1]. Examination of the ways in which teachers responded to the various stimuli prompted the development of the model of outcomes and processes at the end of the paper.

The study had as an aim the documentation and analysis of teacher engagement in the learning experiences, examining whether, as teachers became aware of their mathematical learner selves, this impacted on their view of teaching. Viewing professional development as learning is a comparatively recent development. The language shift to speaking of "professional learning opportunities" reflects this change. Fishman [18] asked, "How do teachers learn?" and contends that this question is necessary if we are ultimately to influence teacher knowledge, which he sees as a core goal of instructional reform effort. Robinson [19] observed that, "It is likely that the innovators of the 1970's did not fully appreciate the nature of change as a learning process" (p 273) and Lerman [20] reminds us that what is usually described as teacher development is in fact teacher learning, a view that is supported by Clarke and Peter: "the process by which teachers change their practices and their knowledge and beliefs about the teacher's role and about their subject is fundamentally a learning process" [21, p. 167]. Lerman further contends that researchers in this area need to acknowledge this and to make their underlying learning theories explicit to become more effective.

2. The situation of the study.

The study was part of a larger initiative working with senior mathematics teachers from eleven schools in a low socio-economic area in Auckland, New Zealand [22]. The teachers in these schools are enigmas: They are successes in that they survive in teaching circumstances that would drive many to leave the profession but their teaching is unsuccessful in that their students do not succeed against the benchmarks society has constructed. In discussions before and during the project it became clear that they attributed their students' lack of success to a number of factors over which they had little or no control such as systemic issues or community constraints. They never raised the question of considering their current teaching practice or reviewing their pedagogies. Indeed the school structure and departmental ethos does not usually lead to an examination of personal pedagogies and consideration of the role that approaches to teaching might play in the students' success. Generally the teachers are teaching as they were taught.

3. Methodology.

My interest in this study and my beliefs grew out of my experiences in secondary teaching and teacher education. As a pragmatic empiricist [23] I was working on what Peshkin [24] calls problem finding. However, in research we need to seek beliefs that have been generated through rigorous enquiry and that are likely to be true, seeking what Dewey referred to as 'warranted assertability' and a post-positivistic philosophy of science offers the best hope of achieving this goal [25]. Post-positivists argue that, since human knowledge is conjectural, the best a researcher can hope to achieve is to provide conjectural knowledge [26]. Conjectures need to be supported by the strongest, albeit possibly imperfect, warrants available and are always subject to reconsideration or refutation in the light of new evidence.

The experimental framework of the study is an instrumental case study located in this post-positivist paradigm. It draws on Pawson and Tilley's [27] realist synthesis approach to attribution of causality. The case is the entire group of participants who attended the professional development workshops. In particular it is the teachers' experiences as learners and their responses to those experiences that are the object of study. These are clearly within the boundaries of the case yet they are impacted on by external factors. For example, the teachers' personal histories are beyond the bounds temporally yet they often determine a teacher's participation. The context of the study is constituted by a multitude of factors: the teachers work in low socio-economic schools, bring different personal histories with them, and the mathematical stimuli varied. In this investigation the effect of an intervention is being examined. A realist approach invokes a generative approach to causation; namely that it is not the programme that works but the underlying reasons or resources that they offer subjects that generate change within a particular context [28]. The approach taken in this study was judged against the criteria formulated by Healy and Perry [29].

4. Data Collection.

Over a period of 2 years teacher talk was recorded at a total of eleven workshops where a total of thirty-one teachers encountered mathematics with which they were not familiar. As reported in Paterson [16] the talks presented by practicing mathematicians and statisticians ranged from the pure mathematics topics like the degree diameter problem to modeling clam populations and the eradication of rats from gulf islands: the proviso was that the topic needed to be accessible but unfamiliar to teachers who had some tertiary mathematics experience. The study was part of a larger professional development project. A core group of fifteen teachers attended five or more of the eleven workshops.

The teachers listened to the talks and then discussed the ideas presented. In addition, they responded to a range of prompts and questions about learning and teaching, individually and in group discussions. This focus on the conversations between teachers draws on the approach used in studies of children's learning in which classroom discourse was examined [30-32] and examinations of teacher dialogues for evidence of their changing understandings of students' thinking [33,34].

All the teachers' oral and written communications during all the phases of each intervention were recorded, transcribed and analysed by the researcher. Selected teachers were interviewed at the end of the study. Observations were triangulated during discussion with other researchers in the team. Practical limitations to the study precluded collection of data from within the teachers' classrooms, other than that which was self-reported during the recorded discussions.

Table 1 provides a glossary of terms used in the discussion of results that follows.

Table 1
Terms used in the Model of Processes and Outcomes

Term	Definition
Intervention	The presentations of unfamiliar mathematics; prompts, questions and discussions that followed
Engagement:	Involved in discussion related to the presentations.
Stimulation	To become interested or excited about the mathematics, increased mathematics-related activity
Re-energised for teaching	An increase in the ability or power to work or make an effort (in relation to teaching) – described by the teachers as a release of energy, a "re-fueling."
Supportive learning community	A group of people who learn together and who feel safe to share ideas about their practice,
Introspection	Mental self-examination of feelings or thoughts about learning mathematics
Identification	A feeling of affinity with another person/s, in this case ones students.
Realisation about teaching	Recognition of the import of what has been realised about self and students as learners for ones teaching.

5. Summary of results.

The subject of this paper is a particular aspect of the findings from the study: the model that emerged and the data from two participants that are used to illustrate this model of teacher behaviour during the professional development encounter. It is useful however to briefly describe the general results and the categorization typology that was developed during the analysis of the data o provide a background to the specific results presented.

In 2007 the data were analysed against a theoretical categorisation developed during the study [17]. This categorisation developed a hierarchy of responses from engagement with the mathematical ideas, responses about teaching and learning, making connections between their experiences and their students' experiences, generalised discussion of teaching practice to an indication of changing viewpoint or reporting of change in practice.

Measured against this categorization thirteen teachers (42%) showed evidence of opening up to the possibility of changing their teaching practice. This figure of 42% may underrate the level of success since, of the teachers who did not show evidence of enacting or considering change, twelve attended three or fewer workshops. It is important to note that all of the teachers who showed evidence of considering or enacting change in their practice also showed evidence of becoming aware of themselves as learners and expressed an empathy with their students as learners. This supported the initial hypothesis that enabling teachers to become aware of their mathematical learner selves is an important phase in change and growth as a professional.

In Paterson [16] reference was made to a number of themes that were beginning to emerge from the data – amongst these were the release of energy when mathematically stimulated and teachers' empathy with their students. While the empathy had been part of the original hypothesis and categorisation the release of energy had not. These are two key components of the model that emerged from the data.

We will now look deeper into the data, at the layers beneath, focusing on those recurrent themes in the teachers' responses. This allows identification of critical constructs in the process of consideration of change and of mechanisms connecting contexts with outcomes. This model grew out of my reading of the data as I attempted to look inside the 'black box' [35] to try to explain how and why the teachers responded to the intervention as they did.

As stated in the methodology the aim of a realist approach is identification of the reasons that a progamme offers subjects to generate change [28].

I will present evidence from two of the teachers who, while they both reported changing their practice, responded in very different ways. I will use their evidence to show that, while the particular nature of teachers' responses will vary, the processes and outcomes may be generalized. The two teachers, Frank and Fiona, were very different from each other both in demeanor and in their response to the intervention. The other teachers for whom evidence of considering or enacting change was found all behaved differently and it is more informative to look in detail at two participants rather than aggregating the data. A detailed examination of their responses serves to elaborate the structure of the model. Table 1 defines the terms as they are used in Figure 1 on page the next page and in the discussion of the teachers' responses.

5.1 Model of processes and outcomes.

The processes and outcomes evidenced in the teacher talk are represented as three strands in a model that links engagement in learning to openness to change in Figure 1 at the end of the paper. These strands emerged from the deeper examination of the teacher talk. The processes identified are: *engagement, introspection, identification* and *discussion*. The outcomes are: *realisation about teaching, mathematical stimulation* and *re-energisation for teaching.*

In Figure 1 (at the end of the article) the *realisation about teaching* strand in connects the processes of engagement, introspection and identification to the outcome realisation about teaching. The *re-energised for teaching* strand connects engagement to the outcomes of mathematical stimulation to the outcome, 're-energised for teaching'. These two strands are the focus of discussion.

The importance of facilitating the *discussion of teaching* strand is recognized, as is the need to create a supportive community [17], but these are not discussed in detail in this paper.

5.2 Teacher talk evidence.

Frank and Fiona have both been teaching for ten years but they are very different from each other, as are their responses to the learning opportunities. Frank has a degree in mathematics and came directly from university to teaching. Fiona completed her degree with a science major after having children. Frank is definite in his opinions and articulate. The most noticeable thing about Fiona was the way her confidence grew over the two years. In addition, Fiona was initially cynical about the effectiveness of professional development.

This evidence of *engagement*, the process that forms the first step in all three strands, is plentiful and unequivocal in the spoken teacher talk. It was found in evidence from 23 of the teachers.

This was true of both Frank and Fiona. Frank appeared very self-assured and was always very engaged and outspoken in discussions. He was prepared to discuss and challenge other members of the group he was sitting with: *Isn't it to do with the space really, not the knots?* While Fiona was initially more tentative, she too took an active part in discussions: *You've got 1,4 already. So this one must go to there?*

Frank's awareness of his own learning, the process of introspection, revealed a fear of being exposed. In this quote he articulates how he identifies with his students:

> *Coming to these sessions in groups – it's made me look at what my students feel when they're in the classroom. Initially the feeling is fear ... For me learning here, and for them learning from me as well. Because anything that's unknown is scary.*

These processes led to his realisation about teaching, namely, that he needed to "*... adapt to our kids, haven't we…often we're trying to teach what we're used to, not what the kids are used to."*

His engagement also led to the second strand in the model: he had been stimulated and his interest in mathematics had been re-kindled: *One of the things I found this year about these talks and stuff is that I think it's kind of enthused me a bit. Into areas of mathematics I have done but I've lost.* This had led on to an awareness that he could make changes in his teaching. In his interview, when asked how the project had impacted on him as a teacher, he showed a very big shift in perspective, from closed and 'blinkered' to allowing new ideas in*:*

> *I wouldn't say changed me as a person … I guess the thing that's kind of changed me is accepting other ideas…I'm headstrong… I sometimes dig my feet in a bit too much and I'm like very blinkered in the way I do things.* He continued*: But the key way that it's made me realise that as a learner – that we're still learning from the students as well…. And I think that's what's made me change a bit.*

As Fiona engaged with the mathematics she realized that she found it impossible to listen and write notes at the same time: *As soon as I start writing stuff down I'm not following what she's saying any more. I do have to be really careful. When I am writing something down I make it really short.* A related insight was the importance of paying attention: *and you're talking to the person beside you about the weekend …you're not giving the subject your proper attention. You're not going to learn it properly.* This awareness of her own needs as a learner (introspection), and her identification with students who had the same difficulties, led to her realizing she should change her teaching practice.

Eighteen months into the study she talked about the way in which her approach to teaching had changed. A researcher, who had spent a few lessons in her class, said to her: *I've noticed in your class that you discourage them from writing while you are talking and you give them time to write notes after you finished talking.* She had also changed where she stood during discussions with the class, moving to join them in examining ideas and concepts written on the board. She said she was more relaxed with her students. It is not possible to establish the exact role of the intervention and project in the development of this and her increased confidence however its development was very clear to all the researchers in the project.

Again, like Frank she refers to being inspired to want to do things differently and attributes her freedom to teach differently to a difference in focus, and a realization of the need for a learner to be interested in the topic, for it to be relevant. She explained how she came to the point of feeling able to teach in different ways, for example by using newspaper-based data:

> *I think the teacher's got to be inspired to do it. It's got to be something you want to do. You can't make yourself make them do it. It's got to be something that inspires everybody. And because I've been trying not to get too stressed, I'm free to do it, be a little bit creative. Too busy getting it all done getting the notes all done, you don't do it.*

She said her new focus enables her to be less stressed and that this has enabled her to be more creative and to find approaches to topics that interest her students. Her re-kindling of interest in mathematics led to her winning a scholarship to return to university the next year.

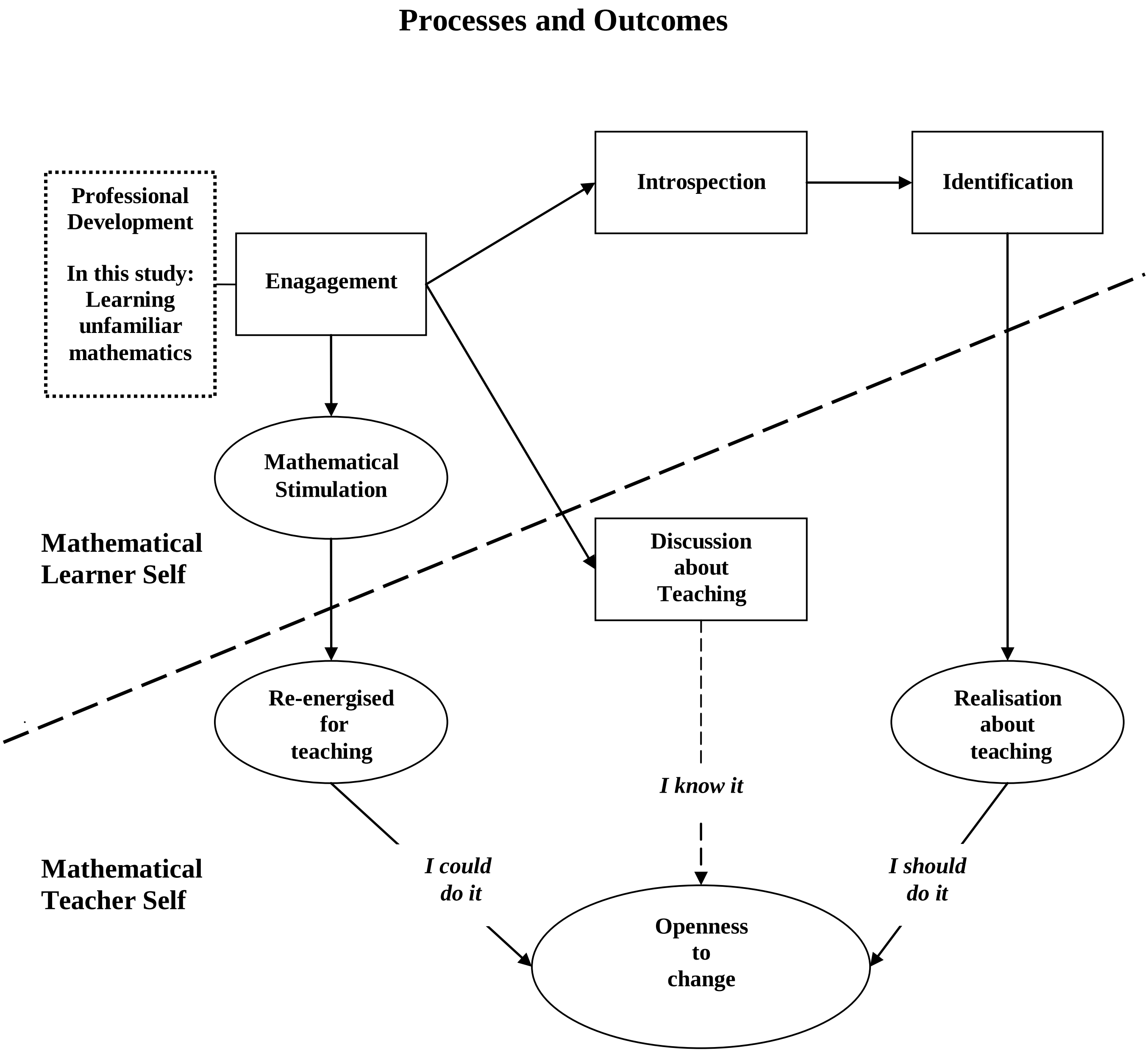
6. Conclusion and Implications.

In the introduction the hypothesis that opportunities to re-visit the experience of being learners of mathematics might encourage the teachers to re-view their own teaching and to consider changes in their pedagogy was put forward. The data from the encounters provided evidence that clearly shows that the processes of engagement, introspection, identification and discussion have led to the outcomes of a number of the participant teachers being mathematically stimulated, re-energised for, and coming to realisations about, teaching. Evidence from Frank and Fiona was presented to elaborate this claim. It is clear that they have come to a point where they decided that they could and should make changes to their teaching practice. The challenge for those who work with teachers to encourage growth and change is to provide opportunities that promote these processes.

From the evidence it is clear that energy plays a key role in enabling the enaction of change. The mechanism that releases the energy is, in this case, I would posit, the teachers' intellectual re-engagement with mathematical thinking. The contexts in which this was able to 'fire' are complex. Multiple factors impacted on teachers' engagement. The teacher's own mathematical knowledge and mind-set are likely to play a role. The data in this study show that re-energising for teaching is a key strand in the consideration and enactment of change. It leads to teachers saying, "I could do it' and considering change in their practice.

The effect of the processes of introspection and identification was to bring teachers to a point where they realised things about their teaching that they would like to change, to thinking "I should do it ….' There could be two bases for this sort of thinking. First, a realisation that aspects of their teaching would be ineffective if they were the learners and consequently may well be ineffective for their students and second, that factors that have made the learning experience effective for them are likely to do the same for their students.

Figure 1


Processes and Outcomes
Professional Development
In this study: Learning unfamiliar mathematics
Enagagement
Introspection
Identification
Mathematical Stimulation
Mathematical Learner Self
Discussion about Teaching
Re-energised for teaching
Realisation about teaching
I know it
Mathematical Teacher Self
I could do it
I should do it
Openness to change

7. **References**

[1] M. Fullan, and A. Hargreaves, *Teacher Change: Improving K-12 Mathematics.* Eisenhower National Clearinghouse, Columbus, OH, 1999.

[2] T.R. Guskey, *Professional Development and Teacher Change.* Teachers and Teaching: theory and practice, 8(2002), pp. 381-391.

[3] M. Kennedy, (1999). *Form and substance in mathematics and science professional development.* National Institute for Science Education Brief, 3(1999), pp. 1-7.

[4] A. F. F. Polettini, *Mathematics teaching life histories in the study of teachers' perceptions of change*, Teaching and Teacher Education. 16 (2000), pp. 765-783.

[5] C. McBeath, *Curriculum dissemination: A problematic issue in educational change*, Australian and New Zealand Journal of Vocational Education Research. 5(1997), pp. 37-55.

[6] J. Adler, *Lessons From and In Curriculum Reform Across Contexts,* (2005) Available at jwilson.coe.uga.edu/DEPT/ TME/issues/v12n2/Adler/Adler.pd

[7] J. Buchanan, and M. Khamis, (1999). *Teacher renewal, peer observations and the pursuit of best practice*, Issues in Educational Research. 9(1999), pp. 1-14.

[8] D. Ball, *Developing mathematics reform: What don't we know about teacher learning - but would make a good working hypothesis?* Paper presented at the Teacher Enhancement in Mathematics K-6, Arlington, VA, 1994.

[9] H. Timperley, A. Wilson, H. Barrar and I. Fung, I, (2007). *Best Evidence Synthesis on Professional Learning and Development.* Wellington: New Zealand Ministry of Education, 2007.

[10] H. Borko, V. Mayfield, S. Marion, R. Flexer, and K. Cumbo, (1997). *Teachers' developing ideas and practices about mathematics performance assessment: successes, stumbling blocks and implications for professional development.* Teaching and Teacher Education, 13(1997), pp. 259-278.

[11] J. Pennell, and W.A. Firestone, (1996). *Changing classroom practices through teacher networks: Matching Program Features with teacher characteristics and circumstances.* Teachers College Record, 98(1996), pp. 46-76.

[12] A. Selden, and J. Selden, (1996). *Of what does mathematical knowledge consist?* Available from from http://www.maa.org/t_and_l/sampler/research_sampler.html

[13] M.L. Maehr, *Paul Pintrich: A Once and Continuing Influence.* Educational Psychologist, 40(2005), pp. 129 -133.

[14] P.R. Pintrich, and D.H. Schunk, *Motivation in Education: Theory, research, and applications*, Merrill, Columbus, OH, 1996.

[15] J.Paterson, *Tertiary Mathematics input: Can it play a role in prompting teachers' thinking about teaching?* Fourth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning, Remarkable Delta:03, Queenstown, New Zealand, 2003.

[16] J. Paterson, J, (2005) *Using Mathematics to open up windows in teachers' minds: Encouraging teacher talk about learning and teaching.* Fifth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning, Kingfisher:05, Fraser Island , Australia, 2005.

[17] J. Paterson, J (2007) *An instrumental case study of a professional development intervention that uses unfamiliar mathematics to prompt secondary teachers' re-thinking about learning and teaching*, Unpublished PhD thesis, University of Auckland, 2007.

[18] B.J. Fishman, (2002). *What Do Teachers Need to Know*? Available from http://prospectassoc.com/NSF/fishman.htm

[19] I. Robinson, *The empowerment paradigm for the professional development of teachers of mathematics*, Deakin University Press, Geelong, 1989.

[20] S. Lerman, *The psychology of mathematics teachers' learning: in search of theory.* Paper presented at the 21st Conference of the International Group for the Psychology of Mathematics Education, Lahti, Finland, 1997.

[21] D.J. Clarke, and A. Peter, (1993). *Modelling Teacher Change.* In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.), Proceedings of the 16th annual conference of the Mathematics Education Research Group of Australasia. Kelvin Grove, Queensland, pp. 167-175, 1993.

[22] H. Bartholomew, B. Barton, B. Kensington-Miller, and J. Paterson, *Mathematics Teacher Development in Low Socio-Economic Areas.* Paper presented at the ICMI 15, Brazil, 2005.

[23] J. Mason, and A. Waywood, (1996). *The Role of Theory in Mathematics Education and Research*. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), International Handbook of Mathematics Education , pp. 1055-1089, Netherlands: Kluwer Academic Publishers.

[24] A. Peshkin, *The Nature of Interpretation in Qualitative Research,* Educational Researcher, 29(2000), pp. 5-9.

[25] D.C. Phillips, and N. Burbules, *Postpositivism and Edcuational Research*. Rowman & Littlefield Publishers, Inc., Lanham, Maryland, 2000.

[26] R. Ackermann, (1976). *The Philosophy of Karl Popper*, University of Massachusetts Press Amherst, 1976.

[27] R. Pawson, and N. Tilley, (1997). *Realistic Evaluation,* Sage, London, 1997.

[28] R. Pawson, *Evidence-based policy: The promise of 'Realist Synthesis'*, Evaluation, 8(2002), pp. 340-358.

[29] M. Healy, and C. Perry, *Comprehensive criteria to judge validity and reliability of qualitative research within the realism paradigm,* Qualitative Market Research: An International Journal, 3(2000), pp. 118-126.

[30] L.M. Anderson, *What Are Students Doing When They Do All That Seatwork.* In C. W. Fisher & D. Berliner (Eds.), Perspectives on Instructional time. Longman, New York, 1985.

[31] P. Cobb, T. Wood, T.,and E. Yaeckel, *Classrooms: a learning environment for teachers and researchers.* In R. B. Davis, C. A. Maher & N. Noddings (Eds.), Constructivist views on the teaching and learning of Mathematics, NCTM, Reston, VA, pp 125-146, 1990.

[32] G. Nuthall, and A. Alton-Lee, (1993). *Predicting students learning from student experience of teaching: A theory of student knowledge construction in classrooms,* American Educational Research Journal, 30(1993), pp. 799-840.

[33] M. Franke, T. Carpenter, E. Fennema, E. Ansell, and J. Behrend, *Understanding Teachers' self-sustaining generative change in the context of professional development,* Teaching and Teacher Education, 14(1998), pp. 67-80.

[34] M. Franke, T. Carpenter, L. Levi, and E. Fennema, *Capturing Teachers' Generative Change: A Follow-up Study of Professional Development in Mathematics,* American Educational Research Journal, 38(2001), pp. 653-689.

[35] M. Gill, and A. Spriggs, *The Development of Realistic Evaluation Theory through the Evaluation of National Crime Prevention Programmes*, Paper presented at European Evaluation Society Conference, Seville, 2002.

Reflections on the method for computing primitives: $u = \tan\frac{x}{2}$

JUAN CARLOS PONCE-CAMPUZANO* AND ANTONIO RIVERA-FIGUEROA

Mathematics Education Department, Centre for Research and Advanced Studies, Cinvestav, Mexico City, Mexico

Abstract: The method of change of variable $u = \tan\frac{x}{2}$ produces "primitives" which cannot be used for computing definite integrals by applying the Fundamental Theorem of Calculus, specifically with integrals of the form $\int R(\sin x, \cos x)dx$ with R as a rational function. Several examples are discussed here in order to point out that in some cases it is important to take certain considerations into account before applying the Fundamental Theorem.

Keywords: primitives, fundamental theorem of calculus, definite integral

1. Introduction

The Fundamental Theorem of Calculus (FTC) specifies the relationship between the two main concepts of Calculus: integration and differentiation. It is well known that it is a powerful tool for computing definite integrals of functions. This relationship plays an important role in the teaching of high school calculus because the problem of computing integrals can be reduced to finding primitives of functions.

2. The Fundamental Theorem of Calculus

Theorem (FTC)**:** Let $f: [a, b] \to \mathbb{R}$ be a continuous function and $F: [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(u)du.$$

Then

1) F is differentiable for all $x \in [a, b]$ and furthermore $F'(x) = f(x)$ for all $x \in [a, b]$.

2) If $G: [a, b] \to \mathbb{R}$ is differentiable such that $G'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(u)du = G(b) - G(a).$$

Apostol [2], Ghorpade & Limaye [4] and Spivak [8] offer general forms and proofs for this theorem. Vajiac & Vajiac [9] offers a method of proof for continuous functions.

* E-mail: jcponce@cinvestav.mx

Recall that a function P, differentiable on an interval, is called a *primitive* function of f if it satisfies $P'(x) = f(x)$ for every x in the interval. From the FTC one can deduce that if f is continuous on $[a, b]$, then it has primitives and one of them is produced by

$$P(x) = \int_a^x f(u)du$$

where $a \leq x \leq b$. Other primitives differ from this in one constant. Let C be a real constant, the equation

$$\int f(x)dx = P(x) + C$$

represents the family of all primitives of f.

Many techniques exist to obtain tables of primitives and the following are just a few:

Integration by substitution
Integration by parts, and
Integration by partial fractions

The method of change of variable $u = \tan\frac{x}{2}$ is frequently used for certain classes of trigonometric functions. In the next section we will discuss some examples using this technique.

3. An example using the change of variable $u = \tan\frac{x}{2}$.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{5 + 3\cos x}.$$

To calculate the definite integral

$$\int_0^{2\pi} \frac{1}{5 + 3\cos x} dx$$

a change of variable $u = \tan\frac{x}{2}$, gives

$$\int \frac{1}{5 + 3\cos x} dx = \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right) + C$$

where C is a real constant.

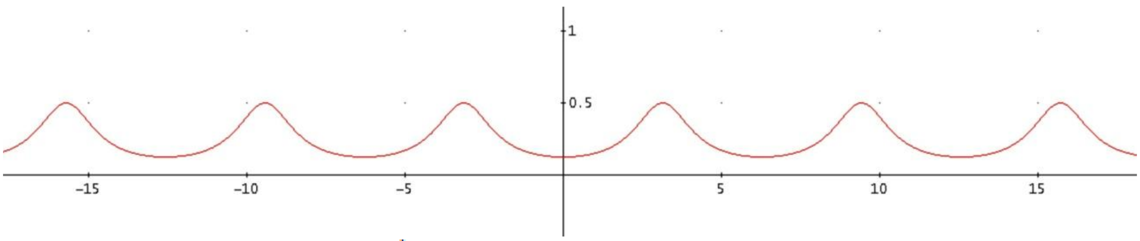


Figure 1. Graph of $f(x) = \frac{1}{5+3\cos x}$.

Thus, using the FTC,

$$\int_0^{2\pi} \frac{1}{5+3\cos x}dx = \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{2\pi}{2}\right) - \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{0}{2}\right) = 0$$

since $\tan\pi = \tan 0 = 0$ and $\arctan 0 = 0$.

This result contradicts the fact that

$$\int_0^{2\pi} \frac{1}{5+3\cos x}dx > 0$$

because $f(x) > 0$ for all $x \in [0,2\pi]$, as can be seen in Figure 2.

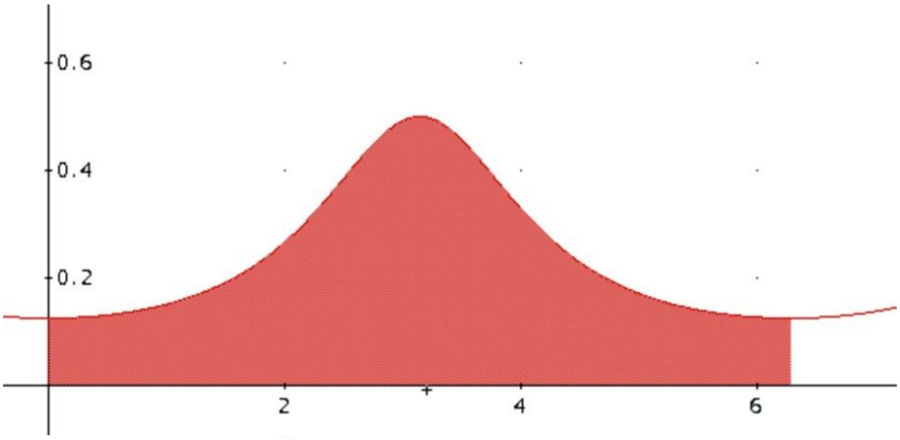


Figure 2: Area under f.

What is the problem here? Is the method of change of variable $u = \tan\frac{x}{2}$ incorrect? The problem with this method is that it works for definite integrals $\int_a^b f(x)dx$, where $a, b \in ((2n-1)\pi, (2n+1)\pi)$, with $n \in \mathbb{Z}$.

Let $F: A \to \mathbb{R}$ be defined as

$$F(x) = \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right).$$

F is a "primitive" of f and has as its domain the union of these opened disjoint intervals, *i. e.*

$$A = \bigcup_{n\in\mathbb{Z}} ((2n-1)\pi, (2n+1)\pi)$$

The above method produces a function F defined in a proper subset of the domain $\mathbb{R}$ of f. $F(x)$ is not defined at $x = (2n+1)\pi$, for every $n \in \mathbb{Z}$, and for this reason it is not true that $F'(x) = f(x)$ for all $x \in [0,2\pi]$. In this case we cannot say that F is primitive of f. That is why we refer to F as a "primitive" in quotation marks.

What can be done about this? Is there a way to find a primitive defined over the interval $[0,2\pi]$, or over all $\mathbb{R}$?

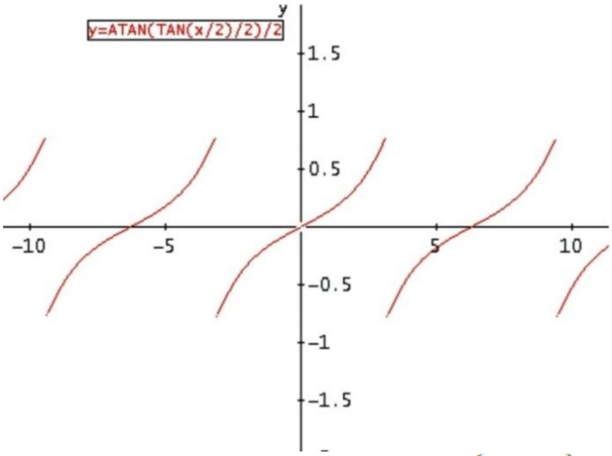


Figure 3. Graph of $F(x) = \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right)$.

WARNING: There is an important concern that we have to underscore. It could simply be a misunderstanding; we could think that the function

$$F(x) = \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right)$$

is not continuous because of what we see in the image at Figure 3. This is of course not true. F **is continuous** (in its domain).

However, if we define F over all $\mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right), & x \neq (2n+1)\pi \\ 0, & x = (2n+1)\pi \end{cases}$$

where $n \in \mathbb{Z}$, then this function **is not continuous.**

Now, using specific strategies and trigonometric identities we can find more “primitives” of the function f. For example:

$$F_1(x) = \frac{1}{4}\arctan\left(\frac{5}{4}\tan x\right) - \frac{1}{4}\arctan\left(\frac{3}{4}\sin x\right)$$
$$F_2(x) = -2\arctan\left(2\cot\frac{x}{2}\right)$$
$$F_3(x) = x - 2\arctan\left(\tan\frac{x}{2}\right) + \frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right)$$
$$F_4(x) = \frac{1}{4}\arctan\left(\frac{4\sin x}{3+5\cos x}\right)$$

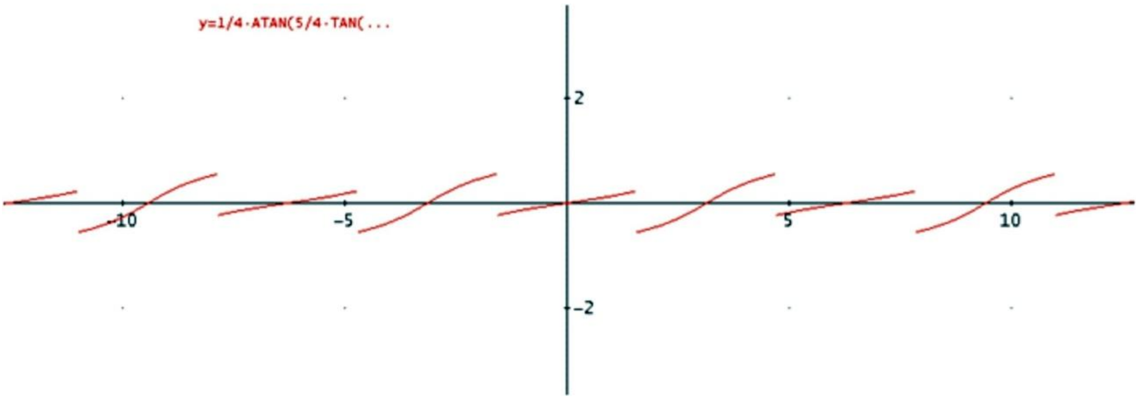


Figure 4. Graph of $F_1(x)$.

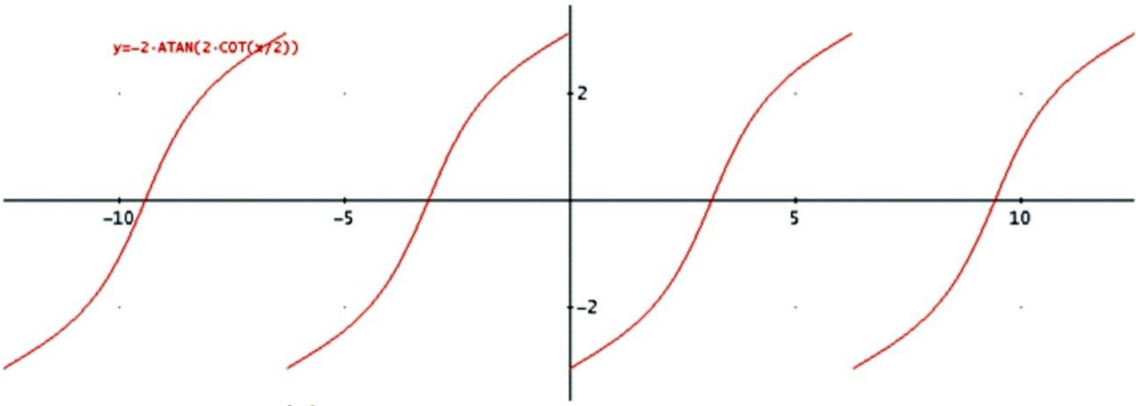


Figure 5. Graph of $F_2(x)$.

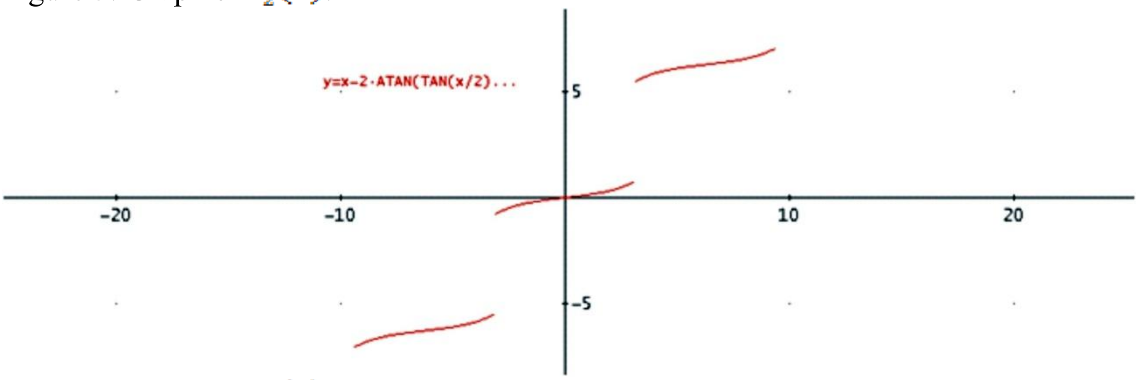


Figure 6. Graph of $F_3(x)$.

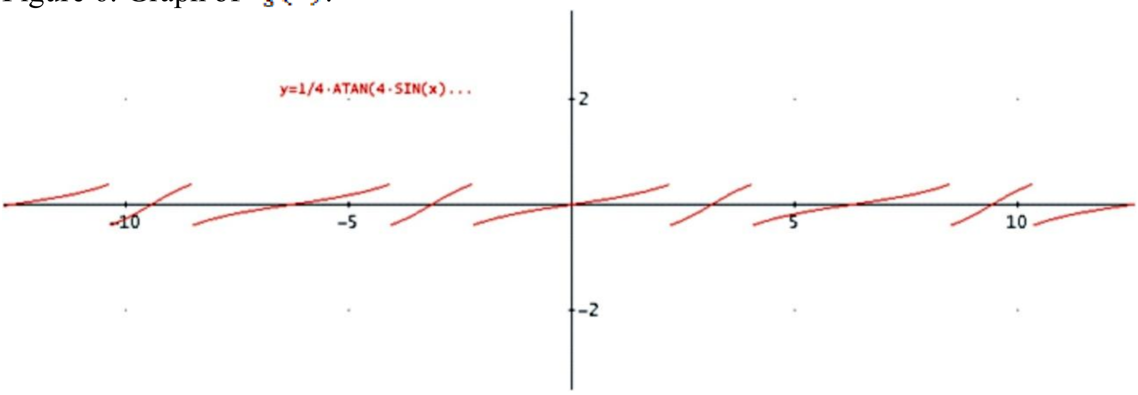


Figure 7. Graph of $F_4(x)$.

Despite $F_i'(x) = f(x)$ for all x in the domain of F_i, none of them can be used to compute the integral on the interval $[0,2\pi]$. Once again, it is not true that $F_i'(x) = f(x)$ for all $x \in [0,2\pi]$. Therefore, we cannot say that the functions F_i are primitives of the function f. Besides, they have different domains as can be appreciated in Figures 4 to 7.

The phenomenon described before is presented in the calculus of integrals of the form $\int R(\sin x, \cos x)dx$ where R is a rational function in sine and cosine.

We want to find a primitive of the function f. Let $G: \mathbb{R} \to \mathbb{R}$ be defined as follow

$$G(x) = \frac{x}{4} - \frac{1}{2}\arctan\left(\frac{\sin x}{\cos x + 3}\right).$$

It is easy to verify that G is a primitive and

$$\int_0^{2\pi} \frac{1}{5 + 3\cos x} dx = G(2\pi) - G(0) = \frac{\pi}{2}$$

which is the correct value.

In this case the primitive G is defined not only in the interval $[0,2\pi]$ but over $\mathbb{R}$ as well (see Figure 8).

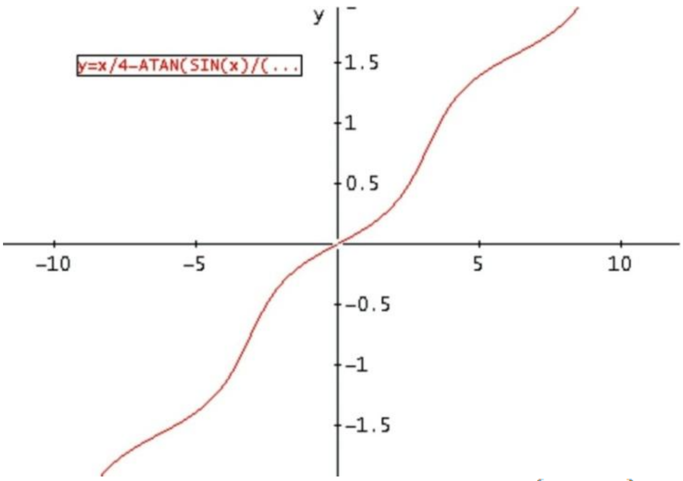


Figure 8. Graph of $G(x) = \frac{x}{4} - \frac{1}{2}\arctan\left(\frac{\sin x}{\cos x+3}\right)$.

4. Another procedure to find $\int \frac{1}{a+b\cos x} dx$ for $a^2 > b^2$.

In this section we present a way of finding another primitive for the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{a + b\cos x}$$

for $a^2 > b^2$.

Because it is convenient, we will seek the next integral

$$\int \frac{\sqrt{a^2 - b^2}}{a + b\cos x} dx.$$

The integrand can be seen as

$$\frac{\sqrt{a^2-b^2}}{a+b\cos x} = \frac{a+b\cos x-(a+b\cos x)+\sqrt{a^2-b^2}}{a+b\cos x}$$
$$= 1-\frac{b\cos x+\left(a-\sqrt{a^2-b^2}\right)}{a+b\cos x}$$

So we have

$$\int\frac{\sqrt{a^2-b^2}}{a+b\cos x}dx = \int dx - \int\frac{b\cos x+\left(a-\sqrt{a^2-b^2}\right)}{a+b\cos x}dx.$$

Using the method of change of variable with $u=\tan\frac{x}{2}$, we have $\cos x=\frac{1-u^2}{1+u^2}$ and $dx=\frac{2}{1+u^2}du$.

Thus

$$\int\frac{b\cos x+\left(a-\sqrt{a^2-b^2}\right)}{a+b\cos x}dx = \int\frac{b\left(\frac{1-u^2}{1+u^2}\right)+\left(a-\sqrt{a^2-b^2}\right)}{a+b\left(\frac{1-u^2}{1+u^2}\right)}\frac{2}{1+u^2}du$$
$$= 2\int\frac{\left(a-\sqrt{a^2-b^2}+b\right)+\left(a-\sqrt{a^2-b^2}-b\right)u^2}{\left((a+b)+(a-b)u^2\right)(1+u^2)}du.$$

In order to facilitate the calculus, we will rename some of the variables:

$$m=a-\sqrt{a^2-b^2}+b$$
$$n=a-\sqrt{a^2-b^2}-b$$
$$p=a+b$$
$$q=a-b$$

Substituting

$$2\int\frac{\left(a-\sqrt{a^2-b^2}+b\right)+\left(a-\sqrt{a^2-b^2}-b\right)u^2}{\left((a+b)+(a-b)u^2\right)(1+u^2)}du = 2\int\frac{m+nu^2}{(p+qu^2)(1+u^2)}du.$$

Now, decomposing the integrand in partial fractions we have

$$\frac{m+nu^2}{(p+qu^2)(1+u^2)} = \frac{\frac{mq+np}{p-q}}{p+qu^2}-\frac{\frac{m-n}{p-q}}{1+u^2}.$$

So that

$$2\int \frac{m+nu^2}{(p+qu^2)(1+u^2)}du = 2\int\left(\frac{\frac{mq+np}{p-q}}{p+qu^2}-\frac{\frac{m-n}{p-q}}{1+u^2}\right)du$$

$$= 2\int \frac{\frac{mq+np}{p-q}}{p+qu^2}du - 2\int \frac{\frac{m-n}{p-q}}{1+u^2}du$$

$$= 2\frac{mq-np}{\sqrt{p}\sqrt{q}(q-p)}\arctan\left(\frac{\sqrt{q}}{\sqrt{p}}u\right) - 2\frac{m-n}{q-p}\arctan u\,.$$

Bringing everything back in terms of a and b, we have

$$2\frac{mq-np}{\sqrt{p}\sqrt{q}(q-p)}\arctan\left(\frac{\sqrt{q}}{\sqrt{p}}u\right) - 2\frac{m-n}{q-p}\,\mathrm{a}\arctan u$$

$$= -2\frac{\sqrt{a^2-b^2}}{\sqrt{(a-b)(a+b)}}\arctan\left(\frac{\sqrt{a-b}}{\sqrt{a+b}}u\right) + 2\arctan u.$$

Since $a^2 > b^2$, the last identity can be reduced to

$$2\arctan u - 2\arctan\left(\frac{a-b}{\sqrt{a^2-b^2}}u\right).$$

Substituting the value of u:

$$2\arctan\left(\tan\frac{x}{2}\right) - 2\arctan\left(\frac{a-b}{\sqrt{a^2-b^2}}\tan\frac{x}{2}\right).$$

Thus

$$\int \frac{\sqrt{a^2-b^2}}{a+b\cos x}dx = x - 2\left(\arctan\left(\tan\frac{x}{2}\right) - \arctan\left(\frac{a-b}{\sqrt{a^2-b^2}}\tan\frac{x}{2}\right)\right).$$

And therefore, without forgetting the constant of integration, we have

$$\int \frac{1}{a+b\cos x}dx = \frac{1}{\sqrt{a^2-b^2}}\left[x - 2\left(\arctan\left(\tan\frac{x}{2}\right) - \arctan\left(\frac{a-b}{\sqrt{a^2-b^2}}\tan\frac{x}{2}\right)\right)\right] + C.$$

since $a^2 > b^2$.

Using the next trigonometric identities:

$$\tan\frac{x}{2} = \frac{\sin x}{\cos x + 1},$$

$\sin^2 x + \cos^2 x = 1$ and

$$\arctan x - \arctan y = \arctan\left(\frac{x-y}{1+xy}\right)$$

we can obtain the trigonometric identity

$$\arctan\left(\tan\frac{x}{2}\right) - \arctan\left(\frac{a-b}{\sqrt{a^2-b^2}}\tan\frac{x}{2}\right) = \arctan\left(\frac{2b\sin x}{2b\cos x + \left(\sqrt{a-b}+\sqrt{a+b}\right)^2}\right).$$

From the above we have

$$\int \frac{1}{a+b\cos x}dx = \frac{1}{\sqrt{a^2-b^2}}\left[x - 2\arctan\left(\frac{2b\sin x}{2b\cos x + \left(\sqrt{a-b}+\sqrt{a+b}\right)^2}\right)\right] + C.$$

Let $H_{a,b}: \mathbb{R} \to \mathbb{R}$ be defined by

$$H_{a,b}(x) = \frac{1}{\sqrt{a^2-b^2}}\left[x - 2\arctan\left(\frac{2b\sin x}{2b\cos x + \left(\sqrt{a-b}+\sqrt{a+b}\right)^2}\right)\right].$$

Using this function, it is easy to verify that

$$\int_0^{2\pi} \frac{1}{a + b\cos x} dx = H_{a,b}(2\pi) - H_{a,b}(0) = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

For the values $a = 5$, $b = 3$, we have $H_{5,3}(x) = G(x)$ and therefore

$$\int_0^{2\pi} \frac{1}{5 + 3\cos x} dx = H_{5,3}(2\pi) - H_{5,3}(0) = \frac{\pi}{2}.$$

The same result can be found by using complex variable for computing this definite integral. In such a case, we must use the Residue Theorem, *i. e.* If we want to evaluate

$$\int_0^{2\pi} R(\sin x, \cos x) dx$$

we make the substitution $z = e^{ix}$ in order to get

$$-i \int_{|z|=1} R\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\frac{dz}{z}$$

It remains only to determine the residues which correspond to the poles of the integrand inside the unit circle. For more information see [1, pp. 155], [3, pp. 117], [5, pp. 114-115] and [6, pp. 236-237]. Nevertheless the tools of differential calculus should still be used.

5. Final remarks

In the teaching of calculus care must be given when computing definite integrals and applying the FTC because situations such as that described in section 3 may arise. The method of change of variable $u = \tan\frac{x}{2}$ produces a function F defined in a proper subset of the domain of f – and consequently naming it a primitive of f is incorrect– and therefore we cannot use this function F in order to apply the FTC.

In the calculus of definite integrals, there are other difficulties such as non elementary integrals – functions whose integrals cannot be expressed in terms of elementary functions – or integrable functions which have infinite discontinuities [8, pp. 274] or functions with a bounded derivative at all points but whose derivative is so discontinuous as to be non integrable [7] and [11].

Hence in high school it is important to note that if the formula

$$\int_a^b f(u)du = G(b) - G(a)$$

is to be used for continuous functions, it will be important to ensure, at the very least, that the function G is a primitive of f, *i. e.* to corroborate that $G'(x) = f(x)$ for all x in the interval $[a, b]$.

6. References

[1] L. V. Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable*, McGraw-Hill, New York, 1979.
[2] T. M. Apostol, *Calculus: One-variable Calculus, with an Introduction to Linear Algebra V. 1*, John Wiley & Sons, New York, 1967.
[3] J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1978.
[4] S. R. Ghorpade, and B. V. Limaye, *A Course in Calculus and Real Analysis*, Springer Science Business Media, New York, 2006.
[5] J. P. Gilman, I. Kra, and R. E. Rodríguez, *Complex Variable: In the Spirit of Lipman Bers*, Springer Science Business Media, LLC. New York, 2007.
[6] J. E. Marsden, and M. Hoffman, *Basic complex analysis*, W. H. Freeman and Company, San Francisco, 1973.
[7] J. C. Ponce, and A. Rivera, *Casos en los que no es aplicable la fórmula* $\int_a^b f(u)du = G(b) - G(a)$, Misc. Mat., 48 (2009), pp. 59-74.
[8] B. Riemann, *Gesammelte Mathematische Werke*, Springer-Verlag, 1990.
[9] M. Spivak, *Calculus*, Publish or Perish, Inc., Houston, 1994.
[10] A. Vajiac, and B. Vajiac, *Areas and the Fundamental Theorem of Calculus*, Int. J. Math. Educ. Sci. Tech., 39:8. (2008), pp. 1023-1036.
[11] V. Volterra, *Sui Principii del Calcolo Integrale.* Giorn. Mat., 19 (1881), pp. 333-372.

Teaching Matrix Algebra in the PC Lab

Karsten Schmidt[§§§§§§§§§]

Schmalkalden University of Applied Sciences, Schmalkalden, Germany

Abstract: When teaching linear algebra we have to deal with the following problem: while the level of mathematical skills required to work with examples is generally low (students only need to add, subtract and multiply), the number of calculations is usually large. Therefore, working with examples is time-consuming and error-prone if done by hand. Students get tired quickly and lose interest in this increasingly important area of mathematics. The Schmalkalden University Faculty of Business and Economics therefore decided to move its introductory linear algebra course from the classroom to the PC lab, and purchased a Computer Algebra System license that also allows its use on the students' own PCs. A collection of functions was then developed to facilitate teaching by providing functions for the computation of zero matrices and vectors, matrices and vectors of ones, identity matrices, as well as idempotent and orthogonal matrices, "just-in-time" whenever they are needed during the course. This paper demonstrates how beneficial it is, from the very basics to the more advanced topics, to sit in front of a PC in an introductory linear algebra course.

Keywords: Computer Algebra System; Matrix Algebra

AMS Subject Classifications: 97U70; 15–01; 15A09

1. Introduction

For many years the introductory linear algebra course at the Schmalkalden University Faculty of Business and Economics was taught in a regular classroom equipped with blackboard and chalk as well as an overhead projector. Basic operations with matrices and vectors, such as adding or subtracting two matrices of the same dimension, or the multiplication of a matrix by a scalar or another conformable matrix, were explained, then demonstrated in examples on either the blackboard or a transparency, and finally the students had to work their way through several examples, using pencil and paper.

In 2000 the faculty began equipping classrooms with beamers which could be connected to the notebook of a teacher who was then able to project the computer screen onto a silver screen. It was now possible to use computer software to highlight certain more advanced topics, e.g. to show that the length of both columns of any 2×2 orthogonal matrix is one (the vectors lie on the unit circle), or that they are orthogonal (form a right angle).

A little later the faculty purchased a license for the Computer Algebra System *Derive* that covered all PCs in the faculty's two PC labs as well as the private PCs of the students. Henceforth the course was taught either in a regular classroom or the PC lab, whatever seemed more appropriate for the topics to be covered during that lesson. As a consequence, the final exam included questions that simulated a session on the PC, using screenshots.

Since 2007 the course has been held in the PC lab only. This limits the number of students per group to 40 (sitting in front of 20 PCs) and therefore comes at a price: the faculty now needs to offer the introductory linear algebra course to three groups, compared to splitting courses into two groups (of up to 70 students in regular

[§§§§§§§§§] E-mail: kschmidt@fh-sm.de

classrooms) as is the case for most of the other subjects. Since then the final exams have been held in the PC labs (which again comes at a price: exams now have to take place in many small groups since naturally only one student can sit in front of a PC during an exam). They contain some questions which can only be solved using a PC, but also questions which take longer to solve with a PC than without. All questions still have to be answered on the exam papers alone; nothing has to be stored electronically.

Reports on the use of a Computer Algebra System (CAS) in tertiary mathematics education have been published in several papers (e.g. [1-5]). Cheung [1] describes a course in elementary number theory for prospective teachers in which the CAS *Maple* was used in an extra computer lab session at the end of the course. Whilst he stresses the fact that CAS use removes the tedium of time-consuming calculations, the final exam of the course was nevertheless obviously in traditional paper-and-pencil style. Fernández et al. [2] also emphasize that the use of CASs (*Derive* and *Mathematica*) avoids having to do tedious computations by hand in algebra and calculus courses for engineering students, but apparently no CAS is used in the final exams either.

Lawson [3] compared the performance of two groups of students of a first-year undergraduate engineering mathematics course: one group was taught conventionally (lectures and tutorials), the other group was taught partially in a computer lab (using the CAS *Derive* and another software package). Two tests and a final exam (all without computers) showed that the results of students from the group that used *Derive* were significantly better than those of the control group. Lawson also recommends using the computer in the assessment process since traditional assessment (paper and pencil) seems to limit to some extent students' interest for computer use in mathematics education.

Stephens and Konvalina [4] investigated the effect of using the CAS *Maple* in two different algebra courses (Intermediate Algebra and College Algebra). For each course there was also a control group of students who received a traditional lecture presentation during the course. For both courses the mean scores of the final exam in the experimental group was higher (though not statistically significant at the 5% level) than in the control group. The authors also mention that the student evaluations from the experimental groups were the best they had ever received in an algebra course for more than twenty years, suggesting that the use of *Maple* created a more positive attitude towards the course and the instructor.

Pecuch-Herrero [5] reports on a course in linear algebra which took place in a computer lab. The main reasons for using a software package developed at the University of Arizona were: to keep the students' attention during this evening class, to be able to work on applied problems without spending time and energy on long calculations, and to free class time to teach students how to write proofs. Instead of doing quizzes and tests students had to write solutions and proofs in their portfolios, and the proportion of failing students declined noticeably.

Note that the faculty's decision to not only use the CAS *Derive* during the lectures but also during the final exam appears to be reasonable in light of the above mentioned papers. Note also that it is not possible to compare the results in the final exam of the introductory linear algebra course under consideration as there is no control group for this course, since all students have been in the experimental group (taught in the PC lab) since 2007.

2. Working with Zero Vectors and Zero Matrices

Early in the course, it is important for students to get a good understanding of matrices and vectors containing only zeros. While students quickly develop a feeling that these zero matrices and vectors behave similarly to the number $0 \in \mathbb{R}$, it is rather tedious to check this in exercises. The function collection therefore includes the following two functions:

O(m,n)	generates an $m \times n$ zero matrix	(1)
o(n)	generates an $n \times 1$ zero vector	(2)

We define a matrix $\boldsymbol{A}$ for which we want to check some of the following properties:

$$\underset{m\times n}{\boldsymbol{A}} - \boldsymbol{A} = \underset{m\times n}{\boldsymbol{O}}\,; \qquad \underset{m\times n}{\boldsymbol{A}} + \underset{m\times n}{\boldsymbol{O}} = \boldsymbol{A}\,;$$

$$\underset{m\times n}{\boldsymbol{A}}\,\underset{n\times l}{\boldsymbol{O}} = \underset{m\times l}{\boldsymbol{O}}\,; \qquad \underset{l\times m}{\boldsymbol{O}}\,\underset{m\times n}{\boldsymbol{A}} = \underset{l\times n}{\boldsymbol{O}}\,;$$

$$\underset{m\times n}{\boldsymbol{O}} = \underset{m\times 1}{\boldsymbol{o}}\,\underset{1\times n}{\boldsymbol{o}'}\,; \qquad \underset{m\times n}{\boldsymbol{O}}\,\underset{n\times p}{\boldsymbol{O}} = \underset{m\times p}{\boldsymbol{O}}\,.$$

If the course is taught in a PC lab students could not only be asked what the result is when a matrix $\boldsymbol{A}$ is subtracted from itself (typically many students find this "too simple" for pencil and paper, as they already guess the answer "zero"), but the computation could actually be done on the PC in order to show that the result is in fact the zero matrix of the very same dimension as $\boldsymbol{A}$.

Note that Screenshot 1 only shows a subset of possible exercises. Note also that the last of the above properties implies that the product of any square zero matrix with itself is a square zero matrix of the same dimension, making square zero matrices a standard example for idempotent matrices (cf. section 5).

#1: LOAD(C:\Program Files\TI Education\Derive 6\Math\EMAD.mth)

#2: $A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

#3: $A - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

#4: $O(2, 3) \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

#5: $A \cdot o(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Screenshot 1

3. Working with Matrices and Vectors of Ones

The understanding of matrices and vectors containing only ones is not as straightforward, as many students expect these matrices and vectors of ones to behave similarly to the number $1 \in \mathbb{R}$. The function collection includes the following two functions:

J(m,n)	generates an $m \times n$ matrix of ones	(3)
l(n)	generates an $n \times 1$ vector of ones	(4)

The function name of function (4) is "lowercase L", since "l" resembles the symbol **1** which is typically used for vectors of ones. For the matrix $\boldsymbol{A}$ defined in the previous section we want to check some of the following properties:

$$\underset{1\times n}{\boldsymbol{a}'}\ \underset{n\times 1}{\mathbf{1}} = \mathbf{1}'\boldsymbol{a} = \sum_{i=1}^{n} a_i\,; \qquad \underset{1\times m}{\mathbf{1}'}\ \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{\mathbf{1}} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\,;$$

$$\underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{\mathbf{1}} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} \end{pmatrix}; \qquad \underset{1\times m}{\mathbf{1}'}\ \underset{m\times n}{\boldsymbol{A}} = \begin{pmatrix} \sum_{i=1}^{m} a_{i1} & \cdots & \sum_{i=1}^{m} a_{in} \end{pmatrix};$$

$$\underset{m\times n}{\boldsymbol{J}} = \underset{m\times 1}{\mathbf{1}}\ \underset{1\times n}{\mathbf{1}'}\,; \qquad \underset{m\times n}{\boldsymbol{J}}\ \underset{n\times p}{\boldsymbol{J}} = n \underset{m\times p}{\boldsymbol{J}}\,.$$

While doing exercises with vectors of ones is a little more rewarding than with zero vectors, it is nevertheless another time-consuming and tiring activity if done with pencil and paper. However, it turns into a pleasant and interesting affair if such exercises are carried out on a PC. Clearly, the first four properties show that vectors of ones are the perfect tool when it comes to the computation of sums, namely of the elements of a vector, of all elements of a matrix, or the row, or column, sums of a matrix.

```
#6:   a := [ a1 ]
           [ a2 ]

#7:                          a'·l(2) = [[a1 + a2]]

#8:                          l(3)'·A·l(3) = [[45]]

                                      [  6 ]
#9:                          A·l(3) = [ 15 ]
                                      [ 24 ]

#10:                         l(3)'·A = [[12, 15, 18]]

#11:                  J(2, 3)·J(3, 4) = [ 3  3  3  3 ]
                                        [ 3  3  3  3 ]
```

Screenshot 2

For a change, the summation of the elements of a vector $\boldsymbol{a}$ in Screenshot 2 is done for a 2×1 vector of arbitrary elements, which are denoted by a1 and a2. The result of the multiplication of two matrices of ones comes as a surprise for many students, as they expect another matrix of ones.

4. Working with Identity Matrices

After realizing that matrices of ones do not behave similarly to the number $1 \in □$, attention can immediately be directed to identity matrices. The function collection includes the function

I(n) generates an $n \times n$ identity matrix (5)

For the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{a}$ defined in the previous two sections we want to check the following property:

$$\underset{m\times n}{\boldsymbol{A}}\ \underset{n\times n}{\boldsymbol{I}} = \underset{m\times m}{\boldsymbol{I}}\ \underset{m\times n}{\boldsymbol{A}} = \boldsymbol{A}.$$

However, we start by looking at the 3×3 identity matrix, and students can see it is a matrix containing ones on the main diagonal and zeros otherwise.

#12: $\mathrm{I}(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

#13: $\mathrm{A}\cdot\mathrm{I}(3) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

#14: $\mathrm{I}(2)\cdot\mathrm{a} = \begin{bmatrix} \mathrm{a1} \\ \mathrm{a2} \end{bmatrix}$

#15: $\mathrm{I}(2)\cdot\mathrm{I}(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Screenshot 3

Screenshot 3 also shows that the product of the 2×2 identity matrix with itself is the 2×2 identity matrix. As this is true for any identity matrix (which follows immediately from the above-mentioned property by choosing $\boldsymbol{A} = \boldsymbol{I}$), identity matrices are another standard example for idempotent matrices.

5. Working with Idempotent Matrices

Now that we have already come across two examples for idempotent matrices (any square matrix $\boldsymbol{A}$ with the property $\boldsymbol{AA} = \boldsymbol{A}$), namely square zero matrices and identity matrices, we should spend a little more time with these matrices. The function collection includes the function

IDEM(n) generates an idempotent $n \times n$ matrix (6)

The formula used in this function,

$$\underset{n\times n}{\boldsymbol{I}} - \tfrac{1}{n}\underset{n\times n}{\boldsymbol{J}},$$

generates a so-called centering matrix using an identity matrix and a square matrix of ones. Centering matrices are used, for example, in multivariate statistics. Let us see in Screenshot 4 what centering matrices look like (and also prove that the 3×3 centering matrix is indeed idempotent).

#16: $$\mathrm{IDEM}(2) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

#17: $$\mathrm{IDEM}(3) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

#18: $$\mathrm{IDEM}(3) - \mathrm{IDEM}(3)\cdot\mathrm{IDEM}(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Screenshot 4

Obviously, the IDEM-function can only generate one particular idempotent matrix of a specific dimension. If more variety is required, we could use a property that holds for the Moore-Penrose inverse $\underset{n\times m}{\boldsymbol{A}^{+}}$ of any matrix $\underset{m\times n}{\boldsymbol{A}}$:

$\boldsymbol{A}^{+}\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^{+}$ (as well as $\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A}$ and $\boldsymbol{I}-\boldsymbol{A}\boldsymbol{A}^{+}$) are idempotent matrices.

If $\boldsymbol{A}$ is a square non-singular matrix, the Moore-Penrose inverse and the inverse $\boldsymbol{A}^{-1}$ coincide, and we have $\boldsymbol{A}^{+}\boldsymbol{A}=\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{I}$ and $\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A}=\boldsymbol{I}-\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{O}$, i.e. we are back to the two standard examples for idempotent matrices from sections 2 and 4. Therefore, the more interesting cases are when $\boldsymbol{A}$ is either square but singular, or non-square. The matrix $\boldsymbol{A}$ from section 2 is such a singular matrix.

```
#19:  LOAD(C:\Program Files\TI Education\Derive 6\Math\MP.mth)
```

#20: $$\mathrm{MPI}(A) = \begin{bmatrix} -\frac{23}{36} & -\frac{1}{6} & \frac{11}{36} \\ -\frac{1}{18} & 0 & \frac{1}{18} \\ \frac{19}{36} & \frac{1}{6} & -\frac{7}{36} \end{bmatrix}$$

#21: $$\mathrm{MPI}(A)\cdot A = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

Screenshot 5

As *Derive* does not include a function for the computation of the Moore-Penrose inverse, we have to use another function collection that includes the function

MPI(A) computes the Moore-Penrose inverse of any $m \times n$ matrix $\boldsymbol{A}$ (6a)

The MPI-function, which is described in detail in [6], is now used in Screenshot 5 for the computation of the Moore-Penrose inverse of the 3×3 matrix $\boldsymbol{A}$, which in turn is used for the computation of a 3×3 idempotent matrix.

Note that finding the rank of an idempotent matrix $\boldsymbol{B}$ is an easy task, as we have in this case $\mathrm{r}(\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{B})$, i.e. rank and trace are identical; here we get $\frac{5}{6}+\frac{1}{3}+\frac{5}{6}=2$, which can also be double-checked on the PC (cf. Screenshot 6).

```
#22:                         RANK(MPI(A)·A) = 2

#23:                         TRACE(MPI(A)·A) = 2
```

Screenshot 6

6. Working with Orthogonal Matrices

Finally, we want to consider orthogonal matrices (any square matrix $\boldsymbol{A}$ with the property $\boldsymbol{A}' = \boldsymbol{A}^{-1}$). The function collection includes the function

ORTH(a) generates an orthogonal $n \times n$ matrix from any $n \times 1$ vector $\boldsymbol{a} \neq \boldsymbol{o}$ (7)

The formula used in this function,

$$\underset{n\times n}{\boldsymbol{I}} - 2\,\underset{n\times 1}{\boldsymbol{a}}\,\underset{1\times n}{\boldsymbol{a}'},$$

#24: $b := \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

#25: $b' \cdot b = [[1]]$

#26: $\mathrm{ORTH}(b) = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{bmatrix}$

Screenshot 7

generates an orthogonal matrix if $\boldsymbol{a}'\boldsymbol{a} = 1$. Hence, the vector $\boldsymbol{a}$ which is passed as parameter will be transformed within the ORTH-function such that it is of length 1. Let us nevertheless start in Screenshot 7 with a vector $\boldsymbol{b}$ that has length 1 anyway.

In order to check if this matrix is indeed orthogonal, we have to compute both its inverse and transpose, and see if these two matrices are identical. Since for any orthogonal matrix we have

$$\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I},$$

a second method to prove that a matrix $\boldsymbol{A}$ is orthogonal is to show that $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}$ (cf. Screenshot 8)

#27: $\text{ORTH}(b)^{-1} - \text{ORTH}(b)' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

#28: $\text{ORTH}(b)' \cdot \text{ORTH}(b) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Screenshot 8

For a second example we choose the 2×1 vector of ones as the parameter of the ORTH-function in Screenshot 9.

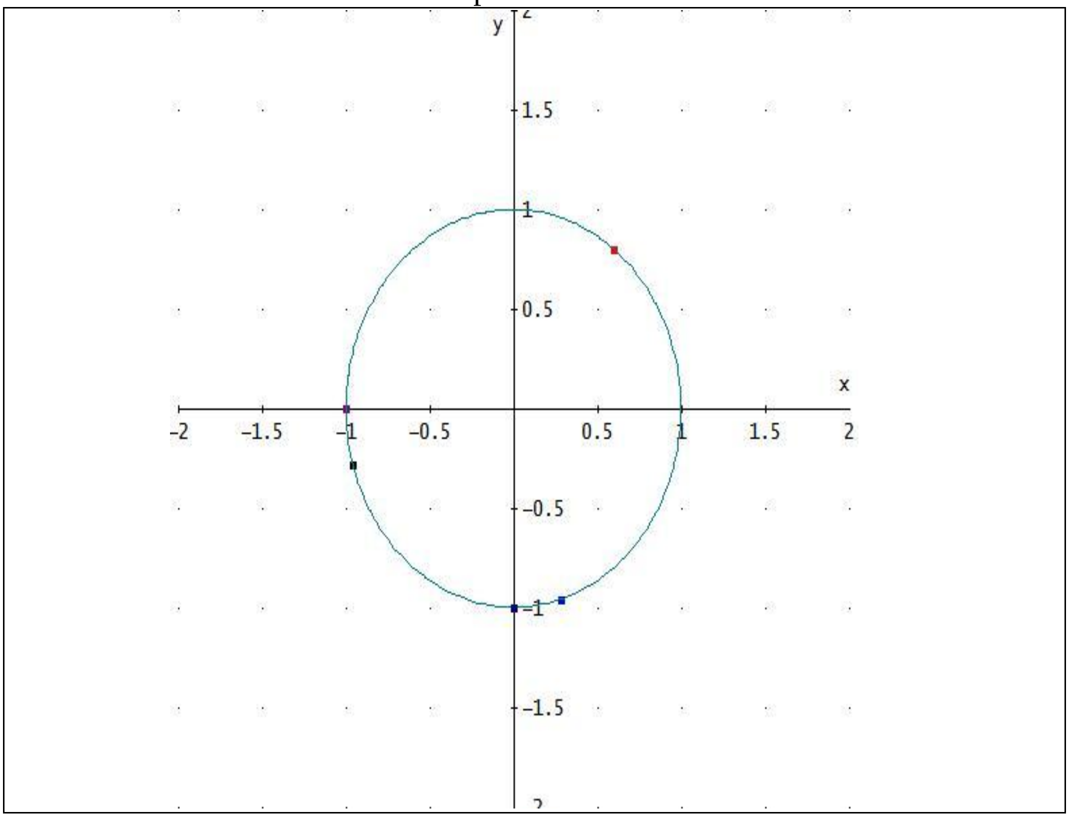
#29: $\text{ORTH}(1(2)) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

#30: $\text{ORTH}(1(2))^{-1} - \text{ORTH}(1(2))' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

#31: $\text{ORTH}(1(2))' \cdot \text{ORTH}(1(2)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Screenshot 9

To complete this section we take advantage of the graphical capabilities of *Derive*. Any column of an orthogonal matrix has length 1, and any two are pairwise orthogonal. Since both examples are two-dimensional, Screenshot 10 shows that its column vectors are lying on the unit circle and form a right angle. The graph also shows the vector ***b*** from the first example.



Screenshot 10

7. References

[1] Cheung, Y.L., Learning Number Theory with a Computer Algebra System. *International Journal of Mathematical Education in Science and Technology*, **27** (1996), 379–385.

[2] Fernández, M., M.A. González, G. Rodríguez and A. de la Villa, Some Examples of Teaching with Computers. *International Journal of Mathematical Education in Science and Technology*, **30** (1999), 309–315.

[3] Lawson, D.A., The Effectiveness of a Computer-Assisted Learning Programme in Engineering Mathematics. *International Journal of Mathematical Education in Science and Technology*, **26** (1995), 567–574.

[4] Stephens, L.J. and J. Konvalina, Use of Computer Algebra Software in Teaching Intermediate and College Algebra. *International Journal of Mathematical Education in Science and Technology*, **30** (1999), 483–488.

[5] Pecuch-Herrero, M., Strategies and Computer Projects for Teaching Linear Algebra. *International Journal of Mathematical Education in Science and Technology*, **31** (2000), 181–186.

[6] Schmidt, K., Computing the Moore-Penrose Inverse of a Matrix with a Computer Algebra System. *International Journal of Mathematical Education in Science and Technology*, **39** (2008), 557–562.

The role of technology in conversational support for learning: lecturer-student communication in a university setting

MOIRA STATHAM[**********] and SHEENA PARNELL

Department of Mathematics, The University of Auckland, Auckland, New Zealand

Abstract: Student profiles and student expectations in the twenty-first century university environment have changed. Modern technology creates an opportunity to develop better communication between lecturers and students at tertiary level. In particular it allows for instant contact via text, and for an extended subject-based electronic dialogue via a web-based Learning Management System. This paper describes a pilot study on the initial effects of introducing these two communication tools into the mathematics courses within the one-year pre-degree Tertiary Foundation Certificate (TFC) Programme at the University of Auckland. It provides some preliminary evidence of the effectiveness of these conversational supports, as gauged from retention data and from the computer-logged dialogue in student reflections.

Keywords: Mathematics Education, Reflections, Foundation Education, Retention, Communication, Web to text communication.

1. Background

1.1 *The Tertiary Foundation Certificate (TFC) Programme at the University of Auckland*

The TFC programme is a full year pre-degree programme designed to prepare students for tertiary study. It is offered each year at the University of Auckland to 200 students who do not have the required entry qualifications for university, but who have the desire to succeed academically. The students may have been out of school for some time, or may be recent school leavers. For this year (2009) there were more than 700 online applicants for the programme with over 500 arriving to sit diagnostic tests in Mathematics and English over the summer break. A smaller number of these were then offered a personal interview with one of the programme's tutors, before final selections were made. There are target groups (mainly under-represented Maori and Pasifika students) that are given priority, subject to satisfactory test results. The student body is diverse, not only in terms of age and ethnicity, but most importantly in background knowledge of mathematics, which is a compulsory subject along with English (or Academic Literacy). Over the last few years there has been a noticeable and rapid change in the nature of students' expectations that is linked to the advent of sophisticated technology.

1.2 *The Tertiary Foundation Certificate Programme's Mathematics Courses*

Mathematics is a compulsory subject within the TFC programme. Students must pass at least one Mathematics course among the eight courses of their programme in order to gain the final certificate. There are two Mathematics courses offered in each

[**********] E-mail: statham@math.auckland.ac.nz

semester, one at a basic level, one at an extension level, and all students are enrolled in one of these. In the first semester the stated aims of the courses are:

To set a mathematical platform that includes

- a knowledge of the accepted conventions of mathematical notation and representation;
- an understanding of basic arithmetic processes;
- the development of algebraic manipulative skills;
- experience in problem solving.

However, the TFC Mathematics courses have an overall goal of fostering deep learning. To give the tutors some indication of students' progress towards this goal, the use of student reflections was introduced as a possible strategy. As part of each of the five assignments per semester, students are asked to reflect on the mathematics they have learnt in the topic and to say how they feel about it. [1]

1.3 *The University of Auckland's web-based Computer Management System (Cecil)*

Cecil is a web based information system available to supplement any course within the University of Auckland. Cecil Explorer is the interface used by University of Auckland staff to administer their courses. It is aimed at supporting academics and their students by providing a highly flexible and reliable system for communication and access to information. The TFC programme uses Cecil as an important communication tool and over the last two years, the Mathematics tutors have taken the opportunity to extend their use of Cecil to take advantage of a new feature that it offers in the area of Personal Journals.

2. A theoretical perspective

The use of reflections as a metacognitive activity is aligned with the "learn to learn" movement. [2]

> This movement attempts to train students, in particular those in the secondary and tertiary levels, to become independent learners. Constructs and strategies developed elsewhere such as concept maps, goal settings, and desirable study habits, are adapted to help students learn mathematics more effectively. [2]

A number of background and affective factors colour students' mathematical performance. Grootenboer and Hemmings [3] identify Positive View, Utilitarian Belief, Traditional Belief, and Maths Confidence, of which Traditional Belief and Maths Confidence are the most significant predictors of mathematical performance. They proposed the following model as a conceptualization of the affective domain.

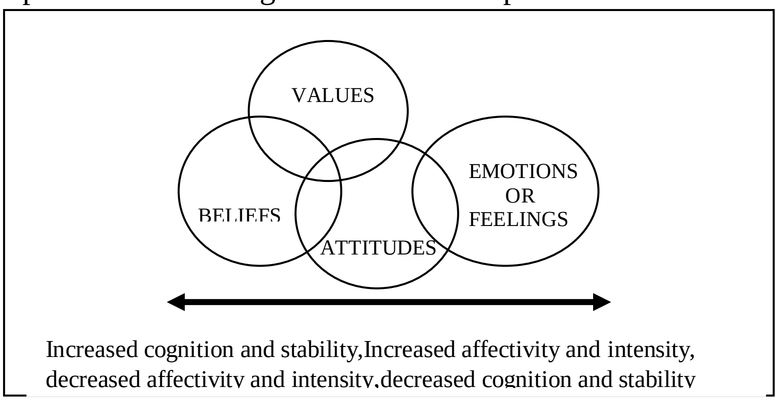


Figure 1. A model of conceptions of the affective domain [3].
Although their particular study was not done at tertiary level, the inherent ideas fit with observations made on TFC students. The students typically experience difficulties in the intersection of attitudes and emotions, which impacts on confidence, and manifests itself in traditional avoidance techniques. At tertiary level, Anthony [4] reported that 'self-motivation was the item rated most likely to influence success by both students and lecturers'. Her study indicated that first–year university mathematics students eventually realized the impact of their own behaviour on successful performance, citing the importance of regular attendance at lectures and tutorials. She also noted that students may be 'neglecting attention to reflective practice designed to consolidate procedures and concepts'. By using student reflections and lecturer responses as a dialogue, it may be possible to focus attention on attitudes and bolster confidence, thus moving students further towards positive beliefs.[1]

3. The impetus for our study

3.1 *Relationship to theoretical background*

This paper is an exploration of some measures to improve relationships between students and lecturers, as a first step towards counteracting some foundation students' traditional negative emotions and low confidence.[1,4] By demonstrating that attendance is a first priority and by boosting the expectation for well-written and considered reflections, it was hoped to focus attention on factors for success.

3.2 *Giving Students a Voice*

Whilst our teaching environment does not support sophisticated technology, our focus is on student-centred approaches to teaching and learning where we want our students to have a voice. The nature of the TFC courses is to provide a foundation; to prepare the students to successfully attempt a mathematics course at degree level. Some of the TFC students come directly from high school, with many demonstrating the negative attitudes towards mathematics mentioned by D'Arcy-Warmington [5],

> 'Mathematics is surrounded by a negative aura in many aspects of life from family and friends to press and film…. Families help to promote the continuance of the negative cycle, therefore it is important to socially give the human touch to mathematics.' [5, p.174]

With this in mind, we considered ways of improving the communication between lecturer and student at a more personal level. Although we had been incorporating student reflections on their learning into each of the course assignments for the last four years, we began to realise that we had lost the original intent of a metacognitive exercise. The students' reflections seemed to be mark-driven because in handwritten form they had become just a minor part of the assignment. Our responses had become rather brief, and easily overlooked by students when their assignments were returned. More importantly, the students had no easily accessible permanent record of the exchange between themselves and their lecturers that charted their own development as they progressed through the year.
Seabright [6] maintains that 'recording of learning provides a means of consolidating learning, moving from initial acquisition to deeper and more meaningful understanding'.

> Initially journal entries may be colloquial, with limited use of language and in the form of a diary or log rather than as entries which are tools for more reflection and comprehension. Once journals are constructed as a record to connect understanding and learning they can be very personal…. one dilemma is that assessing of the journal at all might skew the contents to suit the assessor rather than the author. This relates back to the openness required in the programme in order to engender an honest approach both by the teacher and (students). This is where risk is seen. Teachers have to be aware that the critical analytical approach they are encouraging might raise sensitivities to their own delivery. [6, p.235]

Although she is writing in a different context (professional development of tutors), her basic principles remain valid in the TFC context, even though this operates on a smaller and less sophisticated scale.

3.3 *Retention Problems at Tertiary Level*

Past experience has shown that retention of students is an issue in the TFC programme, and it is well documented [7] that retention levels in foundation courses are low. Our data from 2008 shows a first semester TFC enrolment of 206 students in mathematics, with only 175 remaining to sit the final examination at the end of the first semester, a drop of 13.5%. By the end of the second semester, there were 146 students who sat the final examination, an overall loss of 29% for the year. This pattern is not uncommon, and during 2008 we began thinking about strategies that might ameliorate the losses. Varsavsky [8] highlighted the importance of the student experience in present day universities. She referred to the 'changing student profile' and the 'emerging characteristics of school leavers'. These ideas suggested a possible method of intervention, making use of the observed 'technoliteracy' of the twenty-first century student.

4. Method

In this section we will discuss the measures we took to address the issues raised as the impetus for our study.

4.1 *Reflections go 'on-line'*

In 2008 we moved from handwritten reflections [9] to reflections on-line. Our University CECIL system allowed for the setting up of a 'Personal Journal' where students write their reflection on-line as part of each assignment. As before, they are asked to reflect on their own learning, their difficulties, triumphs and any connections they have made. However, now the writing is saved and can be re-accessed on-line at any time. The lecturer/tutor has access to each student's journal and is able to write a response that can also be saved before the next reflection is added into the journal, creating a computer log of thoughts on each chapter of work completed. Ideally, the students read the tutor's advice or comments, act on them for next time and report on progress.

4.2 *Introduction of Web to Text (Computer Based texting)*

In response to the difficulties in retention, we investigated technological methods of communicating our interest in the student's issues and signaling that we cared about their attendance. Although all students can theoretically be contacted via email, in practice as Carnevale [10] indicates this is dependant on them accessing a computer. Secondly the time delay can be significant.

> As some students reduce their use of e-mail in favor of other means of communication, colleges are trying new technologies to reach them. Among the new techniques: Cell-phone Text Messages. Students live and die by their cellphones [10].

In response to this problem, the TFC Maths tutors applied to Vodaphone to set up a computer web to text system that was trialed in the second semester of 2008. It allowed for instant contact with most students, and it was hoped that it would promote early intervention in problems with attendance before they could escalate into large irreversible issues. Some success in 2008, even beginning at the halfway point of the programme, encouraged us to put the system in place for the beginning of the 2009 year.

4.3 *The Web to Text System*

Vodaphone, when contacted, sub-contracted the work to Bulletin Messenger. 'Bulletin Messenger' is an intra-company messaging tool, developed by Bulletin Wireless Ltd, to be used from a small business or corporate's intranet.

The features of this messaging tool included:

- sending messages as an email to and receiving responses from standard mobile phones using a maximum of 160 characters per message.
- message tracking.
- centralised address books.
- standard web browser access.
- reply functionality delivering return messages back to the originating person.
- one account per organization and messages are free for the students to receive on their mobiles.

The benefit for us was:

- the time saved in communicating with students.
- being able to contact multiple mobiles with one message.
- the fact that there were no changes required to the current computer network setup, just simple browser access.
- it was quick and simple to use.

There was a start up cost of $199NZ, with an on going monthly cost of $65NZ for three logins. One for us (three lecturers in TFC Mathematics have access using one login), one for the IT person in the department and one spare for another course to use. Each text message cost us 17cents.

4.4 *Administration and Use of The Web to Text System*

Before the semester began, a database of cell-phone numbers for all of the students was compiled. To contact a student by text, it is only necessary to type their name in the 'To' line of an email. In the first weeks of the semester, a message in a friendly, concerned tone was sent to any student who missed a class, together with a request for a reply. After the first two weeks of intensive texting, there was an improvement in attendance, but as expected, the mid-semester break upset this pattern. However, the missing students became much better at replying and at explaining their absences. A complete record of all messages sent and received is stored and available for checking on the Bulletin Messenger website. A summary of the numbers of texts sent and replies given, for semester one TFC students for 2009, can be seen on Table 1 below.

Table 1: Number of texts sent and received for first ten weeks of Semester 1 2009.

Semester One 2009 Week	**Number of Texts sent to TFC students**	**Number of replies received by reply text.**
Week 1 (2/3/09 – 6/3/09)	27	8
Week 2 (9/3/09 – 13/3/09)	24	21
Week 3 (16/3/09 – 20/3/09)	8	7
Week 4 (23/3/09 – 27/3/09)	10	5
Week 5 (30/3/09 – 3/4/09)	8	2
Mid-semester break		
Week 6 (20/4/09 – 24/4/09)	11	10
Week 7 (27/4/09 – 1/5/09)	3	2
Week 8 (4/5/09 – 8/5/09)	4	2
Week 9 (11/5/09 – 15/5/09)	4	3
Week 10 (18/5/09 – 22/5/09)	1	1

5. Results

5.1 *Improvements in student reflections*

From the first set of reflections on-line it was noticeable that there was an improvement in the quality of the content when compared to hand-written reflections of the past. Using Cecil appeared to have a positive influence on the requirement for a piece of formal academic writing and students were making thoughtful, relevant comments on their learning. The Personal Journal option can document on-going progress in the form of a valuable log. (see Appendix 1).

The facility for lecturer response allowed for the development of a conversation, and for the opportunity to address pastoral care issues, hopefully improving the student experience. It is possible to see changes in student attitude over the course of the semester where the reflections are used as a communication tool (see Appendix 2).

5.2 *Improvement in Retention (so far!)*

This year (2009) the TFC Mathematics programme had 222 students enrolled at the beginning of Semester One.

By the middle of the semester, there were still 215 students attending, and of these 210 sat the mid-semester test, a drop of 5.5% in participation since the beginning of the semester. At a comparable time in Semester One 2008, we had a drop of 6.6. % in participation. This is a small but encouraging improvement, and we continued to monitor retention during the semester with the final results, in comparison to 2008, shown in the table below.

Table 2: TFC semester one retention figures:

	TFC Maths enrolment	Number sitting final exam	Number of students lost	% retention
Semester One 2008	206	175	31	85.0%
Semester One 2009	222	197	25	88.7%

The administrative difference has been the use of computer texts to students at the first sign of non-attendance in mathematics.

6. Summary

Foundation programmes like TFC offer the opportunity for institutions to tap into a valuable pool of non-traditional students. The interest in such programmes is growing, diversity in the student body is increasing and the lecturers responsible for teaching the students need to keep pace with the demands of a technologically aware clientele. The technology has been shown to be useful in maintaining personal contact with students via text messaging, and in developing support via electronic conversations on Cecil.

> 'The tenacity shown by students, often in the face of compounding negativity, was positively correlated to connections made with teaching staff. An awareness by teaching staff of personal impinging factors appeared to create definitive turning points for individuals, if staff were actively engaged in pastoral as well as academic assistance.' [11, p.25]

Although it is not obvious that there is a major impact on retention at this stage, it is apparent that attendance at lectures is more consistent. By providing opportunities for students to communicate more with their lecturers, it is perhaps possible to alleviate some of the issues in the affective domain between attitudes and emotions. Conversational support is a non-threatening and technologically possible strategy to enhance the experience of foundation students in a university environment.

8. References

[1] Parnell, S. M. (2005). Do student reflections increase undergraduate mathematics learning? In M. Bulmer, H. MacGillivray and C. Varsavsky (Eds.) *Proceedings of the Fifth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning (Delta 05),* Fraser Island, Australia, 95-102.

[2] Wong, K. Y. (2002). Helping your students to become metacognitive in mathematics: A decade later. Published online, April, 2002. Available from http://intranet.moe.edu.sg/maths/Newsletter/FourthIssue/Vol2No5.html (accessed 13 August 2009)

[3] Gootenboer, P. & Hemmings, B., (2007). Mathematics performance and the role played by affective and background factors. In M. Walshaw (Ed.) *Mathematics Education Research Journal*, 19(3), 3-20.

[4] Anthony, G., (2000). Factors influencing first year students' success in mathematics. In Martin Harrison (Ed.) *International Journal of Mathematical Education in Science and Technology*, 31(1), 3-14.

[5] D'Arcy-Warmington, A., (2005). Recreating first mathematical impressions into lasting mathematical impressions. In M. Bulmer, H. MacGillivray and C. Varsavsky (Eds.) *Proceedings of the Fifth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning (Delta 05),* Fraser Island, Australia, 171-178.

[6] Seabright, V., (2005). Reflective practice in adult numeracy professional development. In M. Horne and B. Marr (Eds.) *Proceedings of the Adults Learning Mathematics (ALM) 12th Annual International Conference,* Melbourne, Australia, 232-236.

[7] Cooper,N., Ellis, B. & Sawyer, J., (2000). Expanded future opportunities provided by a bridging course at a regional university campus. *The Fourth Pacific Rim – First Year in Higher Education Conference: creating Futures for a New*

Millennium, Queensland University of Technology, Brisbane, 5-7 July. Available from: http://www.fyhe.qut.edu.au/past_papers/abstracts/CooperAbstract.htm (accessed 28 May, 2009)
[8] Varsavaky, C., (2007). The experience of Australian mathematics and statistics undergraduate students. In A. D'Arcy-Warmington, V. Martinez Luaces, G. Oates, & C. Varsavsky (Eds.) *Proceedings of the Sixth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning (DELTA 06)*, El Calafate, Argentina 169-178.
[9] Parnell, S. & Statham, M. (2007). An Effective preparation for tertiary mathematics. In Martin Harrison (Ed.) *International Journal of Mathematical Education in Science and Technology*, 38(7), 869-879.
[10] Carnevale, D., (2006). E-mail is for old people. *The Chronicle of Higher Education,* Published online, October 6, 2006. Available from: http://chronicle.com/free/v53/i07/07a02701.htm (accessed 22 May, 2009)
[11] Morgan, J., (2004). Challenges for educators in meeting the needs of students bridging into tertiary education. In V. Harford & B. Kiernan (Eds.) *Proceedings of the Fourth Conference of the New Zealand Association of Bridging Educators,* Auckland, New Zealand, 21-30.

9. Appendices

Appendix 1 A Student's reflections as a log. (unedited)

Reflection One 16/03/2009
I have really enjoyed coming back to mathematics and found chapter 1 at just the right level to begin with. Many of the things we covered (such as operations with fractions and standard form) I was very rusty on but was able to recall and work out problems using these aspects of maths quite quickly. The work required me to think and I had most trouble just working out how to solve word problems when they were not telling you how to go about getting the answer, but most of the time I managed to figure it out and did not feel out of my depth at any stage. I did appreciate coving how to get prime factors and tell if a large number is a Prime number very much as it saved me so much time. So far really looking forward to more maths.
Reflection Two 30/03/2009
This latest section on percentages, ratios and proportion has been really useful for me. Throughout my education I have been talked to about and given problems involving ratios, and yet I never really understood what they were and how to use them. It was very satisfying to be able to read a word problem involving simplifying or using a ratio and know what they were asking and what I needed to do to get the answer. Percentages I have always felt reasonably confident with, but learning GST was a big thing- as I felt very similarly about GST as I did about ratios. I am enjoying the coursework and the content, and while some of it feels very easy and I would like to extend the level- I find there is enough work to do as it is and I always have something to revise or solve or go over so boredom is never near. Still finding maths is one of my favourite subjects and doing the worksheets and exercises have become a comforting way to relax for me.
Reflection Three 6/05/2009
The Metric System was not a topic I can say I was overly enthused about at the beginning. I have never got as much enjoyment out of measuring lines or edges or

height and working out areas and volumes as I have in the past in topics on algebra or proportion. It has been a thankfull surprise then that I have actually had a lot of fun doing this section of the course and I had much less difficulty learning the various fourmulae needed for calculating area, volume and capacity. It turned out that what might seem one of the easiest processes (converting from one unit of measure to another) was actually the one that provided the most trouble and resulted in mistakes. Learning the best way to covert from square meters to square kilometers or cubic centimeters to cubic milimeters is something I can see that if I had been taught earlier in my mathematics lessons I would have put to great use in my time a school and saved myself many lost marks in exams. I still struggle to convert answers for questions like 3C in the assignment into time that involves years,months,weeks,days,hours, ect and would probably benefit from some examples or methods to continue to practice with. Apart from that I continue to look forward to my math tutorials very much and feel keen to continue with expanding my knowledge of it - hopefully into next year also. Just wanted to add that I offer my greatest sympathies Sheena for your brother and it really is nice to have you back.

Reflection Four 20/05/2009

Introduction to algebra was a really good refresher chapter and I got much enjoyment out of expanding and factorising expressions containing variables. It did feel too basic and easy at times, but given the diversity in mathematical experience within the group i thought the section catered well. I did not find myself bored at any time- as it seemed the easier the problems given, the more there were to do! The extension and extra practice worksheets handed out within the tutorial were very handy for practice and highlighted shaky or absent parts of my knowledge on algebra- especially concerning how to expand expressions that were to the power of three and above (I only struck gold in the co-lab itself which gave great delight but unfortunately was not long enough for me to finish the exercise). I find most of all discovering patterns or relationships with algebra the most exciting and fufilling part- understanding why the operations or methods I use work interests me just as much as having the correct result at the end. Am keen to continue learning.

Appendix 2 – A student's reflections as a communication tool. (unedited)

Reflection One 16/03/2009

During the first 2 weeks of this course I felt degraded while doing the math work. The work we have been doing is the equivalent of what I learnt in year 5 and 6 of primary school. It was not challenging aside from the collab assessment we did because the group I was in did not work well together and the first who thought they were leading it was useless. I feel that this course is not challenging at all and I hope that it will at the end of this maybe go past my year 11 math work. I showed the homework I was given to my friends and family, and they all laughed at it, wondering how I could have been put into this course. I don't want to be in it, but it is a necessity for me to go into my Bachelor of Science or Bachelor of Architectural studies next year. I am also doing a course through open polytechnic while studying the TFC just so I dont die from boredom.

Reply Reflection One 18/03/2009

I think your challenge is to develop the best study skills and test techniques that you can, so that you can achieve the best grades. I am sorry that you didn't achieve university entrance last year, and that you don't see this programme as the opportunity that it is. Try to keep an open mind and be as positive as you can.

Reflection Two 29/03/2009

Over the last week or so, i have been contemplating the differences between PSP's and Nintendo DS's while in class because it seems more logical then the math, as I want to buy one soon. I worked out that PSP's are better then DS's because they have a better quality screen and RPG's are better played on them. If i did buy one it means that I could kill the time between classes at uni, instead of counting down the minutes till the 50 hours of math I dont want to go into. The work we have been doing is horrific and at least this week I was prepared for the collab assessment because the first one we never actually got told that it was a collab or that we had a time limit of 25 mins for each part. That would have been helpful instead of emailing everyone the day before at 5pm.... Thanks for that. If i didn't have to do this course then I would much rather spend my time working so that I could pay for my entire first year doing Architecture. Hopefully in the next few weeks it'll get more difficult...

Reply Reflection Two 2/04/2009

I have been pleased with your apparent interest in class in some of the work Kyle, and in the way you are sharing some of your knowledge with classmates. Your test results were below the class average, and I would be sad to see your potential grades deteriorate. Next time, try to write a reflection on your learning, think about the questions you didn't get right, why that might be. What you have written here doesn't address the issue and is inappropriate at university level. Your challenge is still to keep positive and to score the best grade that you can, as in all your subjects. Moira

Reflection Three 4/05/2009

Over the last two weeks I have noticed the work gradually becoming more difficult and I am enjoying this. The assignment that we had was challenging and I hopefully did well in it. The new work we have, Algebra, is what I enjoy in math and am hoping it will gradually get more difficult. I did have trouble in the assignment converting a few things like metres pre second to kilometres per hour but that may have been due to me being tired while doing it. I am happy with the work we are going into and hope my grades will get higher.

Reply Reflection Three 7/05/2009

I think you just need to clarify your thinking Kyle. Try to 'self-talk' the processes so that you are not remembering things ,but rather arguing the logic. Units are always a bit of a challenge, but I hope that you enjoy a challenge. You have been very responsive in class and I hope that the algebra assignment will be enjoyable.

Reflection Four 18/05/2009

I enjoyed the work we have been doing in class over the last few weeks. I have always enjoyed Algebra and find it a challenge. I do admit that the one question at the end of the assignment stumped me and was glad that I could get your help. I do find a few of the questions in the book to be quite easy but I realise that looking back, not paying enough attention to sign changes in equations will result in my end score being lower. I can't wait for this work to get slower harder because we are now into the area of math that I enjoy most. Working out the unknowns and building equations from information given. I must keep my eyes out for change in signs in equations and also for thinking harder about the obvious answers.

Reply Reflection Four 21/05/2009

Yes, you are being let down by aiming for speed over attention to detail. I don't want to be in a building you design that collapses because of an algebraic miscalculation. You need this tool kit of skills to be really useful, and for that to happen you need to sort out your common errors (which you have started to do) and mentally flag them for extra attention.

Promoting contingent teaching and active learning in lectures with classroom response systems

Sepideh Stewart* and Wayne S. Stewart
Department of Mathematics, Mathematics Education Unit,
Department of Statistics, The University of Auckland, Auckland, New Zealand

Abstract: It would be almost impossible to measure the level of students' understanding of the material presented in any particular lecture. However, it is possible to have a sense of students' thinking by means of creating a feedback loop in which a question is asked or an issue is raised. Each student then indicates his/her response from a set of options provided on the slide by using a personal data entry device sometimes referred to as a "clicker". This process encourages responses from all students and can be used in a completely anonymous fashion. The benefits could include exploring students' prior knowledge as the course starts; raising question as a trigger for discussion or raising interest; giving the lecturer the ability to assess levels of understanding of a particular concept. Moreover the instant response will allow the lecturer to identify problem areas immediately and thus revisit those areas or adjust the pacing of the course. This paper highlights authors' experiences with the use of classroom response systems in creating an environment that fosters contingent teaching based on students' needs, helps to support, deepen and enhance learning through an innovative interaction between students and the lecturer.

Keywords: Contingent teaching, classroom response systems, interactive classroom situation.

1. Introduction

The architecture of the human mind has been described as multidimensional utilising both general-purpose and specialised processes [1, cited in 2]. The general-processes are organised from a directive-executive function (DEF) which makes mental and behavioural goals and monitors their progress until they are accomplished. The specialised processes "refer to mental operations and problem solving skills that are suitable for the handling of different types of information, relations and problems" [2, p. 7]. Demetriou believes domain-specific thought systems satisfy criteria such as orientation in space, communication, types of relations biased to different symbol systems. In a study they identified the following six domains of thought that satisfy these criteria: "categorical, quantitative, special, causal, verbal, and social thought" [3 cited in 2, p. 7].

Savion and Middendorf [4] also acknowledge the importance of the categorical domain-specific thought system. By which they mean that the brain does not record information like a tape recorder. Instead, it processes information by reducing it into meaningful portions or categories. In Skemp's view "to understand a concept, group of concepts, or symbols, is to connect it with an appropriate schema" [5, p. 148]. He believes that often we don't know what an object is, because we cannot adequately categorise it. Thus, categorisation is a first step in connecting a new idea with a suitable schema. However categorisation is not sufficient, as "there are some states which we can correctly classify by an available concept, and for which we have an appropriate schema; but we cannot make the necessary connections between

*Email: stewart@math.auckland.ac.nz

the two, so we are unable to construct any path from these states to chosen goal states" [5, p. 146]. This may happen when the instructor introduces new concepts. Each person has unique schemas, new ideas have to be connected with these and provided that the new idea is categorised and realised, a learner would more easily make the necessary connections and hence understand. Once the categories are in place the DEF and domain-specific thought processes can function by suitable interaction within the learning environment.

1.1. *Classroom response systems and contingent teaching*

Based on the above theoretical framework, it only seems sensible to think that our teaching methodology ought to originate from the way the mind functions. One implication for teaching is that, the information must be delivered in suitable categories and portions which are recognised by the learner and connected to an existing schema. The challenge is how to best draw "students into the world of the lecturer" [6, p. 2] in harmony with the way the brain operates. As Mason declares the lecturer needs to re-structure students' attention to her/his world. This can be achieved by discovering what it is that students are attending to, through posing appropriate questions at appropriate times. Based on this information the teaching can be adjusted to suit students' needs. This is the heart of contingent teaching, which happens to some extent in any good lecture. Since categorisation is not sufficient for understanding, the instructor may consider using a formative method of assessment to instantly access students' state of understanding.

Classroom response systems are technologies that allow instructors to rapidly collect and analyse responses during class [7]. They are often used to create an environment that helps to support, deepen and enhance learning through promoting interactivity in large lectures. This is achieved by means of a feedback loop in which a question is asked or an issue is raised (typically via PowerPoint). Then each participant indicates their response from a set of options provided on the slide by using a personal handheld wireless device sometimes referred to as a "clicker". Specialised software on the computer then processes the acquired data and displays the results via a data projector [8]. This process encourages responses from all participants and can be used in a completely anonymous fashion. It means the shy students or weaker students who otherwise would be reluctant to participate can remain anonymous in a chosen activity.

These systems have been and still are used in business, as well as at all levels of training and education. According to Banks, "the technology, in itself, does not offer some wonderful new "magic bullet" that will offer learning gains simply by its adoption. It can certainly provide novelty and fun for all participants, but must be used within the context of the teaching and learning process for its full promise to be achieved" [8, p. ix].

Researchers have examined many aspects of this technology from its history, effects, pedagogy and implications to some practical lessons from four years of using clickers in every lecture of a large class [9]. For example, Abrahamson [10, p. 1] poses the question that "why, for an idea apparently more than forty years old, it took this long to happen!". Cutts [9] describes his experience in implementing clickers in a computer science classroom solving small computing problems and translation of those solutions into programming code. Students responses showed that they were more awake and attentive in classes and as classroom response systems break up a lecture into manageable chunks, students report that they appreciate the change in pace. In a study by Durbin and Durbin [11], the authors presented a case study of implementation of clickers in an engineering tutorial environment. They found that student response was consistent and generally positive at both the beginning and end of the semester and the opportunity for live (anonymous) assessment proved useful from an instructional perspective, despite some limitations of the system. Moreover, their research showed that the new tutorial format appeared to encourage discussion after each question was concluded, driven in part by the fact that a bar graph of responses to the multiple-choice questions was displayed.

Whether for fun, graded quizzes, games or class engagements in an activity, certainly clickers can be used in many different settings and for different purposes [7]. However, in line with the functionality of the brain it is possible to implement contingent teaching and pose a diagnostic question or questions after a topic has been introduced. Learners respond through the clicker technology and depending on their response a new path through the presentation can be made. This will optimise their categorisation and subsequently understanding.

In this paper we will illustrate the use of this technology and will give some insights on how contingent teaching can be employed to find out students' state of thinking and act accordingly to maximise their understanding.

2. Method

This research is a report based on the authors' experiences with the use of classroom response systems at the University of Auckland in 2009 in one semester. The participants were two groups of second year students. The first group were general mathematics students and the second group were Biometry students (150 and 130 students respectively). These two courses were part of a trial run for the Qwizdom system at the university. The teacher console and radio frequency (RF) technology were two deciding factors in choosing the Qwizdom system. The clickers, which act in much the same way as a hand-free phone, were issued to the students for the semester. As part of the implementation of contingent teaching in each lecture, diagnostic questions were asked through the PowerPoint slides at appropriate places. The answers given by students were retrieved dynamically by the use of clickers and thus instant feedback was achieved. A decision was then made in accordance with a special designed criterion as to whether a new path of teaching should be taken or not. This meant that the content displayed to the students was contingent on their needs. So in one lesson there were many more PowerPoint slides which did not necessarily get used. Although the routes were contingent upon student response to the questions, each route was created with the goal to move students to the main aim of the concept taught. The clickers (Q2's) were used in response to multiple-choice questions, true/false, or yes/no statements. Graphs of responses were then made visible, giving students the opportunity to see how they were doing in comparison with the rest of the class. The "teacher remote" (Q5), enabled the instructor to create spontaneous questions, choose when to show results to students, control the PowerPoint slides and to roam freely. Figure 1 illustrates the contingent movement through the slides in any particular lesson.

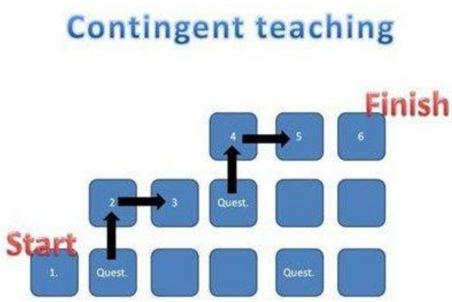


Figure 1. A dynamic contingent slide illustration.

3. Results

Since the software captures students' responses, the data is made available to the lecturer at the end of each lecture. The following data based on the experiences of the authors consist of examples of contingent teaching; general day to day questions in lectures; spontaneous questions and finally students' attitude.

3.1. *An example of using contingent teaching with a second year Biometry class*

A series of PowerPoint slides were used to develop the notion of a standardised histogram. Knowing that this might be a difficult concept a contingent paradigm of slides was prepared using the hyperlink facility in PowerPoint. After a typical presentation was given the second-name author projected the following question and obtained the results shown using Qwizdom Q2 clickers. In response to the question: "Do you understand the development of the standardised histogram?", 22% gave a *No* response (see Figure 2). Therefore, a more concrete development was shown by way of the "Fix the problem" icon.

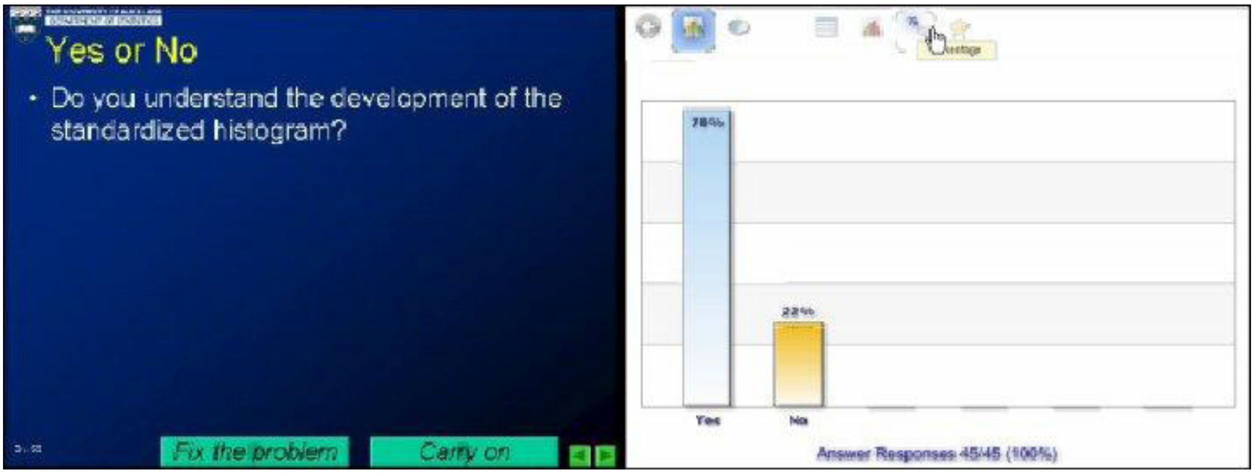


Figure 2. First diagnostic question.

The hyperlink took the students to the following slide and subsequently through another hyperlink to an Excel sheet where the problem was then developed step by step in much more detail (see Figure 3).

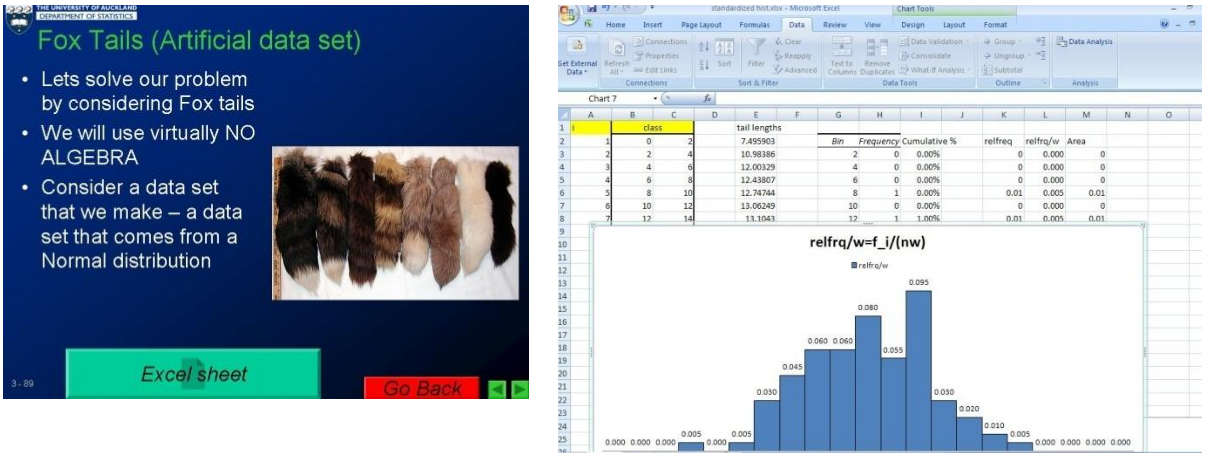


Figure 3. First slide of the contingent explanation.

After the further explanation in more concrete terms the *Go Back* button was clicked and the students were asked the original question again.

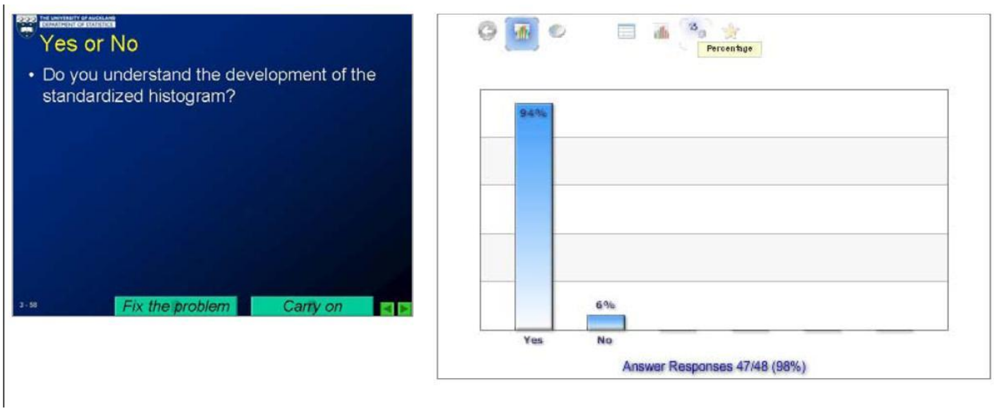


Figure 4. First diagnostic question reposed.

The results showed that this time only 6% did not understand the standardised histogram development (Figure 4). The mood of the class was clearly excited in a favourable way and many commented on how effective it was for them. This kind of intervention is clearly active and engages the student in a way that puts their needs first and allows them to see a presentation that takes their weaknesses into account. Thus students' level of understanding can be constantly diagnosed and monitored. Ideally, all contingent branches are tailored to the categorisation that the students' minds can cope with.

3.2. *An example of using contingent teaching with a second year Mathematics class*

In recent years many mathematics education researchers have been concerned with students' difficulties related to undergraduate linear algebra courses. The extensive evidence by Stewart [12] revealed that the majority of students had major problems understanding the concepts that are the essence and foundation of a linear algebra course.

The study also revealed that the possession of a rich schema allows the mathematician to tie together all the pieces of his knowledge in a way that the student may not be able to [13]. To test students' existing schema, in summer 2009, before introducing the topic of differential equations, the first-named author asked the class whether they were able to recall any material from the linear algebra section taught right at the beginning of the course. Hence, she posed several diagnostic questions on students' understanding of linear independence, linear combination, span, etc. The results showed (see Figure 5) almost half of those who responded, admitted that they could do certain calculations but did not understand the theory.

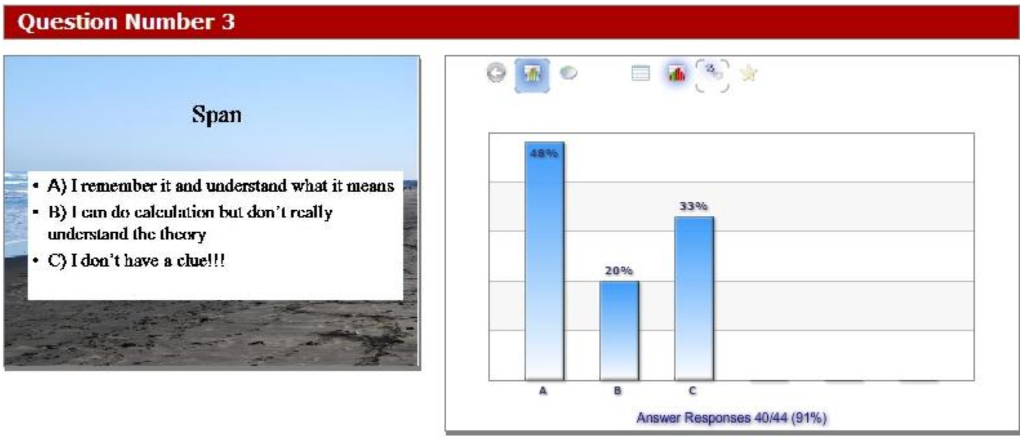


Figure 5. Students responses to the term span.

In another occasion the first-named author presented the class with a slide showing a student's responses in defining the linear algebra terms: Invertible matrix, eigenvector, basis, spanning set, linear combination, subspace, linearly independent. Then she asked the class whether they felt the same way. Surprisingly, 73% of the students said yes (see Figure 6). Contingent on this observation, she made special effort to revise the concepts in some detail, as the lessons approached concepts such as linear combination of the solutions of the DE's and in the context of Wronskian.

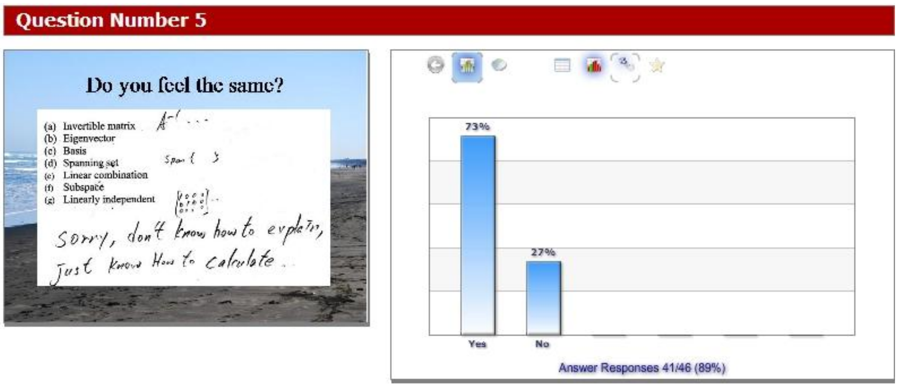


Figure 6. Class responses to feeling the same about the definitions in linear algebra.

3.3. General usage of clickers on a day to day basis in lectures

On a day to day basis it was useful to use clickers to find out students' previous knowledge; how they felt about a particular lesson and where their difficulties laid. This was particularly important in the lectures where obtaining such information would not be otherwise feasible. It was also possible to check if the students have done tasks which were previously assigned to them. Figure 7 shows mathematics students describing their level of understanding of the DE's at the end of the lecture. This valuable and instant information, gave the lecturer helpful insights to be used in preparation for the start of the next lecture.

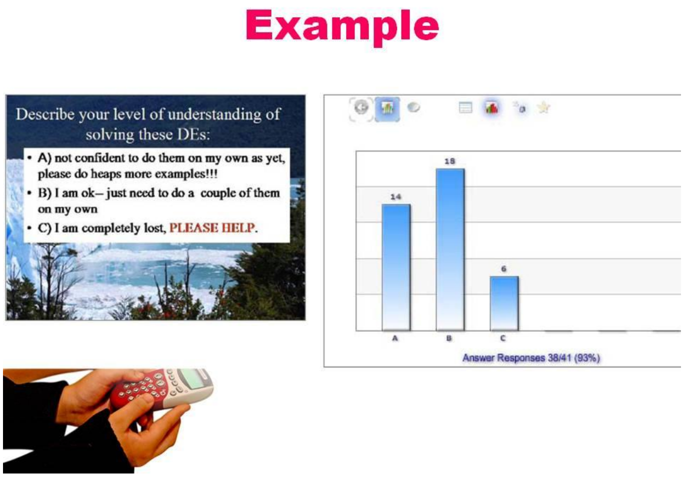


Figure 7. Students' level of understanding of DE's at the end of one lecture.

It was also useful to ask spontaneous questions. It is not possible to always plan questions beforehand. The spontaneous questions facility in Qwizdom Q5 "teacher remote" allowed the lecturer to pose questions at any time, in response to unforseen situations. Figure 8 illustrate one such question.

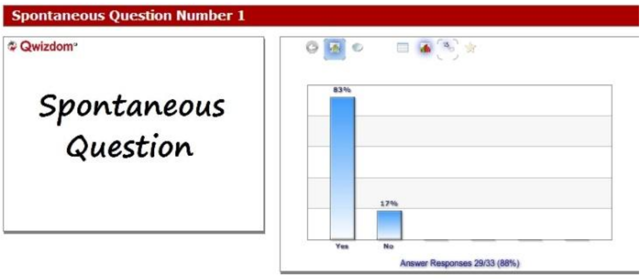


Figure 8. A spontaneous question was posed by the lecturer on the spot.

Overall students from both classes gave positive feed back concerning the use of clickers in lectures. It was interesting to see that many students liked being ranked within the class.

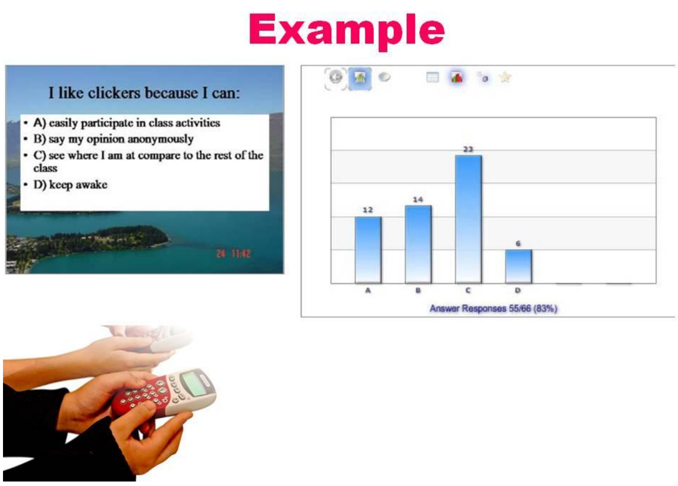


Figure 9. Students overall opinion on clickers.

4. Discussion and Conclusions

According to the report presented, the learning theories may benefit from a careful understanding of the way the mind functions and absorbs information. In such a model the study of the brain is essential to how we teach and learn. Ignoring this connection may result in poor uptake of new concepts. In regard to deep understanding of concepts, Skemp [5] shows the necessity of categorisation and connecting the pieces of information to the existing schemas to build new and richer schemas. The classroom response system can allow the instructor to have access to the learners' state of understanding and create a way forward in implementing the brain centered paradigm of teaching. This is possible through PowerPoint hyper linking to construct a contingent teaching slide series, based on students' needs. Though this can be accomplished by a single instructor a complete implementation may be time consuming and would be more easily facilitated by a teaching team. Although research on the use of clickers is on the rise and many universities are now using them, it would be helpful to see more research aligned with contingent teaching. Our experience with clickers has been positive in the sense that our lessons were more engagingly active and less passive. There was much interest shown by students as their presence and diverse needs were not ignored. As instructors we left the lecture rooms having a better idea of how well the lessons were received and understood. We had an up to date knowledge and sense of our students' level of learning and thus were better able to prepare for the next lecture**.**

5. References

[1] J. B. Carroll, *Human Cognitive Abilities: A Survey of Factor-analytic Studies*. New York: Cambridge University Press, 1993.

[2] A. Demetriou, *The architecture and development of mathematical thought. Plenary address*, in *Proceedings of the 33rd International Conference for the Psychology of Mathematics Education*, Thessaloniki, Greece, 2009, pp. 5-22.

[3] A. Demetriou and S. Kazi, *Unity and modularity in the mind and the self: Studies on the relationships self-awareness, personality, and intellectual development from childhood to adolescence.* London:Routledge, 2001**.**

[4] L. Savion and J. Middendorf, Enhancing concept comprehension and retention. *National Teaching and Learning Forum, 1994, 3*(4), 6–8.

[5] R. Skemp, *Intelligence, learning, and action* in *The Psychology of Learning Mathematics*, The Gresham Press, UK, 1979.

[6] J. H. Mason, *Teaching for flexibility in mathematics: Being aware of the structures of attention and intention*, Quaestiones Mathematicae, Suppl.1(2001), Supplement, pp. 1- 15.

[7] D. Bruff, *Teaching with Classroom Response systems: Creating Active Learning Environments.* Published by Jossey-Bass, USA, 2009.

[8] D. A. Banks, *Audience Response Systems in Higher Education Application and Cases*. Information Science Publishing, 2006.

[9] Q. Cutts, *Practical lessons from years of using an ARS in every lecture of a large class*, in *Audience Response Systems in Higher Education Application and Cases*, D. Banks, eds., Information Science Publishing, *2006*, pp. 65- 79.

[10] A. L. Abrahmson, *A brief history of networked classrooms: Effects, cases, pedagogy, and implications*, in *Audience Response Systems in Higher*

Education Application and Cases,D. Banks, eds., Information Science, Publishing, pp. 1-25.

[11] K. Durbin and S. Durbin, *Anonymous polling in an engineering tutorial environment: A case study*, in *Audience Response Systems in Higher Education Application and Cases*, D. Banks, eds., Information Science Publishing, *2006*,pp. 116-126.

[12] S. Stewart, *Understanding linear algebra concepts through the embodied symbolic and formal worlds of mathematical thinking.* Unpublished doctoral thesis, Auckland University, 2008. Available from: http://hdl.handle.net/2292/2912.

[13] S. Stewart and M.O.J. Thomas, *A framework for mathematical thinking: The case of linear algebra*, International Journal of Mathematical Education in Science and technology, 40(7) (2009), pp. 951-961.

Teaching the concept of kurtosis in introductory statistics courses using *Mathematica*: Searching for platypuses and kangaroos beneath the cloth of Table Mountain

Paul J. van Staden[a,*] and M.T. (Theodor) Loots[a,b]

[a]Department of Statistics, University of Pretoria, Pretoria, 0002, South Africa
[b]*TransUnion, Analytic & Decision Services, Pretoria, 0040, South Africa*

Abstract: The shape of a statistical distribution is described via the skewness and the kurtosis of the distribution. Like location and spread, the concept of skewness is easily taught to first-year students in introductory statistical courses. Unfortunately kurtosis is not a simple characteristic in that it is related to both the tails and the peak of the distribution under consideration. The topic of kurtosis is therefore avoided in both introductory courses and textbooks. In this paper we briefly present the history of kurtosis. We then focus on Pearson's measure of kurtosis and clarify two misconceptions regarding its use and interpretation. Using a new function in *Mathematica*, we show how the concept of kurtosis can be illustrated graphically to students and combine our proposed technique with a recently proposed method for calculating the excess tail and peak areas. We end with some final conclusions.

Keywords*:* Gaussian distribution; generalized secant hyperbolic distribution; Kotz-Seier method; kurtosis measures; mesokurtic, platykurtic and leptokurtic distributions; Pearson's kurtosis moment ratio; statistical education; tail and peak areas; uniform distribution;

MSC2010 Subject Classification: 97K30; 62E10

1. The birth and history of kurtosis

Kurtosis was first defined by the 'father of kurtosis', Karl Pearson, [1], in June 1905 as a measure of departure from normality. Kurtosis is now more than 100 years old, yet it still remains an enigma for many scholars, students and educators in statistics. For an excellent account of the history of kurtosis, readers are referred to [2,3]. We will only briefly highlight the most important 'birthdays' below.

Pearson made significant contributions to the mathematical theory of evolution. In his research he realised that there were often significant physical differences between actual frequency distributions and the Gaussian (normal) distribution. It is well known that the Gaussian distribution

- has its mode equal to its mean,
- is symmetric (in effect, its skewness moment ratio is zero),
- and has a kurtosis moment ratio equal to 3.

[*] Email: paul.vanstaden@up.ac.za. Web: www.up.ac.za/pauljvanstaden.

Pearson, [1], noted that symmetric frequency distributions with mode equal to mean often have kurtosis moment ratios different from 3. Accordingly he labelled a symmetric distribution

- *mesokurtic* if it has a normal kurtosis moment ratio,
- *platykurtic* if it is more flat-topped and hence has shorter tails than the Gaussian distribution,
- and *leptokurtic* if it is less flat-topped and thus has longer (heavier) tails than the Gaussian distribution.

In 1927 'Student', [4], gave an amusing mnemonic for the last two terms above. As illustrated in (Figure 1), platykurtic distributions are squat with short tails like the platypus, while leptokurtic distributions are high and have long tails like kangaroos, which are known for 'lepping'.

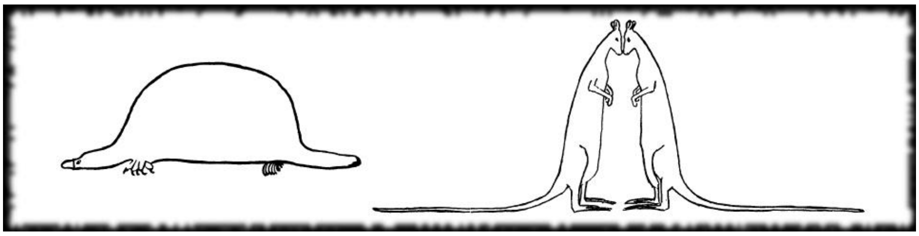


Figure 1. A short-tailed platypus and two kangaroos with long tails (taken from [4]).

Since the 1920s Pearson's measure of kurtosis appeared, together with measures for location, spread and skewness, in most statistic textbooks. However, in the 1940s researchers – see for example [5–7] – started to point out misconceptions about the explanation and interpretation of kurtosis in introductory textbooks. For instance, in some textbooks such as on p. 165 in [8] – see [7] for more listed texts – it was explained that, compared to the Gaussian distribution, a distribution with density curve higher (lower) in the region of the mean will have higher (lower) kurtosis. In Section 3 we will clarify this misconception and also discuss another misconception related to terminology as (mis)used by many statistical educators.

In 1964 van Zwet, [9], defined a criterion for ordering symmetric distributions based on their cumulative distribution functions and proved the following orderings for symmetric distributions:

(1) U-shaped $\leq_S$ Uniform $\leq_S$ Gaussian $\leq_S$ Logístic $\leq_S$ Laplace

(2) Logistic $\leq_S$ Cauchy

Since the 1980s, various measures of kurtosis have been defined, including a statistic robust to outliers, a measure based on octiles, a measure defined as the ratio of spread functions and the *L*-kurtosis ratio – see [10–13] respectively.

2. What is kurtosis?

Well, it depends who you ask. The term kurtosis comes from the Greek word 'kurtos' (curvature). Kurtosis is thus an apt name in that its interpretation is linked to the curve of the density function of a distribution. But there has never been general consensus among the statistical community regarding the interpretation of kurtosis and its role in describing the shape characteristics of a distribution. This is highlighted if one looks at some of the titles of published articles on kurtosis:

- 'Is kurtosis really "peakedness?"', [14]
- 'Kurtosis measures bimodality?', [15]
- 'What is kurtosis?', [16]

One of the main debates on kurtosis centred on whether kurtosis is related to the peakedness of a distribution under consideration, the tails of the distribution or both the peak and the tails. It is now well accepted in the statistical literature that kurtosis is indeed related to both the peak and the tails – see [16].

Another controversial question was raised in the 1970s – see [14,15,17]. Should measures of kurtosis, apart from measuring the peak and tails of a distribution, also be able to detect bimodality? The current viewpoint is that they do not have to detect bimodality – see the explanation in [18].

In terms of a definition, we concur with Balanda and MacGillivray on p. 111 in the abstract of [19] that

> 'it is best to define kurtosis vaguely as the location- and scale-free movement of probability mass from the shoulders of a distribution into its center and tails and to recognize that it can be formalized in many ways.'

3. Pearson's measure of kurtosis

Pearson, [1], provided the first measure of kurtosis. He suggested measuring kurtosis with the standardized 4$^{\text{th}}$ central moment, also commonly referred to as the kurtosis moment ratio and given by

$$\alpha_4 = \frac{\mu_4}{\mu_2^2}, \tag{1}$$

where

$$\mu_r = E\left[(X-\mu_X)^r\right] = \begin{cases} \sum_x \left[(x-\mu_X)^r P(X=x)\right], & X \text{ a discrete variable,} \\ \int_{-\infty}^{\infty} (x-\mu_X)^r f(x)dx, & X \text{ a continuous variable,} \end{cases} \tag{2}$$

is the r^{th} central moment for a random variable *X*. In effect, μ_r is the r^{th} moment about the mean of *X*, $E(X)=\mu_X$. In this paper we will be focusing on continuous random variables and distributions. Note that $\mu_2 = \sigma_X^2 = Var(X)$ is the variance of *X*, while the skewness moment ratio (the standardized 3$^{\text{rd}}$ central moment) is given by

$$\alpha_3 = \frac{\mu_3}{\mu_2^{1.5}}. \tag{3}$$

For details regarding the above expressions for the mean, variance and skewness and kurtosis moment ratios, see [20].

If *X* is a Gaussian distributed random variable, then $\alpha_3 = 0$ and $\alpha_4 = 3$. The Gaussian distribution therefore plays a central role in moment theory as suggested in [1].

There exist two misconceptions regarding α_4 among many educators in statistics:

(1) α_4 is kurtosis.
(2) A random variable *X* with $\alpha_3 = 0$ and $\alpha_4 = 3$ is Gaussian.

Let us first consider Misconception (1). The kurtosis moment ratio is *not* kurtosis. Calling α_4 kurtosis is equivalent to stating that the mean *is* location or that the variance *is* spread. The mean is one of many measures of location, others being the mode and the median. Similarly the variance is a measure of spread as is the inter-decile range and the quartile deviation to name just two other measures. The kurtosis moment ratio is a *measure* of kurtosis. Many alternative measures of kurtosis have been developed since 1905. It is not the purpose of this paper to compare the properties of these various measures and we will only focus on Pearson's measure of kurtosis in the rest of the paper.

Consider next Misconception (2). When Pearson formulated the concept of kurtosis in 1905, he was focusing on departures from normality. Subsequently many more statistical distributions have been developed. In the left panel of (Figure 2) the density curves of the standard Gaussian distribution and a special case of the standard generalized secant hyperbolic (SGSH) distribution are plotted. The properties of the SGSH distribution are given in [21] and will be discussed in more detail in Section 6. Both plotted distributions are unimodal, have infinite support, are standardized to have zero mean and unit variance, that is, $\mu_X = 0$ and $\sigma_X^2 = 1$, and have $\alpha_3 = 0$ and $\alpha_4 = 3$. However, the special case of the SGSH distribution clearly departs from normality. Even distributions that are not unimodal or that have bounded support can have $\alpha_3 = 0$ and $\alpha_4 = 3$. This is illustrated in the right panel of (Figure 2) where the density curves of the double-gamma distribution with parameters $\alpha = 2.941$ and $\beta = 3.134$ (see [15]) and Tukey's lambda distribution with parameter $\lambda = 5.2$ (see [22,23] for details) are plotted along with the standard Gaussian distribution's density curve.

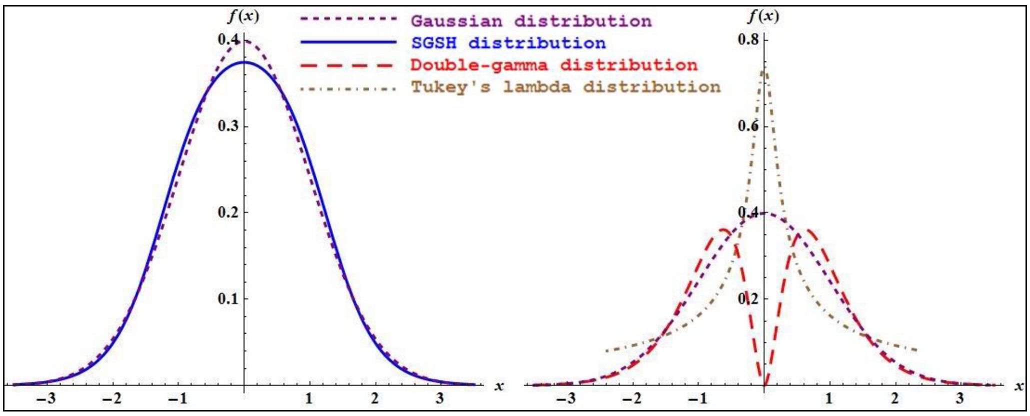


Figure 2. Density curves of standardized symmetric distributions with kurtosis moment ratios equal to three.

It should be clear from the above examples that a higher (lower) density curve in the region of the mean does not imply higher (lower) kurtosis as measured by the kurtosis moment ratio. The relationship between the density curve and kurtosis is complex and this is true for all measures of kurtosis, not just the kurtosis moment ratio.

4. Statistics Lecturer versus Undergraduate Student

It is Round 1 of the boxing match between the Statistics Lecturer and the Undergraduate Student, hereafter referred to as Paul[††††††††††] and Groentjie[‡‡‡‡‡‡‡‡‡‡] respectively. Groentjie has prepared well for the match through rigorous training in mathematics at school level, so he deals well with the various introductory statistical concepts and techniques delivered to his head by Paul (the latter is aiming for the gray matter inside Groentjie's head). But then Paul lets loose with a devastating uppercut (kurtosis), knocking the fragile student down.

A survey of introductory statistics courses at universities in South Africa, [24], revealed that the majority of BCom students at first-year level are not exposed to the concept of kurtosis. At some universities the first-year BSc students, who in general have a sounder mathematical background, come into brief contact with kurtosis when moments and moment generating functions are taught and it is mentioned to them that the expression in Eq. (1) is the kurtosis moment ratio. It should be noted that Paul is not currently lecturing first-year statistics in the Department of Statistics at the University of Pretoria (UP).

The concept of kurtosis is also conspicuously absent from the textbooks currently used in introductory statistics courses, see for instance [25,26]. In fact, [27] was the only relatively recent textbook that we could find that provides a brief discussion on the interpretation of kurtosis – see pp. 147–150. This textbook, which is out of print and no longer prescribed at any South African university as far as we could gather, also gives two measures for kurtosis, namely the kurtosis moment ratio (referred to as the coefficient of kurtosis) and an unnamed measure based on quantiles. This second measure was defined in [28] and appeared along with Pearson's measure in many statistics textbooks from the 1920s onwards.

Let us return to the boxing match. Recall that Groentjie was knocked down in Round 1. He was however not knocked out and he actually managed to survive Round 2 as well. Fast-forward to Round 3 and Paul is relentlessly punching Groentjie with concepts like 'heavy-tailed stock return data' and 'autoregressive conditional skewness and kurtosis' within the context of financial time series analysis (note that Paul currently lectures the third-year module on time series analysis at UP). The constant beating to the head leaves Groentjie dazed and confused and chances are that he will be either knocked out before the end of Round 3 or barely manage to survive the contest.

Now as any boxer will tell you, if you were knocked down in Round 1, then it is very difficult to fight back in later rounds. The avoidance of the concept of kurtosis at first-

[††††††††††] Paul is the first author of this paper.
[‡‡‡‡‡‡‡‡‡‡] Groentjie is an Afrikaans term for freshman, but he is not the second author of this paper.

year level creates problems for students in their later years in that they struggle to master more complex ideas in advanced undergraduate statistics courses at third-year level. An example of this in the context of financial time series analysis was given in the previous paragraph. A second example is in actuarial statistics, in which students must be able to estimate the parameters of a generalized loss distribution, such as the four-parameter Pareto distribution, using method of moments or method of percentiles estimation. These two estimation methodologies both utilize measures of location, spread, skewness and kurtosis. We therefore strongly advocate the teaching of kurtosis at first-year level. In Section 6 we will illustrate how this can be done using *Mathematica*.

5. Location, spread and skewness

Kurtosis is of course not the only important characteristic of a distribution. The other important moment and shape characteristics, namely location, spread and skewness, as well as the appropriate measures thereof, are covered in nearly all introductory statistics courses and textbooks. This is not surprising, since these concepts and measures are easy to teach and comprehend. For instance, graphs similar to those in (Figure 3) are typically used as visual aids for illustrating location, spread and skewness respectively.

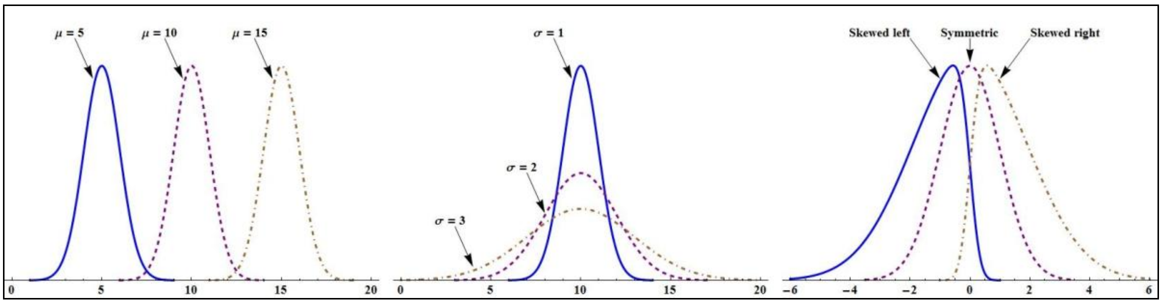


Figure 3. Graphs for illustrating the mean as measure of location, the variance as measure of spread and the concept of skewness.

6. **`Manipulate`** in *Mathematica* for illustrating kurtosis graphically

In Version 6 of *Mathematica* a new function named **`Manipulate`** was introduced. This function was updated in Version 7 of *Mathematica*, the version we used for creating the graphs in this paper. As explained in [29],

> '**`Manipulate[`***expr*, {*u*, u_{min}, u_{max}}**`]`** generates a version of ***expr*** with controls added to allow interactive manipulation of the value of ***u***.'

We propose illustrating the concept of kurtosis visually with *Mathematica* by manipulating an appropriate measure of kurtosis (or alternatively the kurtosis parameter) of a chosen distribution and, by doing so, changing the appearance of the plot of the distribution's density curve. One could use any applicable measure of kurtosis, but we suggest Pearson's measure of kurtosis, since other measures may require background knowledge that first-year students have not acquired yet. The chosen distribution must have the following properties:

- It is preferable that the mean of the distribution is zero.
- The variance of the distribution must stay constant if the kurtosis moment ratio is changed.
- The distribution must be symmetric.

- Short tails and heavy tails must be accommodated.
- It is preferable that the distribution has a single shape parameter.

The standard generalized secant hyperbolic (SGSH) distribution satisfies all the above criteria. It has density function

$$f_X(x) = c_1 \frac{e^{c_2 x}}{e^{2c_2 x} + 2ae^{c_2 x} + 1}, \tag{4}$$

where

$$a = \cos t, \quad c_2 = \sqrt{\tfrac{1}{3}\left(\pi^2 - t^2\right)}, \quad c_1 = \frac{\sin t}{t} c_2, \quad \text{for} \quad -\pi < t < 0, \tag{5}$$

$$a = 1, \quad c_1 = c_2 = \frac{\pi}{\sqrt{3}}, \quad \text{for} \quad t = 0, \tag{6}$$

$$a = \cosh t, \quad c_2 = \sqrt{\tfrac{1}{3}\left(\pi^2 + t^2\right)}, \quad c_1 = \frac{\sinh t}{t} c_2, \quad \text{for} \quad t > 0. \tag{7}$$

The cumulative distribution function is given by

$$F_X(x) = \begin{cases} 1 + \frac{1}{t}\cot^{-1}\left(-\dfrac{e^{c_2 x} + \cos t}{\sin t}\right), & -\pi < t < 0, \\ \dfrac{e^{\pi x/\sqrt{3}}}{1 + e^{\pi x/\sqrt{3}}}, & t = 0, \\ 1 - \frac{1}{t}\coth^{-1}\left(\dfrac{e^{c_2 x} + \cosh t}{\sinh t}\right), & t > 0. \end{cases} \tag{8}$$

The standard secant hyperbolic distribution and the standard logistic distribution are special cases of the SGSH distribution with $t = -\frac{\pi}{2}$ and $t = 0$ respectively, while the SGSH distribution tends towards the standard uniform distribution as $t \to \infty$. The standard Gaussian distribution is not a special case of the SGSH distribution, but can be approximated by the SGSH distribution – see [21].

It was proven in [30] that the shape parameter *t* can be interpreted as a kurtosis parameter in the sense of van Zwet’s ordering, [9]. The moment kurtosis ratio of the SGSH distribution is given by

$$\alpha_4 = \begin{cases} \dfrac{21\pi^2 - 9t^2}{5\pi^2 - 5t^2}, & -\pi < t < 0, \\ 4.2, & t = 0, \\ \dfrac{21\pi^2 + 9t^2}{5\pi^2 + 5t^2}, & t > 0. \end{cases} \quad (9)$$

As illustrated in (Figure 4), an inverse relation exists between α_4 and *t*. It follows that $\alpha_4 \to \infty$ as $t \to -\pi$, while $\alpha_4 \to 1.8$ as $t \to \infty$. If $t = \pi$, then $\alpha_4 = 3$, which is the special case of the SGSH distribution depicted in (Figure 2). The SGSH distribution has short tails for $t > \pi$ and heavy tails for $-\pi < t < \pi$. It is straightforward to show that

$$t = \begin{cases} \sqrt{\dfrac{21\pi^2 - 5\pi^2\alpha_4}{5\alpha_4 - 9}}, & 1.8 < \alpha_4 < 4.2, \\ 0, & \alpha_4 = 4.2, \\ -\sqrt{\dfrac{5\pi^2\alpha_4 - 21\pi^2}{5\alpha_4 - 9}}, & \alpha_4 > 4.2. \end{cases} \quad (10)$$

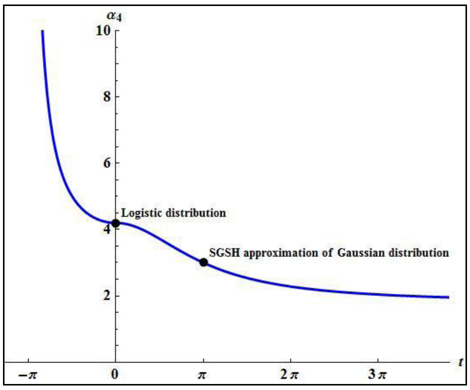


Figure 4. A plot of the relation between the kurtosis moment ratio, α_4, and the kurtosis parameter *t* of the SGSH distribution.

In (Figure 5) the graphical output obtained from manipulating the value of $\alpha_4 \in (1.8, 5]$ for the SGSH distribution is shown (note that the upper limit of α_4 can be taken as any value greater than 1.8). Looking at the plots from left to right and top to bottom, the increase in peakedness of the SGSH distribution, as the value of α_4 is increased, is evident. The density curves of the standard uniform and the standard Gaussian distributions are added in the plots in (Figure 5) as reference curves.

Recently Kotz and Seier, [31], proposed a new method for evaluating the kurtosis of a continuous distribution. With their method, which we will call the Kotz-Seier method, the density curve of a distribution with median, me, and variance, σ_X^2, is compared with the density curve of the uniform distribution with equal median and variance. Unimodal symmetric or skew distributions will have excess areas in the tails and in the peak, while U-shaped distributions will exhibit 'missing' areas. Note that symmetric distributions have $me = \mu_X$. As shown in (Figure 6), the density curve of a unimodal symmetric distribution will cross the density curve of the corresponding uniform distribution four times at $x_1 < x_2 < x_3 < x_4$, where

$$x_1 = \mu_X - \sigma_X \sqrt{3}, \tag{11}$$

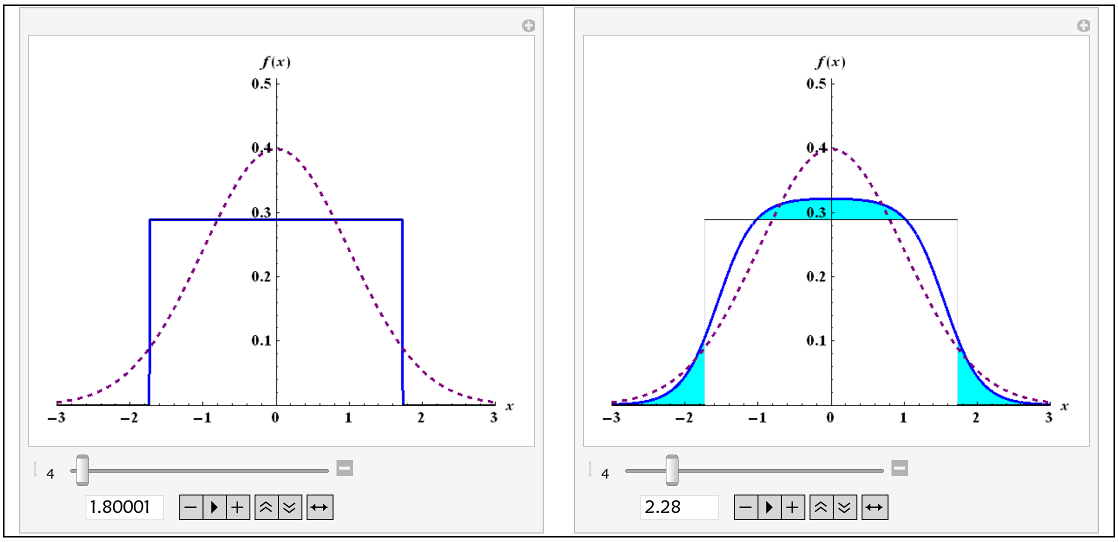
$$x_4 = \mu_X + \sigma_X \sqrt{3}, \tag{12}$$

and x_2 and x_3 are values on the x-axis such that

$$f(x_2) = f(x_3) = \frac{1}{2\sigma_X \sqrt{3}}. \tag{13}$$

The excess areas can then be calculated using

$$\text{Area} = \begin{cases} F_X(x_1), & \text{in the left tail,} \\ F_X(x_3) - F_X(x_2) - \dfrac{x_3 - x_2}{2\sigma_X\sqrt{3}}, & \text{in the peak,} \\ 1 - F_X(x_4), & \text{in the right tail.} \end{cases} \tag{14}$$



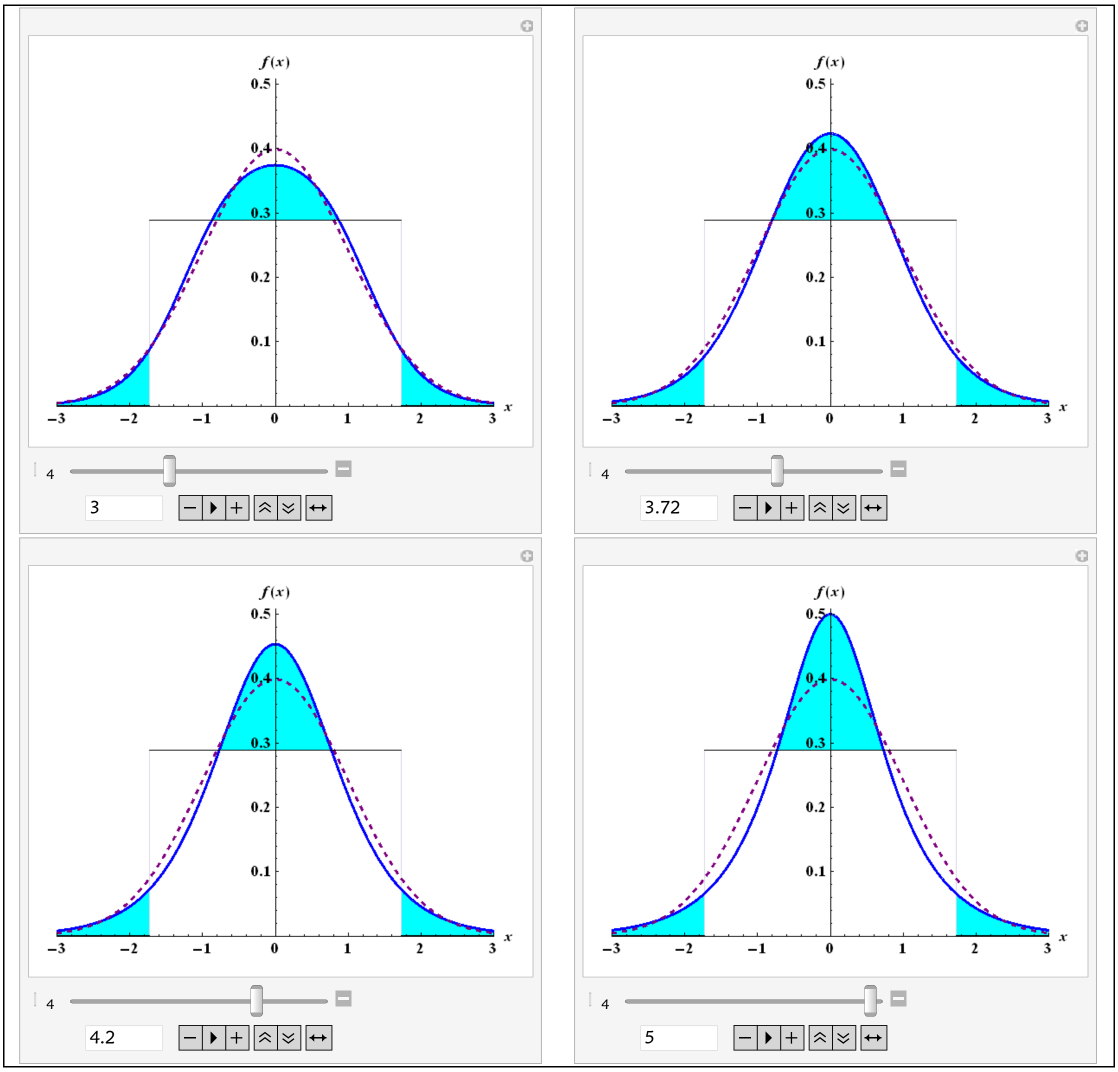


Figure 5. Sequential plots, from top left to bottom right, of the manipulation of the kurtosis moment ratio, α_4, and the resulting change in the density curve of the SGSH distribution (thick line) compared to that of the standard uniform distribution (thin line) and the standard Gaussian distribution (dashed line).

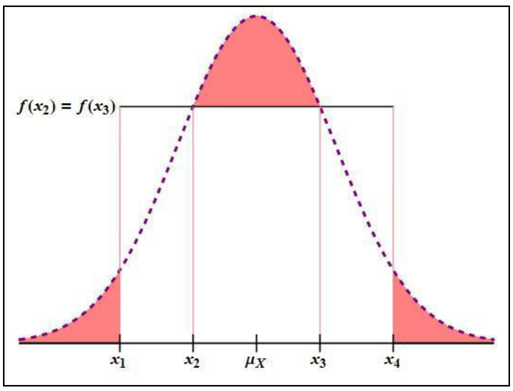


Figure 6. A plot illustrating the Kotz-Seier method for calculating excess tail and peak areas for a unimodal symmetric distribution.

The Kotz-Seier method was applied to various distributions in [31], but not to the SGSH distribution. We did so by indicating the excess areas in the plots in (Figure 5). Recall that $\mu_X = 0$ and $\sigma_X^2 = 1$ for the SGSH distribution. For the six chosen values of α_4 in (Figure 5), the tail and peak areas as well as the sum thereof are given in (Table 1) and represented graphically in (Figure 7).

Table 1. Tail and peak areas for the SGSH distribution.

α_4	t	$-x_2 = x_3$	Left Tail = Right Tail	Left Tail + Right Tail	Peak	Tails + Peak
1.80	∞	-	0.0000	0.0000	0.0000	0.0000
2.28	2π	1.0223	0.0310	0.0620	0.0519	0.1139
3.00	π	0.8725	0.0382	0.0765	0.1019	0.1783
3.72	$\pi/2$	0.8011	0.0407	0.0814	0.1386	0.2201
4.20	0	0.7692	0.0414	0.0828	0.1587	0.2416
5.00	$-\pi/2$	0.7297	0.0418	0.0837	0.1869	0.2706

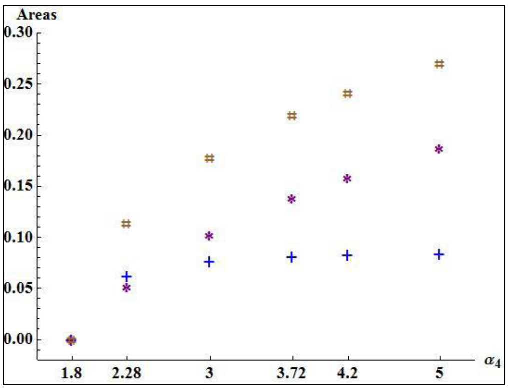


Figure 7. A plot of the tail areas (plus signs), peak areas (star signs) and the sum thereof (hash signs) for various values of α_4 for the SGSH distribution.

7. Conclusion

In this paper we considered the teaching of the concept of kurtosis, which is currently a neglected topic in introductory statistical courses. Contrary to the concepts of location, spread and skewness, the topic of kurtosis is typically not taught to first-year students. The impact of this is not immediate. Only in later years, usually at third-year level, students are again confronted with kurtosis when dealing with advanced topics.

The first author of this paper currently lectures the third-year module on time series analysis at the University of Pretoria (UP). Annually, before dealing with heavy-tailed financial time series and the analysis thereof, a lecture on kurtosis must first be presented. This year was the first time that *Mathematica* and the techniques described in this paper were used to graphically illustrate the concept of kurtosis. Unfortunately at the time of preparation of this manuscript, no results were yet available regarding changes, if any, in the students' studies on a cognitive level. However, during and after the lecture on kurtosis, students expressed their appreciation for the innovative way in which mathematical software was used to illustrate this difficult concept to them. Afterwards numerous students wanted to know where and how they could acquire *Mathematica* – at this stage undergraduate students at UP do not have access to *Mathematica*, but this will hopefully be rectified soon!

Although the first author of this paper really enjoys teaching kurtosis to his third-year students, it does require time which should ideally be spent on the prescribed topics in the module. The concept of kurtosis is not a prescribed topic, but forms part of learning presumed to be in place. Ideally this learning should occur in introductory courses at first-year level. We believe that the techniques proposed in this paper are at a level suitable for first year BCom and BSc students. Depending on the mathematical background of the students, the lecturer can decide how much of the mathematical details to include. For instance, for BCom students with limited mathematical skills, showing (Figure 5) should suffice in at least introducing them to the concept of kurtosis.

Advances in mathematical software have changed the way in which we communicate with our students in the classroom. Through the use of *Mathematica* we are hopefully lifting the table cloth and revealing the platypuses and kangaroos hiding beneath…

8. References

[1] K. Pearson, *'Das Fehlergesetz und Seine Verallgemeinerungen Durch Fechner und Pearson.', a rejoinder (Skew variation, a rejoinder)*, Biometrika, 4 (1905), pp. 169–212.

[2] E. Seier, *Curtosis (in Spanish)*, Pesquimat, 6 (2004), pp. 1–26.

[3] E. Seier, *Celebrating 100 years of kurtosis*, (2004). Available at: www.etsu.edu/math/seier/seier.htm.

[4] 'Student', *Errors of routine analysis*, Biometrika, 19 (1927), pp. 151–164.

[5] F.J. Dyson, *A note on kurtosis*, J. Roy. Statist. Soc., 106 (1943), pp. 360–361.

[6] D.E. Scates, 1943, *Characteristics of kurtosis*, J. Exp. Educ., 11 (1943), pp. 226–237.

[7] I. Kaplansky, *A common error concerning kurtosis*, J. Amer. Statist. Assoc., 40 (1945), p. 259.

[8] G.U. Yule and M.G. Kendall, *An Introduction to the Theory of Statistics*, 12th rev. ed., Griffin, London, 1940.

[9] W.R. van Zwet, *Convex Transformations of Random Variables*, Mathematics Centre Tract 7, Mathematisch Centrum, Amsterdam, 1964.

[10] G.R. Stavig, *A robust measure of kurtosis*, Percept. Motor Skill, 55 (1982), p. 666.

[11] J.J.A. Moors, *A quantile alternative for kurtosis*, J. Roy. Statist. Soc. Ser. D, 37 (1988), pp. 25–32.

[12] K.P. Balanda and H.L. MacGillivray, *Kurtosis and spread*, Canad. J. Statist., 18 (1990), pp. 17–30.

[13] J.R.M. Hosking, *L-moments: Analysis and estimation of distributions using linear combinations of order statistics*, J. R. Stat. Soc. Ser. B Stat. Methodol., 52 (1990), pp. 105–124.

[14] R.B. Darlington, *Is kurtosis really "peakedness?"*, Amer. Statist., 24 (1970), pp. 19–22.

[15] D.K. Hildebrand, *Kurtosis measures bimodality?*, Amer. Statist., 25 (1971), pp. 42–43.

[16] D. Ruppert, *What is kurtosis? An influence function approach*, Amer. Statist., 41 (1987), pp. 1–5.

[17] B.S. Chissom, *Interpretation of the kurtosis statistic*, Amer. Statist., 24 (1970), pp. 19–23.

[18] J.J.A. Moors, *The meaning of kurtosis: Darlington reexamined*, Amer. Statist., 40 (1986), pp. 283–284.

[19] K.P. Balanda and H.L. MacGillivray, *Kurtosis: A critical review*, Amer. Statist., 42 (1988), pp. 111–119.

[20] N.L. Johnson, A.W. Kemp, and S. Kotz, *Univariate Discrete Distributions*, 3rd ed., John Wiley & Sons Inc., Hoboken, New Jersey, 2005.

[21] D.C. Vaughan, *The generalized secant hyperbolic distribution and its properties*, Comm. Statist. Theory Methods, 31 (2002), pp. 219–238.

[22] J.W. Tukey, *The future of data analysis*, Ann. Math. Stat., 33 (1962), pp. 1–67.

[23] B.L. Joiner and J.R. Rosenblatt, *Some properties of the range in samples from Tukey's symmetric lambda distributions*, J. Amer. Statist. Assoc., 66 (1971), pp. 394–399.

[24] Various lecturers at South African Universities, *Personal and electronic-mail correspondence*, (2009).

[25] T.A. Williams, D.J. Sweeney, and D.A. Anderson, *Contemporary Business Statistics with Custom Selections*, 3rd ed., South-Western Cengage Learning, Australia, 2009.

[26] D. Wackerly, W. Mendenhall, and R.L. Scheaffer, *Mathematical Statistics with Applications*, 7th ed., Thomson, Belmont, California, 2008.

[27] A.G.W. Steyn, C.F. Smit, S.H.C. du Toit, and C. Strasheim, *Modern Statistics in Practice*, Van Schaik, Pretoria, South Africa, 1994.

[28] T.L. Kelley, 1921, *A new measure of dispersion*, Quar. Pub. Amer. Statist. Assoc., 17 (1921), pp. 743–749.

[29] Wolfram *Mathematica* Documentation Center, `Manipulate`, (2009). Available at: http://reference.wolfram.com/mathematica/ref/Manipulate.html.

[30] I. Klein and M. Fischer, *A note on the kurtosis ordering of the generalized secant hyperbolic distribution*, Comm. Statist. Theory Methods, 37 (2008), pp. 1–7.

[31] S. Kotz and E. Seier, *Visualizing peak and tails to introduce kurtosis*, Amer. Statist., 62 (2008), pp. 348–354.

Undergraduate mathematics education: Tales of two worlds

Cristina Varsavsky[§§§§§§§§§§], Marta Anaya[⬩]

[*]School of Mathematical Sciences, Monash University, Melbourne, Australia;
[⬩]Engineering Faculty, Buenos Aires University, Buenos Aires, Argentina

Abstract: This paper provides an insight into the state of undergraduate mathematics education of two countries of the DELTA network: Australia and Argentina. It attempts to compare the two education systems in which undergraduate education takes place, the curriculum content, the teaching workforce, and the graduate outcomes.

Keywords: undergraduate mathematics education, Argentina, Australia

1. Introduction

Mathematics underpins our rapidly changing and highly technological world. However, despite its strategic importance, in the past two decades there has been a growing concern worldwide about the mathematics education of its citizens. The decline of graduates in mathematics and other fundamental sciences has been documented by the Organization for Economic Cooperation and Development (OECD) at the Global Science Forum held in 2005 [1]. This forum raised concerns about the skill base necessary to sustain innovation, and urged governments to take steps to address the shortages. Similar reports have been produced in the United States [2], the United Kingdom [3] and in Australia [4,5].

This urgent need to address the shortage of the mathematical skill base necessary to support innovation calls for a greater understanding of the issues underlying the downward trends in graduate outcomes. A small but fundamental step towards this understanding is to investigate the approaches taken around the world in the preparation of mathematics graduates, and the career opportunities and employment pathways of these graduates.

The recent review of Mathematical Sciences in Australia [5] recommended that "the mathematical sciences components of courses must be comparable to those in other nations". This suggests that it is timely and relevant to understand and compare undergraduate mathematics curricula and graduate outcomes in different parts of the world. For this paper we have chosen two Southern Hemisphere countries of the DELTA network, Australia and Argentina, experimenting graduate numbers below the OECD averages, but in different contexts and taking different approaches. The paper aims to provide some insight into the state of undergraduate mathematics education of these two countries; it attempts to compare the contexts in which undergraduate mathematics education takes place, their respective curricula and their graduate outcomes.

These comparisons are made drawing from information made available by the university sectors and the education departments of the two countries, either directly to the public or through international organizations such as OECD and UNESCO. It should be noted at the onset that although in recent years the flow of this official information has become more regular, it still presents many gaps and some

[§§§§§§§§§§] E-mail: Cristina.Varsavsky@sci.monash.edu.au

inconsistencies; consequently direct comparisons are not always possible, and some of the figures presented are only the best estimates available.

The context for mathematics education of the two countries is outlined first, summarising the two education systems, the profile of the teaching workforce, and the main characteristics of the primary and secondary mathematics education, including the level of achievement and the level of preparedness for university level mathematics. This is followed by a comparison of the undergraduate course structures and the curricula of two typical universities. Finally, an analysis of the graduate outcomes is made.

The results of this comparison will depict the current status of the mathematics undergraduate education of the two countries in the two different contexts and their relationship with the problem posed.

2. The context for mathematics education

The education systems

The education systems of the two countries are structurally very similar. Although changes are occurring in their governance, the pre-primary, primary and secondary sectors are still largely decentralized and are administered by state or provincial governments; the university sectors are the responsibility of the federal governments, and the tertiary non-university sectors are administered jointly by provincial and federal governments. Primary and junior secondary schooling is compulsory in both countries. Senior secondary school also became compulsory in Argentina with the new education law introduced in 2006, and although it is not mandatory in Australia, expectations have been raised across the different states aiming for a minimum of 13 years of education for all its citizens (including pre-primary) [6].

There is a big gap in the levels of public investment in education of these two countries; however, both fall below the OECD average. In Australia this is 4.8% of the gross domestic product (GDP), of which 25% is devoted to tertiary education. In Argentina it is 3.8% of GDP, with 17% allocated to the tertiary sector [7]. Although this difference does not appear very pronounced, it should be noted that the GDP purchase power per capita in Australia is more than double of that in Argentina. In addition, Australia has always had a stable political and economic environment, while Argentina had suffered several economic crises—the most recent one being during the first years of this century—which had affected investment and strategic innovation in education.

The university systems of both countries are largely publicly funded, but while Argentine students have access to free undergraduate education, Australian students must repay part of their education cost after they graduate. In both countries most students need to engage in paid work to support themselves, and generally struggle to juggle both work and studies.

The teaching workforce in the primary and secondary sectors

Like in many other countries, the teachers of the two countries are an ageing population with a female majority. In both countries teaching is perceived as a low status profession. In Argentina these perceptions are founded on the lack of resources, poor access to professional development, and low salaries [11, 12]. In Australia this view is based on the low demand for teaching training places by high achieving students, and perceptions of poor salaries and career pathways [13].

Teacher training in Argentina takes place mostly within the tertiary non-university sector, but a teacher qualification can also be pursued through a university degree. In both cases the length of the programme is at least 4 years, and students specialize in mathematics right from the start. At least 40% of any of these programmes is allocated to mathematics discipline studies, and the rest is for general and discipline-specific studies on pedagogy and for teaching practical placements [6]. Although there are no firm data about career pathways of teacher graduates, there appears to be no shortage of supply of mathematics teachers in Argentina.

On the other hand, teacher training in Australia is conducted by universities and involves three or four years of undergraduate studies, or a three year bachelor of science degree followed by a one-year diploma in education. Compared to the Argentine model where the curricula are highly prescribed, the Australian programmes give students much more choice, and usually involve specialization in another discipline in addition to mathematics. This results in a diverse range of skills of mathematics teachers, which combined with a concerning decline in the number of student teachers taking mathematics studies [14] led to the current shortage and a poorly trained mathematics teaching workforce. Although over 90% of secondary mathematics teachers have a tertiary qualification, more than 20% have only studied level-one university mathematics, and 8% have done none at all [15].

Primary and secondary mathematics

In both countries the primary and secondary school curricula at each year level are set in terms of learning outcomes rather than being prescribed, and the number of hours per week devoted to mathematics is similar to the World Economic Indicators (WEI) average [8].

At senior secondary level there is, however, a fundamental difference which has important consequences for the supply of graduates with mathematical skills. While Australian students are given the choice to do (or not) mathematics in their last two years of their secondary schooling, learning mathematics is a core activity for Argentine students, taking between 2 and 4 hours per week (out of 25), depending on their specialization. About 20% of senior secondary Australian students decide at a very young age to close the pathway to mathematically based disciplines. Furthermore, the remaining 80% are increasingly gravitating towards the elementary level mathematics subjects; in 2006 only 10% undertook the advanced level studies [9].

Another consequence of this choice given to students in Australian schools is the gender difference in the uptake of mathematics studies. When given a choice, girls are less likely to choose mathematics studies [10]. This has a significant flow on effect in the participation of Australian women in science, engineering and technology professions, and in the supply of teachers with a strong mathematics background.

Despite the weaker preparation of Australian mathematics teachers compared to teachers in Argentina, the Australian primary and secondary students significantly outperform Argentine students. The common and most reliable points of comparison are the results of the Program for the International Student Assessment (PISA), as both countries are participants in this programme for testing the performance of 15-year old students in mathematics, science and language [16]. The 2006 results show Australian students performing above the OECD average, while Argentine students were placed more than one standard deviation bellow this average. Analysis of the proficiency level of the participating students also shows that while in Australia only 3.3% of students fall below the basic proficiency level (Level 1), almost 40% of

Argentine students belong to this category, with a further 25% reaching only Level 1 (Figure 1).

This significant difference in the performance of 15-year old students confirms that although the mathematical preparedness of the teaching workforce is important, all other socio economic factors of these two countries appear to have a much stronger effect on what and how students learn mathematics and on their levels of achievement.

In summary, due to the choice available in senior years, not all Australian students who complete secondary schooling can progress to undergraduate studies that require a strong mathematics foundation. On the other hand, even though all Argentine students undertake mathematics in senior secondary school, their level of preparedness for undergraduate mathematics is weak. This is acknowledged by universities, who require all applicants for undergraduate degrees to undertake and pass a bridging year of studies before they enter university. This bridging year effectively adds one year to the students' education, and acts as a filter to access undergraduate mathematics.

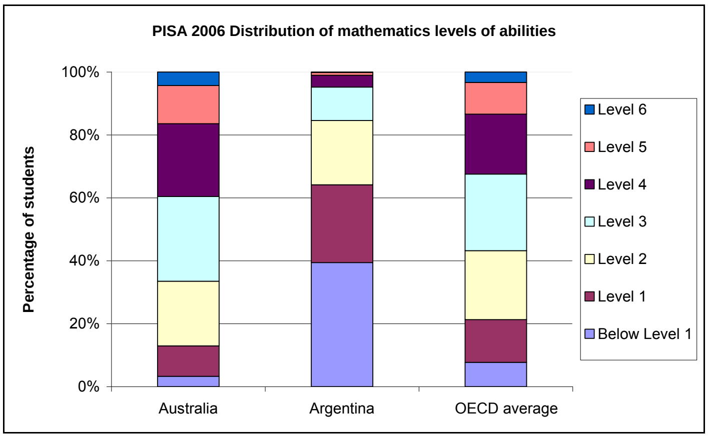


Figure 1. PISA 2006 Distribution of the level of mathematics abilities. Source: [16]

3. Undergraduate education

Mathematics in undergraduate programmes

The structures of the undergraduate programmes within which students can specialize in mathematics are very different. In Argentina, in addition to the teaching training programmes, mathematics studies are offered through a *licenciatura*, specializing in either pure mathematics or applied mathematics. *Licenciaturas* are normally five year long programmes which include a thesis in the final year; they involve studies related only to the area of specialization (that is, a graduate with a *licenciatura* in mathematics would have completed all five years studying only mathematics subjects). This is in sharp contrast with the programme structure in Australia, where students specialize in mathematics within a three-year bachelor of science. Although some bachelor of science programmes with a greater focus on mathematics (normally called tagged degrees) still exist, the majority of graduates hold a generic bachelor of science degree. Within the generic bachelor of science students have much flexibility in the choice of subjects; in order to graduate with a specialization in mathematics, they would normally need to complete only a minimum of the equivalent to one year of studies in mathematics. High achieving graduates who wish to undertake further discipline studies can then continue with a one-year long honours programme, which has a strong emphasis on research. The combination of a bachelor degree and an honours degree in mathematics is somewhat equivalent to the Argentine *licenciatura*, although given the flexibility referred to above, the Australian qualifications do not guarantee that every graduate has the same amount of mathematics studies. Both the Australian honours degree and the Argentine *licenciatura* are the point of entry for a doctorate programme.

Mathematics and statistics are also compulsory studies in courses such as sciences, engineering, information technology and business. While over the last years the trend in Australia has been towards reducing the amount of mathematics studies in these programmes, this has not been the case in Argentina. To make a comparison, an Argentine graduate in chemistry would have undertaken four mathematics subjects including multivariable calculus, differential equations and numerical methods, while most Australian graduates in chemistry would have done none or very little university level mathematics. Similar differences are observed in biology, geosciences, engineering and economics graduates of the two countries.

Curriculum comparison of university mathematics programmes

Tables 1 and 2 show the curriculum content of two typical pure mathematics programmes offered by large universities in the respective countries, one is the major in pure mathematics taken as part of a bachelor of science at Monash University*********** [17], and the other is the *licenciatura* in mathematics programme of Buenos Aires University [18]. The prerequisite knowledge for these programmes is very similar,

Monash University, Major in pure mathematics within a bachelor of science
FIRST YEAR **Functions and their applications:** one variable calculus, vectors, complex numbers. **Techniques for modelling**. Vectors, linear equations, ordinary differential equations of first and second order (constant coefficients) and applications, infinite series, probability.
SECOND YEAR **Multivariable Calculus:** Partial derivatives, multiple integrals, line and surface integrals, vector analysis, theorems of Green, Gauss and Stokes *At least one of* • **Differential geometry:** explores the metric structure of curves and surfaces, primarily in R^3. • **Real Analysis**: the Cantor set; compactness and convergence; sequences and series; applications .
THIRD YEAR *At least 4 from the following:* • **Differential** Equations with modelling: Particularly heat conduction and oscillations, using computer simulation, laboratory experiments and mathematical analysis. It includes the use of a computer algebra package. • **Linear Algebra with Applications,** Eigenvalues. Quadratic forms. Jacobi and Gauss-Seidel iteration. Applications to coding, computer graphics, geometry, dynamical systems, Markov chains, differential equations. • **Algebra and Number Theory (if not already taken): Groups.** The theorems of Fermat, Euler and Wilson, Chinese remainder theorem; rings, fields and abelian groups in number theory. • **Mathematics of uncertainty:** Probability models. Introduction to stochastic processes. Application to statistical models. Mathematical principles of inference. • **Partial differential equations:** First-order and Second-order PDEs. The wave equation, the heat equation, Laplace's equation. Exact solution, numerical methods, stability analysis. Well-posed problems, boundary and/or initial conditions. • **Complex Analysis and integral transforms,** Differentiation and integration of complex functions. Taylor and Laurent series. Fourier and Laplace transforms. • **Differential geometry:** explores the metric structure of curves and surfaces, primarily in 3-dimensional Euclidean space.

*********** This is the program that would normally be taken by students who only do intermediate mathematics in secondary schools, which is the case for the majority of students. Stronger programs exist at Monash University and at some other Australian universities for the minority of high achieving students.

- **Algebra and Number Theory II:** Rings, fields, ideals, algebraic extension fields. Coding theory and cryptographic applications of finite fields. Gaussian integers, Hamilton's quaternions, Chinese Remainder Theorem. Euclidean Algorithm in further fields.
- **Analysis and Topology:** abstract metric spaces. Topological vector spaces, Hilbert spaces.

Time series and random processes in linear systems, Multivariate distributions. Estimation. Confidence intervals. Analysis in the time domain and in the frequency domain: stationary models ARMA and ARIMA models. Spectral analysis. State-space models. Kalman filter. Use of ITSM. (Information Technology Service Management)

Table 1. Undergraduate pure mathematics programme at Monash University. Source: [17]

although Argentine students achieve this through a compulsory bridging course they must pass to be admitted to university. On average, the number of contact hours involved in each of the subjects listed are 10 and 5 for Argentina and Australia respectively. At Monash University students are normally assessed with assignments and a final examination, each of them contributing a certain percentage to the final grade. At Buenos Aires University the assessment for compulsory pure mathematics subjects involves two partial examinations, which students must pass before they are allowed to take the final examination; only some applied subjects such as computational mathematics involve completing assignments using software.

Buenos Aires University, *licenciatura* in mathematics, specialization in pure mathematics

FIRST YEAR
- **Mathematical Analysis I**: Multivariable calculus. Fubini's theorem.
- **Algebra I** : Sets. Relations. Induction. Combinatorics. Complex numbers. Polynomials,divisibility.
- **Mathematical Analysis II**: Vector Analysis ,Theorems of Green, Gauss and Stokes. Applications.
- **Linear Algebra**: Abstract vector spaces. Dual space. Linear transformations. Definite positive bilinear forms. Diagonalization of symmetric matrices. Otrhogonal Transformations, Jordan form.
- **Advanced Calculus Workshop**: Cauchy sequences. Topology de R y R^n. Uniform continuous functions. Finite variation functions. Riemann-Stieljes integral. Series

SECOND YEAR
- **Advanced Calculus**: Sequences. Cardinals. Metric and normed vector spaces, differentiation in euclidean spaces.
- **Numerical Analysis**: Number Systems. Numerical solution of: difference equations, linear and non linear equations, Differential equations. Use of Matlab.
- **Complex Analysis**: Complex functions. Holomorphic and analytic functions. Infinite products. Conformal maps.
- **Probability and Statistics**: Sample Space. Probability Space. Random variables. Almost everywhere and probability convergence. punto. Weak and strong Law of large numbers de los grandes números. Weak convergence. Central limit theorem. Applications

THIRD YEAR
- **Real Analysis:** Lebesgue measure. Lebesgue Integral. Fubini's theorem. Lp spaces. Measure and integration in abstract spaces. Signed measures. Radon- Nikodym Theorem.
- **Algebra II:** Groups. Rings. Modules over a ring.
- **Topology**: General topology. Function spaces. Algebraic Topology: Homotopy. Fibrations. Singular and simplicial Homology.
- **Projective Geometry**: Projective spaces. Parametrized curves and surfaces. Differential forms. Orientation. Triangulation of surfaces.

FOURTH YEAR
- **Functional Analysis**: Normed spaces. Hahn-Banach Theorem. Linear Operators.Riesz representation theorem. Hilbert spaces. Compact linear operators. Spectral theorem for self-adjoint operators.
- **Differential geometry**: Differentiable manifolds. Tangent space of a differentiable manifold. Lie groups. Tensors. Riemann Manifolds. Gauss equations.
- **Differential Equations A**: Calculus of variations. Partial differential equations. The wave equation, the heat equation, Laplace's equation. Solution of Dirichlet problem in R^n . Fourier transform. Sobolev spaces. Variational formulation of boundary value problems.
- **Algebra III**: Extensions of fields. Galois theory. Finite fields. Cyclotomic fields.

FIFTH YEAR
- Remaining elective units chosen from the following topics: algebraic geometry, differential topology, Lie groups and Lie algebras, number theory, commutative algebra, harmonic analysis, algebraic topology.
- **Research Thesis**

Table 2. Undergraduate pure mathematics programme at Buenos Aires University. Source: [18]

Monash University, Honours in pure Mathematics within a bachelor of science (length: 1 year)
• **Seven lecture topics from:** General Relativity , Topology, Banach Algebras , Lie Groups, Diophantine Equations, Partial Differential Equations, Differential Geometry, M4072 Galois Theory • **Research Thesis**

Table 3. Honours in pure mathematics programme at Monash University. Source: [19]

Inspection of these programmes leads to the conclusion that the approach in Argentina is unquestionably more theoretical and more rigorous. There is also a stark contrast in the knowledge base of graduates from the two systems: while all Argentine pure mathematics students graduate with a good grounding in algebra, real and complex analysis, differential equations, topology, differential geometry and probability theory, given the choices available through a bachelor degree there is no guarantee that Australian graduates would have studied more than two or three of these sub-disciplines. Only the minority of graduates who take up more than the minimum required mathematics subjects and who continue their studies with an honours program would be comparable to Argentine graduates (See Table 3). Similar comparisons can be made between applied mathematics programmes of the two institutions, with similar conclusions (See Tables 4 and 5). In particular, it should be noted that Argentine applied mathematics graduates have a good grounding in statistics and operations research, but this cannot be guaranteed for an Australian graduate.

Monash University, Major in applied mathematics within a bachelor of science
FIRST YEAR As for pure mathematics major.
SECOND YEAR • **Multivariable calculus** • **Differential equations with modelling.**
THIRD YEAR *At least one of* • **Partial Differential equations** • **Introduction to computational mathematics:** Error analysis, the solution of algebraic equations; approximations of functions: curve fitting, least squares and interpolation; analysis of data by Fourier Transforms and Fast Fourier Transforms; numerical differentiation and integration; ordinary differential equations. Case studies drawn from physical sciences. Use of Matlab. *At least three from* • **Linear algebra with applications** • **Real Analysis** • **Mathematics of uncertainty** • **Complex analysis and integral transforms** • **Advanced Ordinary Differential Equations**: dynamical systems and boundary-value problems.. Sturm-Liouville eigenvalue problems and orthogonal polynomials matrix methods for the numerical investigation of boundary-value problems Dynamical systems analytical and numerical methods for planar autonomous systems. Use of Matlab. • **Time series and random processes in linear systems** • **Fluid Dynamics** The basic equations of fluid dynamics; vorticity and circulation; rotational and irrotational flow; two dimensional homogenous incompressible flow; viscous effects and boundary layers; separation from a ball, with applications to cricket, golf and baseball; flow in a rotating reference frame, geostrophic flow.

Table 4. Undergraduate applied mathematics programme at Monash University. Further subject details are provided in Table 2. Source: [17]

As noted above, teacher training in Argentina takes place mainly within the tertiary non-university sector, but universities also offer undergraduate degrees for mathematics teachers. Such programmes are normally of 4.5 to 5 years duration and qualify graduates to teach in both secondary and tertiary sectors. Normally at least the equivalent to 2.5 to 3 years of these programmes are allocated mathematics discipline studies, and the rest for general and discipline specific pedagogical subjects and teaching practice in schools. At the University of Buenos Aires, for example, the first two years of the teacher training programme are the same as for pure mathematics students pursuing the *licenciatura*, plus a geometry subject and additional electives chosen from the third-and forth year subjects listed in Table 2.

The teaching workforce

There are insufficient data to compare the teaching workforces of the two countries, and in particular the academic staff profile and the student/staff ratio. Students undertaking a mathematics programme are normally taught by staff in a mathematics department, but different practices apply for mathematics subjects taken as part of another type of programme.

Buenos Aires University, *licenciatura* in mathematics, specialization in applied mathematics
FIRST and SECOND YEAR **As per pure mathematics specailaization.**
THIRD YEAR • **Masure and Probability**: **:** Lebesgue measure. Lebesgue Integral. Fubini's theorem. Lp spaces. Measure and integration in abstract spaces. Signed measures. Radon- Nikodym Theorem. • **Introduction to Computacional Mathematics**: Algorithms . Variables. Data structure. Algorithms design techniques. • **Statistics**: Introduction to statistics inference. Bayesian and minimum estimators. Confidence intervals and regions. Hypothesis test. Robust Estimation. Use of software. • **Operations research**: Linear Programming . Minimum Flow cost Integer Linear Programming. Use of software
FOURTH YEAR • **Physics** • **Optimization**: Optimization with linear equality restrictions and non linear inequality restrictions. Non deterministic methods. Discrete methods . Use of Matlab • **Differential Equations** • **Numerical Analysis**: Finite differences: Parabolic and hyperbolic equations. Finite elements, 1-D and 2-D elliptic equations.
FIFTH YEAR • Remaining elective units chosen from the following topics: linear models, methods of final elements and applications, non-parametric methods, stochastic porcesses. • **Research Thesis**

Table 5. Undergraduate applied mathematics programme at Buenos Aires University. Further subject details are provided in Table 2. Source: [18]

There is a significant difference in staffing arrangements: while in Australia staff is appointed to a department and may be expected to teach across different subjects, in Argentina staff is normally appointed to chair or support a particular subject (such positions are normally re-advertised every 7 years). Also, unlike in Australia, academic staff in Argentina constitute largely a part-time workforce; this is mainly the combined effect of poorly resourced universities and the low salaries on offer. There are no specific data for mathematics academics, but it is known that across the Argentine university sector only about 14% of teaching staff are full time, 22 % are categorised as semi-full time and the rest as part-time. Subject chairs are included in all these three work modes (21%, 15 % and 8% for full-time, semi full-time and part-time respectively) [20]. Full-time academic staff normally teach the higher-level specialist subjects, leaving the lower years to their part-time colleagues.

Enrolments and graduates

Due to the different programme structures and the discipline classification and reporting used in Australia, an accurate comparison of mathematics students and graduates of the two countries cannot be made. Analysis of the internal data of the two countries indicate that while in Argentina the number of students who pursue a degree in mathematical sciences remained stable over the last decade [20], in Australia in the 1989-2005 period the number of mathematical sciences enrolments decreased by one third [4]. The decline of mathematics students in Australia has been a matter of grave concern over the last decade [5], which led to the closure and downsizing of several mathematics departments around the country.

In Argentina, for every 100 law graduates, there are only two mathematics and one physics graduate. The percentage of students undertaking engineering, agricultural science, chemistry, physics or mathematics represents less than 10% of students in public universities, and only about 2.5% in private institutions [21].

Tables 6 and 7 provide some detail about student enrolments and graduates for the most recent years where comparisons are meaningful. When reading these figures, it should be noted that the percentages for Argentina represent the students enrolled in a *licenciatura* or a university teaching programme (ie studying only mathematics or statistics subjects), while Australian figures represent the proportion of students pursuing a bachelor of science and enrolled in mathematics/statistics subjects (these students may not be necessarily pursuing a mathematical sciences specialization, but may be studying mathematics to complement another science area o study). It should also be remembered that only a minority of Australian graduates hold an honours degree. Graduate figures in Table 7 imply that the percentages of university graduates who graduate every year with a mathematics qualification are very similar for the two countries (0.3 for Argentina and 0.4 for Australia) and lower than the OECD average of 1% [1]; but as argued by Dobson, due to the reporting mechanisms used in Australia, these figures need to be taken with a grain of salt [4].

	Argentina		**Australia**	
	Mathematical sciences % of natural and physical sciences	Natural and physical sciences % of all students	Mathematical sciences % of natural and physical sciences	Natural and physical sciences % of all students
1989	-	-	23.8	-
2001	19.6	2.7	13.5	-
2002	20.0	2.9	13.6	-
2003	21.6	3.1	13.5	6.7
2004	21.3	3.0	13.4	6.8
2005	21.4	3.0	13.1	6.8
2006	22.4	2.8	-	-

Table 6. Percentage of mathematical sciences students of the two countries compared to all students and to all natural and physical sciences students. Source: [4, 17].

The gender differences are worth noting. Over 65% of the Argentine undergraduate students undertaking mathematical are female; this figure is significantly higher than the average across all disciplines (57% in 2005) [17]. This is in large contrast with the Australian student population which also has a large proportion of female students (54.5% in 2005), but typically less than 40% of those who engage with mathematical sciences are female. It might be speculated that this gender difference in the uptake of mathematics in the two countries is related to the perceived poor career prospects and low salaries, but it is more likely to be a direct consequence of the compulsory nature of mathematics studies in senior secondary school.

	Argentina		**Australia**	
	Mathematical sciences % of natural and physical sciences	Natural and physical sciences % of all graduates	Mathematical sciences % of natural and physical sciences	Natural and physical sciences % of all graduates
2003	14.0	2.4	5.2	7.0
2004	13.0	2.4	5.4	6.7
2005	13.5	2.5	5.7	7.0

Table 7. Percentage of mathematical sciences graduates of the two countries compared to all graduates and to all natural and physical sciences graduates. Source: [4, 17].

Employment outcomes

According to the latest statistics available, the median Australian salary for mathematics graduates is higher than the average for all graduates, and about 86.6% of the graduates under 25 years of age are employed [22]. The top three occupations of these graduates are in business and computing (38.5%), sales and services (12.6%); 9.8% work in education, 2.2% in health 17% in government organizations, and 85.2% in the private sector. However, despite these positive statistics, the perception of poor career outcomes amongst prospective students when making decisions about their future is still very common.

That same perception is evident in Argentina, but there is no data about what Argentine graduates in mathematical sciences do to prove it incorrect. The closest available comparison is the starting salary of an engineer; this is only 2.5 times of the minimum wage [23], not providing enough motivation to put in the effort required to obtain a university qualification.

University providers describe the career outcomes for a *licenciatura* in mathematics as leading primarily to teaching and research jobs, but also noting that graduate attributes developed will prepare graduates for work in private and government organizations where high level problem solving skills are required [24]. Twelve years ago the Ministry for Science and Technology identified a shortage of applied mathematics skill-based workforce to support science and technology development, and urged the university and industry sectors to collaborate for the further development of applied mathematics programmes [25]. A 2002 study of the mathematical sciences in Argentina indicated that, with some exceptions, the mathematics departments operated in isolation, with little interaction with industry, reporting that their programs are focused almost exclusively on the discipline, with very little emphasis on applications and on interactions with other disciplines [26]. Although the number of applied mathematics programme offerings has increased since the release of this report, there is not much evidence of an increased contribution of mathematics graduates to the private business and industry sectors. In fact, only about 12.5% of science and technology researchers are employed by the private sector, while in Australia this is close to 30%; furthermore, these figures are almost reversed for the government sector [27]. These statistics do not provide details for mathematics researchers but could serve as indicators of the contribution mathematics graduates make towards science and technology development.

4. Final remarks

Argentina and Australia are amongst the countries in great need of making a substantial investment in the strategic development of the mathematics education of its citizens required to underpin and sustain their technological development. Despite the big gap in the socio economic realities of these two countries, the percentage of mathematics graduates is comparable and much lower than the OECD average.

In both cases, the secondary school systems seem to be at the heart of the problem. The number of students undertaking undergraduate mathematics could increase only if more students develop a sound foundation in mathematics at the end of their secondary schooling. However, the strategies for achieving this would need to be different. In the case of Australia, the efforts should focus on engaging more students with mathematics at senior level, and even consider the bold proposal of making mathematics a compulsory subject throughout the secondary school. On the other hand, in Argentina the emphasis should be placed on raising the level of achievement of the secondary students to OECD standards, which will require a review of curricula and pedagogical approaches. This would also make very good economic sense, as with higher secondary school standards, a bridging university year would no longer be required.

Although the proportion of mathematics graduates is comparable, their quality differs. The Argentine curriculum appears to be more rigorous and prepares graduates with a strong theoretical foundation. The limited number of students may reflect the perception of careers in mathematics being unattractive, either because of poor preparation in secondary school, or limited employability prospects. Increased interaction of mathematics departments with the world outside their universities would facilitate the forging of links between the mathematics discipline and the scientific and technological applications relevant to the Argentine industry, opening opportunities to provide greater motivation to students to engage with mathematics studies.

In the Australian context, given the flexibility to choose subjects outside the area of specialization, the quality of mathematics graduates is not uniform. In addition, the Australian curriculum has a stronger emphasis on applications. One may argue that a more applied focus makes the graduates more employable, but there is not evidence to support this argument; further investigation of the relationship between the kind of undergraduate mathematics programs students undertake and their likelihood to find employment will be needed.

The career prospects of mathematics graduates are questioned by potential students of both countries. However, graduate destination surveys show that Australian mathematics graduates are employable; this cannot be said of the Argentine mathematics graduates as there is very little data of what they do after they graduate. The perception of poor career prospects are linked to the values assigned to terms such as “successful professional” and “personal satisfaction”; more research needs to be done on how mathematics graduates see their own careers and the satisfaction they receive from it.

Finally, the examples presented here are evidence that although graduate outcomes are affected by the level of government investment in education, it is however, how the different parts of the whole education system come together and support each other that may be more important. These countries need education systems where the value of mathematics is embedded throughout the curriculum from primary, through secondary, and to university levels. They need systems that support the development of inspiring mathematics teachers, and value their jobs. They need systems that interact with industry to ensure that mathematics is perceived as a rewarding and valuable career.

5. References

[1] OECD, 2006, *Evolution of student interest in science and technology studies: Policy Report*, Paris: OECD Global System Forum.
[2] National Academy of Sciences (NAS), 2006, *Rising above the gathering storm: Energizing and employing America for a brighter economic future*. Washington DC: NAS Committee on Science, Engineering and Public Policy.
[3] United Kingdom (UK), 2002, *Review of Scientific and Engineering Labour* (Chair: Roberts, G), SET for success: the supply of people with science, engineering and mathematics skills. London: HMSO.
[4] I. Dobson, 2007, *Sustaining Science: University science in the twenty first century*, Melbourne: Australian Council of Deans of Science
[5] Australian Academy of Science (ACS), 2006, *Mathematics and Statistics: Critical Skills for Australia's Future. The National Strategic Review of the Mathematical Sciences Research in Australia*, http://www.review.ms.unimelb.edu.au/ (accessed 11/5/2009).
[6] C. Varsavsky, and M. Anaya, 2009, Mathematics education in Argentina, *Int. J. of Math. Edu. in Sci. and Tech.*, Vol. 40, No. 1, 15 January 2009, 87–100.
[7] UNESCO Institute of Statistics (UIS), 2009, *UIS statistics in brief*, http://stats.uis.unesco.org/unesco/ (accessed 1/5/2009).
[8] C. Martyniuk, 2008, Susana Vior, educator, *Clarín*, 3/8/08, pp. 34–35.
[9] A. Abramowski, 2007, Interview to the pedagogue Justa Ezpeleta, *El Monitor*, Revista del Ministerio de Educación Ciencia y Tecnología de la Nación, N 13, 5a Época Julio/Agosto 2007, pp. 42–45.
[10] R. Tytler et al, 2008, *Opening up pathways: Engagement in STEM across the primary-secondary school transition*, Australian Department of Education, Employment and Workplace relation.
[11] MCEETYA, 2004, *Demand and supply of primary and secondary school teachers in Australia*. Melbourne: Ministerial Council on Education, Employment, Training and Youth Affairs.
[12] K. Harris, F. Jensz, G. Baldwin, 2005, *Who is teaching science? Meeting the demand for qualified science teachers in Australian Secondary Schools*. Melbourne: Australian Council of Deans of Science.
[13] UNESCO Institute of Statistics (UIS), *World Economic Indicators 2006*, UNESCO Institute of Statistics, Available at http://www.uis.unesco.org/ (accessed 13/8/2008).
[14] F. Barrington, 2006, *Participation in Year 12 Mathematics Across Australia*, International Centre of Excellence for Education in Mathematics and Australian Mathematical Science Institute. An update is available at http://www.amsi.org.au/pdfs/Yr12_Update06.pdf (accessed 11/5/ 2009).
[15] S. Fullarton, M. Walker, J. Ainley, K. Hillman, 2003, *Patterns of Participation in Year 12*, Australian Council of Educational Research, Melbourne.
[16] PISA, 2007, *PISA 2006. Science Competencies for Tomorrow's World*, OECD.
[17] Monash University, 2009, *Undergraduate Handbook 2009*, available at http://www.monash.edu.au/pubs/handbooks/aos/sci-ug-study-clayton-maths.html (accessed 18/5/2009).

[18] Universidad de Buenos Aires, 2009, *Licenciatura en Matemáticas,* available at http://cms.dm.uba.ar/academico/carreras/licenciatura/ (accessed 18/5/2009).
[19] Monash University, 2009, *Mathematics Honours Handbook*, available at http://www.maths.monash.edu.au/honours/ (accessed 18/5/2009).
[20] DiNIECE, 2008, *Anuario de Estadísticas Universitarias 2006,* Buenos Aires: DiNIECE.
[21] Guadagni L "¿Qué universidad queremos?" *LA NACION* 11-25- 2008
[22] *Graduate Careers Australia*, http://www.graduatecareers.com.au/ (accessed 11/5/ 2009).
[23] Remuneraciones de los graduados universitarios, *Clarín*, 16 August 2009. Source: Mercer (Argentina)
[24] Universidad de Buenos Aires, *Carreras de grado*, http://www.uba.ar/download/academicos/o_academica/carreras/lic-csmatematicas.pdf (accessed 11/5/2009)
[25] L. Caffarelli, H. Foellmer, P. Griffiths, W. Pulleyblank, 2002, La matemática en la Argentina, *Ciencia Hoy* -Volumen 12 - Nº 67 Febrero/Marzo
[26] MCTIP, *Informe de Comisiones del Ministerio de Ciencia y Tecnología,* http://www.mincyt.gov.ar/Planplur4/matematica_aplicada.htm (accessed 11/5/2009).
[27] MCTIP, *Indicadores del Ministerio de Ciencia y Política de Innovación Productiva,* http://www.mincyt.gov.ar/index.php?contenido=indicadores/banco_indicadores/comparaciones_internac (accessed 11/5/2009).

Exploring the Sources of Student Motivation in Learning Mathematics at the Preparatory Year Level: The Case of Bilingual Male Arabs Students

B. Yushau[††††††††††††1] , M. H. Omar[2], H. Al-Attas[3], and B. Al-Absi[4]
[1 & 4] Prep Year Mathematics Program
[2 & 3] Mathematics and Statistics Department
King Fahd University of Petroleum & Minerals

Abstract. The paper explores the sources of students' motivation in learning mathematics at the Preparatory Year Mathematics Program at King Fahd University of Petroleum and Minerals. The process used to develop the scales, and the reliability and validity issues of the scales are also elaborated. It was found that all the ten selected variables contributed in different degree to students' motivation level. However, the influences of "family" as a factor top the list. The role of other factors and educational implication of the results are discussed.

Keywords: motivation, mathematics, bilingual Arabs, preparatory year program

1. Background of the Study

Motivation has been defined as an internal state, condition, need, desire or want that serves to activate or energize behavior and gives it direction [1]. It is a driving force for most human endeavors, and has been considered as one of the most important psychological concepts in education that is related to various learning and performance outcomes [2]. Consequently, motivation has attracted the attention of many researchers in the area of the psychology of teaching and learning [3]. In particular, the question of how to motivate students and develop their interest in the mathematics classroom has become a leading concern for mathematics teachers and educators. Some researchers [4,5,6] have been investigating the possible factors that contribute in motivating students. Some of the identified factors that motivate students to study include interest in the subject matter, perception of its usefulness, general desire to achieve, self-confidence and self-esteem, as well as patience and persistence. Bell in [7] summarized them as a zeal to "to create things, to make things work, to obtain recognition, and to find personal satisfaction" (p.33).

Many conceptual perspectives have been suggested in order to have a clear understanding of academic motivation. One such perspective states that behavior can be extrinsically motivated, intrinsically motivated, or amotivated [2]. Extrinsically motivated students are students that are motivated by external factors such as society, family and friends. The intrinsically motivated students derived their energy from safe satisfaction, such as personal enjoyment and fun. On the other hand, *amotivation* is relatively a new term in the literature that refers to students that are neither intrinsically nor extrinsically motivated. According to Lavery in [2], *amotivated* individuals are people who could not "perceive contingencies between outcomes and their own actions, and usually experience feelings of incompetence and lack of control". Individuals with amotivated behaviors can be seen as the least self-determined and they have no expectation of reward or some fear of punishment.

[††††††††††††††††††††] E-mail: byushau@kfupm.edu.sa

However, the factors that affect motivation positively or negatively are multivariable and vary from person to person. For instance, some students seem naturally enthusiastic about learning, but many need to be inspired, challenged, and stimulated by some forces [4], while others defy these internal and external factors. Also, not all students are motivated by the same values, needs, desires, or wants.

A lot of researchers [2, 8, 9, 10] have taken interest in finding the sources of students' motivation and difficulties at the university level. However, the findings are pointing to the fact that academic motivation cannot be understood unless through lenses of the social fabric in which it is embedded [2]. As a result, in recent times, researchers have begun to consider motivation as a cultural phenomenon. According to Lavery in [2], "there has been a huge focus over the years, as to why certain ethnic groups outperform others academically". This indicates that when considering academic achievement, particularly in a University context, it is likely that academic ability is only one determinant of success. Another significant determinant of success is academic motivation [2]. However, much less attention has been given to measuring this construct. It seems imperative to consider the academic motivation of students, not only to investigate the possibility of this construct being related to achievement, but also in order to better structure learning environments and support systems for these students [2].

Although, previous studies [2] have examined university students' motivation as a function of ethnicity and culture in general, not much is known about motivation in Arab student culture, especially at preparatory year university level. The present study is an attempt to extract the role of some selected factors as a motivating/demotivating force for students' learning of mathematics at the university preparatory year level. It is expected that the differing motivational profiles of these students will provide valuable information that may add to the available literature.

2. Methodology

Operational Definition of Motivation

In this study, motivation means the drive to academic achievement, and is measured here by a questionnaire. The students were also categorized into three performance levels based on their previous mathematics examination with a view to measure their response to motivation questions.

Participants

Around one thousand seven hundred students of the preparatory year students of King Fahd University of Petroleum & Minerals (KFUPM) participated in this study. KFUPM is an all male Saudi Arabian University. These are predominantly Arab students who were fresh from High school, and were undergoing a compulsory one-year preparatory program. The aims of this mandatory preparatory year program include: 1) developing students' English skills as almost all the students were from Arabic medium high school, and 2) creating a smooth transitional program from Arabic medium high school to English medium University. In addition to English, students also take two pre-calculus algebra courses, designated as Math 001 and Math 002. Students taking Math 001 cover topics such as properties of real numbers and different types of algebraic functions and equations. Students taking Math 002 cover topics such as logarithms, exponential and trigonometric functions, as well as Matrix Algebra. Participants in this study were Math 001 male students whose ages ranged between 17 and 18.

Selected factors

Intrinsic and Extrinsic motivation are two classical categories of sources of motivation, and they serve as our theoretical framework in this work. In a study conducted by Achoui in [10], the impact of family as a motivating factor for students at KFUPM was investigated. Among his finding were that the family factor has both a positive and negative role to play in students' motivation. Based on the literature review [2], and considering the peculiar nature of the participants in this study, ten factors were considered to be critical in contributing to our students' motivation or lack of it. They were: (1) Language of Instruction, (2) Teacher, (3) Students, (4) Friends, (5) Family, (6) High school background, (7) KFUPM, (8) Help, (9) Mathematic subject and (10) Textbook. Each of these factors was selected for practical reasons. For instance, the students were fresh from Arabic medium high school and now learning mathematics in English for the first time. Some students with weak background in English found it difficult to follow lectures or read the textbook, hence the reason for including English language (the language of instruction) as a subscale. Since the mathematics textbook is written in English, the factor 'textbook' was also included. Another extra factor considered is the students' willingness to work with others, and willingness to help each other for the achievement of their common goals. The other factors were selected due to their intrinsic or extrinsic role in motivating this group of students.

Development of the Instrument

The instrument went through two rounds of pilot testing before it was finalized in the current format. In the first stage, the questions were constructed in English. This was due to the fact that the first two authors who developed the initial questionnaire were not Arabs. The questions were translated into Arabic, and two people who were proficient in both Arabic and English reviewed the translations. Forward and backward stages of the translation process were used to make sure that the translations were accurate. A pilot study was carried out with the Arabic version of the questionnaire, the aim of which was to test the validity and reliability of the items. Many of the items were discarded after the analysis, as they were found not suitable due to lack of validity or reliability. We are then left with only the satisfactory items that were shuffled and used for the final data collection. Details of these items are in Appendix. It should be noted that the questionnaire was developed on a Likert-type scale coded as "Strongly Agree" = 1, "Agree" = 2, "Neutral" = 3, "Disagree" = 4, and "Strongly Disagree" = 5.

It is also customary in questionnaire development to ask a few questions with directly opposite meaning to check if respondents are serious enough to answer questions. Students' seriousness levels in answering the questionnaire items were assessed by including a set of three identical but negatively rephrased questions that were placed in different parts of the instrument. If students were seriously answering the questionnaire, their answers to these questions would be consistently identical. Otherwise, their answers were very different which would signify their ambivalence.

Data Analysis

The collected data were analyzed using Statistical Package for Social Science (SPSS) version 14. The package was chosen as it is more versatile than others for reliability and factor analyses of this paper. After data entry and cleaning, a number of analyses were conducted. This included results on descriptive statistics for all the items, factor analysis, validity and reliability of the scale and its subscales, and correlation analysis between all the variables. In addition, the data were analyzed with special interest on three categories of students: high performing, average and low performing students.

3. Results

Data Cleaning

Students' seriousness levels in answering the questionnaire items were assessed by three sets of questions. These are:

1) "I like Mathematics" versus "I don't like Mathematics"
2) "I like KFUPM" versus "I don't like KFUPM" and
3) "I want Engineering as my major" versus "I don't want Engineering as my major"

Note that there are no other questions with direct opposites that may provide any spurious effects in the analysis of the survey. In addition three such questions would be more than enough for data cleaning purposes. Furthermore, the oppositely worded questions such as "I don't like Mathematics" will obviously use the likert scale in an opposite manner for serious respondents. As such, it is customary for data analysis purposes to reverse the scale for oppositely worded items. That is, unlike other items, the Likert-type scale is reversely coded so that "Strongly Agree" = 5, "Agree" = 4, "Neutral" = 3, "Disagree" = 2, and "Strongly Disagree" = 1 for these few items.

Table 1 shows an example where the diagonal of the table shows the majority of the students answering identically to the questions "I like Mathematics" and "I don't like Mathematics". There are, however, five students who contradictorily say they like and don't like mathematics. As highlighted in the Table, these students and those whose answers were very contradictory (i.e., those with more than 1 point difference) on these questions were not included in the analyses.

Table 1. Student Seriousness Level in Answering Questionnaire Questions.

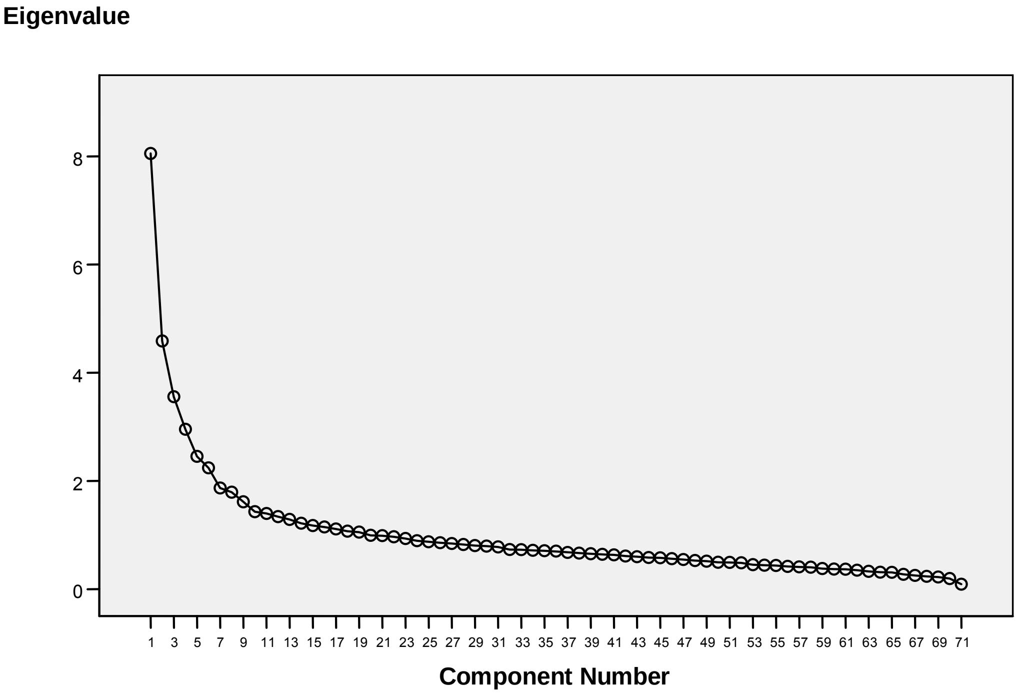
		I like mathematics					Total
		1.00	2.00	3.00	4.00	5.00	
I do not like mathematics (reversely coded)	1.00	332	105	20	5	5	467
	2.00	70	362	62	23	10	527
	3.00	16	82	194	34	12	338
	4.00	6	31	31	91	25	184
	5.00	5	7	9	24	78	123
Total		429	587	316	177	130	1639

Only 1490 students were serious enough with their answers on the survey. These students are the only students included in the remainder of the analyses.

Factor Analysis

Figure 1 and Table 2 provide the eigenvalues of factors in the survey. Based on the usual cut-off of at least 60% [11], 21 factors are needed to explain the student responses. Thirty four factors can be extracted based on Kaiser's criterion of eigenvalues exceeding 1. However, based on the scree plot below and with the elbow criterion [12], only 10 factors seem to be important. Ten factors are also recommended if Bentler and Yuan's [13] criterion of linear excess factors are used.

Figure 1. Scree Plot of Eigenvalues in the Survey.



We thus fit 10 factors. The results of the rotated factor analyses are reported in Table A2 in the Appendix where loadings above an absolute value of 0.3 are bolded. It

Table 2. Percentage of Total Variance Represented by Principal Component Factors.

Component	Initial Eigenvalues		
	Total	% of Variance	Cumulative %
1	8.053	11.342	11.342
2	4.586	6.459	17.801
3	3.556	5.009	22.811
4	2.955	4.161	26.972
5	2.453	3.455	30.427
6	2.241	3.157	33.584
7	1.869	2.632	36.217
8	1.790	2.522	38.738
9	1.614	2.273	41.011
10	**1.431**	**2.016**	**43.027**
11	1.398	1.969	44.997
12	1.339	1.886	46.883
13	1.288	1.813	48.696
14	1.216	1.713	50.409

15	1.174	1.654	52.064
16	1.148	1.617	53.680
17	1.111	1.564	55.245
18	1.070	1.507	56.752
19	1.053	1.483	58.235
20	0.994	1.400	59.635
21	**0.987**	**1.390**	**61.025**
22	0.967	1.362	62.387
23	0.935	1.316	63.703
24	0.894	1.259	64.962
25	0.875	1.232	66.194
26	0.857	1.207	67.402
27	0.842	1.187	68.588
28	0.824	1.161	69.749
29	0.804	1.133	70.881
30	0.795	1.120	72.001
31	0.778	1.096	73.098
32	0.731	1.030	74.128
33	0.728	1.026	75.153
34	0.716	1.009	76.162
35	0.709	0.998	77.160
Similar entries are removed to conserve space			
62	0.350	0.493	96.874
63	0.327	0.461	97.336
64	0.313	0.440	97.776
65	0.309	0.435	98.211
66	0.273	0.384	98.596
67	0.253	0.356	98.952
68	0.236	0.332	99.284
69	0.223	0.314	99.597
70	0.195	0.275	99.872
71	0.091	0.128	100.000

Extraction Method: Principal Component Analysis.

generally appears from the table that most of these items are loading on the same factors. Although there are some exceptions, we can fairly say that the items group together to define ten basic common factors.

Reliability Analyses for Scales

We further conducted reliability analyses for each of the scales represented by the ten factors. The Cronbach alpha reliability for each of the scale is reported in Table 3 below. The overall motivation scale meets the minimum acceptable reliability of 0.80. For short scales, the individual scales generally have decent reliability values considering the length of the scales. "Teacher", "English", and Mathematics "Subjects" scales are exceptionally high for scales with less than 10 questionnaire items. This indicates that the questionnaire items are consistently measuring the same trait as other questions in the scale.

Table 3. Cronbach alpha reliability coefficient and descriptive statistics for motivation questionnaire scales

Scale	n	reliability	M	SD
Student	10	0.648	20.270	5.403
Text	5	0.453	21.403	3.827
Teacher	8	**0.867**	21.467	7.147

English	4	**0.884**	10.637	4.710
Family	5	0.375	9.381	2.874
Friends	7	0.563	20.100	3.941
KFUPM	10	0.678	26.591	5.595
Subjects	9	**0.872**	21.275	7.384
High School	7	0.625	14.947	4.348
Help	4	0.424	8.311	2.587
Total	71	**0.854**	174.381	24.191

N= 1490 serious students out of 1692

Motivation Levels for Groups of Students with Different Achievement Levels

We further use these scales to study the motivation levels of different groups of students by their mathematics achievement. Specifically, student groups were broken down by performance levels (High, Medium, and Low) and the means and standard deviations for each scale were reported in Table 4 to compare their motivation levels.

As can be seen in Table 4, as a group, low achieving students tend to register the highest scores (highlighted in bold fonts) on each of the motivation scale except the subscales *friend* and *help*. Since the responses to questionnaire items were from "strongly agree" (coded as 1) to "strongly disagree" (coded as 5), it appears that the low achieving students tended to disagree more as compared to the high or medium ability students. The high achieving students tended to register the highest score on the *friend* subscale as the motivator for studying preparatory year mathematics. It can also be seen that on the basis of the standard deviations, the low achieving students are generally more variable or different from each other in their responses as compared to the other groups.

Table 4. Motivation Scores by Scale and Student Achievement Levels

	Performance levels			
Scale	Low	Medium	High	All
	Mean			
Stud	**20.80**	19.97	19.77	20.30
engl	**11.00**	10.92	10.30	10.87
friend	19.74	19.92	**20.42**	19.91
HiSchool	**16.08**	14.61	12.74	14.98
teacher	**23.66**	21.24	17.86	21.80
KFUPM	**27.68**	26.82	24.80	26.91
mathsubj	**24.06**	20.52	16.29	21.45
text	**21.60**	21.34	20.87	21.38
family	**9.85**	9.37	9.09	9.54
help	7.25	8.66	**9.96**	8.24
	Std. Deviation			
Stud	5.26	5.30	**5.48**	5.32
engl	**4.68**	4.62	4.52	4.63
friend	4.13	3.85	**4.48**	4.07
HiSchool	**4.56**	4.18	3.41	4.40
teacher	**6.72**	6.57	5.62	6.79
KFUPM	**5.95**	5.43	5.53	5.74
mathsubj	**7.68**	6.73	5.63	7.49
text	**4.05**	3.88	3.85	3.95

family	**3.24**	3.05	2.80	3.11
help	2.39	2.54	**2.57**	2.65
	n			
	625	638	202	1465

We further investigated the correlation between motivation scales for the three different student achievement groups. Table 5 provides comparisons between all students and the medium achieving students. Here, the *medium* students (below the diagonal of the table) are reporting similar correlation profile as *all* students (above diagonal). All highlighted Pearson correlation are significant at $\alpha = 0.05$. In addition, correlation with absolute values more than 0.3 are given in bold fonts. In addition, if the same kind of correlation is not also found to be above an absolute value of 0.3 in both groups, the correlation is given in bolded italics instead of bolded fonts. In particular, *all* students are showing the same kind of significant correlation as *medium* students with exception of additional correlations between *help* and *student* motivation factors and between *high school* and *family* for medium performing students. Like all students, the medium achieving students are reporting a medium correlation between the motivation factors *high school* mathematics and *preparatory year* mathematics. In addition, the medium achieving students concurred with all students by reporting that *family* motivational scale is the most highly correlated to *students* intrinsic motivational scale when compared to other scales studied.

Table 5. Correlation between Motivation Scores by Scale for All Students (above diagonal) and Medium Achieving Students (below diagonal)

	Stud	engl	friend	HS	teacher	KFUPM	math	text	family	help
Stud	1	0.054	0.150	0.242	0.109	0.250	0.171	0.205	**0.358**	0.222
engl	0.023	1	0.035	0.062	0.292	0.173	0.043	0.264	0.115	-0.154
friend	0.149	0.012	1	0.100	0.103	0.186	0.099	0.148	0.113	0.119
HiSchool	0.214	0.093	0.120	1	0.164	0.272	**0.582**	0.116	0.222	-0.036
teacher	0.127	0.297	0.136	0.135	1	0.203	0.243	0.142	0.156	-0.169
KFUPM	0.222	0.170	0.255	0.294	0.228	1	**0.421**	0.238	**0.315**	-0.063
math	0.124	0.093	0.132	**0.546**	0.173	**0.394**	1	0.168	0.208	-0.113
text	0.200	0.224	0.181	0.097	0.143	0.263	0.142	1	0.055	0.077
family	**0.372**	0.123	0.132	***0.302***	0.174	**0.310**	0.244	0.095	1	0.015
help	***0.327***	-0.226	0.115	0.069	-0.075	-0.055	0.001	0.089	0.067	1

Table 6 provides comparisons between and the high achieving and the low achieving students. Significant (at α=0.05) correlations are highlighted and correlations with absolute values more than 0.3 are bolded. If the absolute value is above 0.3 for only one group, the correlation is also italicized. As can be noted from the table, the correlation profile of low achieving students is not similar to that of high achieving students.

Table 6. Correlation between Motivation Scores by Scale for High Achieving Students (above diagonal) and Low Achieving Students (below diagonal)

	Stud	engl	friend	HS	teacher	KFUPM	math	text	family	help
Stud	1	-0.048	0.094	0.286	0.136	0.225	0.247	0.117	**0.334**	0.293
engl	0.114	1	0.008	0.128	***0.401***	***0.330***	0.110	0.247	0.074	-0.246
friend	0.182	0.073	1	0.095	0.055	0.155	0.056	0.040	0.053	0.137
HiSchool	0.236	-0.003	0.122	1	0.229	0.267	**0.608**	0.026	***0.307***	0.049
teacher	0.044	0.258	0.133	0.038	1	0.253	0.280	0.127	0.207	-0.030
KFUPM	0.269	0.119	0.163	0.195	0.090	1	**0.387**	0.211	***0.342***	-0.077

mathsubj	0.162	-0.044	0.139	**0.530**	0.124	**0.402**	1	0.180	***0.344***	0.052
text	0.232	***0.303***	0.161	0.131	0.119	0.211	0.166	1	-0.015	0.139
family	**0.343**	0.111	0.125	0.103	0.086	0.295	0.113	0.025	1	0.084
help	0.198	-0.035	0.093	0.056	-0.091	0.059	0.017	0.112	0.026	1

Nevertheless, both groups concurred by reporting that the *family* motivational scale is the most relatively highly correlated to *students* scale when compared to other scales studied. For the low achieving students, motivation on other scales is related to the *student* scale except for *teacher*. For the high achieving students, *student* scale is not related to *teacher*, *English*, *friends*, and the preparatory year *mathematics* subject scale. For the low achieving students on the other hand, *English* is more related to the *textbook* scale which is opposite for the high achieving students. Also, *English* and *Family* subscales have no correlation with preparatory year *mathematics* subject for the low achieving students. On the other hand, the subscales *student*, *English*, and *friends* do not correlate with *mathematics* subject for the high achieving students.

4. Discussion

From the results above, it can be seen that ten factors suffice to describe preparatory year students' motivation levels. It can be deduced that motivational level of students at the preparatory year level of KFUPM in general is directly correlated with all the variables we selected in this study with the exception of English (language of instruction). The magnitude of the correlation varies, with *Family* (r =.358) having relatively stronger correlation followed by *KFUPM* (r =.250), *High School* (r =.242), *Help* (r=.222), *Textbook* (r =.205), *Mathematics* (r =.171), *Friends* (r =.150), and *Teacher* (r =.109) in descending order of magnitude.

Further analysis shows that the correlation of *students* with *English* language is indirect. Amazingly this indirect correlation is with all the variables except Friends. This indicates that students are normally not motivated to speak English between themselves and even while discussing mathematics. Similarly, although in other results teacher's correlation is low, but it turns out to be relatively stronger with English Language, followed by *Textbook*. While *Mathematics* and *High school* have the least correlations with *English* language, *Family* and *KFUPM* are moderately correlated, which seems to make intuitive sense.

As for the *High performing* students, the correlation is relatively the highest with *Family* (r =.334), followed by *High School* (r =.283), *Help* (r =.247), *KFUPM* (r =.225), *Teacher* (r=.136), and *Textbook* (r =.117). Like in *all* students, the *Family* remains the dominant motivation factor.

For the *average* students, *Family* (r =.372) having relatively stronger correlation followed by *Help* (r = .327), *KFUPM* (r =.222), *High School* (r =.214), *Textbook* (r =.200), *Friends* (r =.149), *Teacher* (r =.127), and *Mathematics* (r =.124).

For the *Low performing* students, *Family* (r =.343) have relatively stronger correlation followed by *KFUPM* (r =.269), *High School* (r =.236), *Textbook* (r =.232), *Help* (r=.198), *Friends* (r =.182), *Mathematics* (r =.162), *English* language (r = .114), and *Teacher* (r =.044).

It is interesting to note that regardless of the grouping, the *Family* motivation factor has relatively the strongest correlations with the *Student*s. And this is even more so with *medium* performing students. This seems to indicate that the role of the parents in the success and failure of the students is critical. Therefore, there should be a formal means of collaborating with the parents in order to help the students attain their maximum potential. Although KFUPM typically invites parents and other family members during student graduation parties, more formal alliance with the students' families is needed as early as in the preparatory year level. For instance, progress reports such as midterm or pre-midterm grades can be sent to parents to alert them of the academic progress of their wards. This is necessary as students are relatively new to the university system, and to the new language of instruction. Similarly, the parents of the student could be invited for discussion on how they can collaborate with the university to help the students in their academic pursuits.

After the *Family*, the second factor for all the students is *Help*. The same result is obtained if the analysis is restricted to *High* and *Medium* performing students. While, for the Low performing students, *Help* is fifth. This indicates that students with some mastery of mathematics tend to be ready offer their help to other students to consolidate their own command of the subject matter.

After *Help*, the third factor for all students is *KFUPM*. The same result is obtained if the analysis is restricted to *medium* performing students. For high performing students, *KFUPM* comes after Mathematics while for low performing students, *KFUPM* comes second after *Family*. This indicates that students are coming to KFUPM by choice, and are very conscious of what it takes to be at KFUPM, and possibly the challenges ahead of them. Therefore, they are directly or indirectly, consciously or subconsciously much concerned with the academic prestige and career outlook attached to the name, KFUPM.

The *Textbook* is interestingly correlated with the *students*. In particular, for most students, it is at least fourth after other scales. Textbook is the fifth factor for all students, and fourth for the low performing students. This finding is not far from our assumptions that on the average, students struggle to read the book, and may try to avoid it if they can find any other alternatives. That is why many times after the exams students found it hard to believe that some questions were similar to or directly from the textbook. Therefore, there is a need to develop students' confidence on the studying the recommended text, and encourage them to read it by all means.

It is generally believed that student's prior background is critical for their future performance. This is more in a hierarchical subject like mathematics than others. In this study *High school* is found to be correlated with *student*. It is the third in strength of relationship with *students* in general, and *Low* performing students in particular. However, it is fourth for *average* students and sixth for the *High* performing students. The result appears to show that while the Average and Low performing students are struggling with the *Textbook*, the High performing students have an advantage due to their High school background.

Mathematics as a subject is sixth in strength of relationship with all students, and it is eighth and seventh respectively for the Average and Low performing students. It is interesting that mathematics is correlated third for *High* performing students. A possible explanation for this is that although most students don't see mathematics as an interesting subject, the High performing students appear to have increasingly more interest in this subject and regard it as important.

Another interesting thing in the result is the fact that the Low performing students are the least correlated with the *teacher*. A plausible argument for this is that, perhaps for the Low performing students, the lack of understanding of the material from High school is still hurting them and demotivating them in the preparatory year mathematics. In addition, the language of instruction being English might have contributed to their lack of following the classroom discourse. This is further seen in this study by the fact that the Low performing students is the only group that is correlated with English language.

It was initially thought that the influence of friends is significant in students learning behavior. However, the results have shown that this is true only for the students in general, and Average and Low performing students in particular. But it should be noted that the strength of the correlation is low in all categories compared with other factors. As for the High performing students, they are not significantly correlated with *Friend*. This might be due to the fact that they are generally more intrinsically motivated and focused. In addition, with their strong High school mathematics background they may not allow friendship to easily distract them from their studies.

5. Conclusion

In this study the role of some selected variables as motivational factors for learning mathematics at the preparatory year level was investigated. These factors were: Language of Instruction (in this case English), Teacher, Students, Friends, Family, High school background, KFUPM, Help, Mathematics subject, and Textbook. Each of these factors was selected for some practical reasons.
Among the findings of the study are:

1. Exploratory factor analyses showed that ten factors should be extracted from the survey.
2. Reliability of the survey is adequate.
3. Low ability students tend to disagree with the survey questions more often than average and high performing students.
4. English language factor as the medium of instruction is indirectly connected with all other motivation variables selected for this study. This is consistent with findings by Lavery [2] on the differential effects of social culture on motivation.
5. Similarly, regardless of the performance category, the *student* subscale is directly correlated with the *family* factor. It is interesting to note that regardless of the grouping, the *Family* has relatively the strongest correlations with the *Student*s. This seems to indicate that the role of the parents in the success and failure of the students is critical. Therefore, there should be a formal means of collaborating with the parent in order to help the students attain their maximum potential. Some possible avenues of involving the parents in preparatory year students academic pursuit at the university have been discussed in this study.

6. Limitations

Like any other experiment and empirical studies, the present study has some identifiable limitations.

1. First, the sample comprised of preparatory year students at KFUPM; therefore, it may not be generalized to other class of students at the university.

2. The variables examined here are certainly not exhaustive of those identified as related with students' motivation. We, however, selected these variables

because of their practical relevance in the KFUPM context. The limitations notwithstanding, the findings presented here provide some valuable information for university authorities by allowing for a synergistic interaction between practice and research findings.

3. There is a need for follow up in order to further validate the findings.

Acknowledgements

The authors would like to acknowledge with thanks the research facilities at King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

7. References

[1] Kleinginna, P., Jr., & Kleinginna A. (1981a). A categorized list of motivation definitions, with suggestions for a consensual definition. *Motivation and Emotion, 5*, 263-291.

[2] Lavery L. (1999). Ethnic Group Differences in the Academic Motivation of University Students. Paper presented for the symposium "Cultural Issues in Motivation" at the AARE-NZARE Conference, Melbourne,

[3] Menec, V. H., Perry, R. P., & Struthers, C. W. (1995). The effects of adverse learning conditions on action-oriented and state-oriented college students. *Journal of Experimental Education*, 63, 281-302.

[4] Davis, B.G., (1993). *Tools for Teaching*, Jossey-Bass; San Francisco.

[5] Bligh, D. A. (1971). *What's the Use of Lecturing?* Devon, England: Teaching Services Centre, University of Exeter.

[6] Sass, E. J. "Motivation in the College Classroom: What Students Tell Us." *Teaching of Psychology*, 1989, 16(2), 86-88.

[7] Bell, F. H. (1978). *Teaching and learning mathematics*. Dubuque, Iowa: Wm. C. Brown Company.

[8] Stipek, D. (1988). *Motivation to learn: From theory to practice.* Englewood Cliffs, NJ: Prentice Hall.

[9] Vallerand, R. J., & O'Connor, B. P. (1989). Motivation in the elderly: A theoretical framework and some promising findings. *Canadian Psychology*, 30, 538-550.

[10] Huitt, W. (2001). Motivation to learn: An overview. *Educational Psychology Interactive*. Valdosta, GA: Valdosta State University. Retrieved [date], from http://chiron.valdosta.edu/whuitt/col/motivation/motivate.html November/December.

[11] Achoui, M. M. (2004). Family impact on students’ motivation. Proceedings of the Second Conference on Adminstrative Sciences, Volume 2, 643 – 663, held in KFUPM, 19 – 21 April.

[12] Malhotra, N.K. & Birks, D.F. (2007). *Marketing research: an applied orientation.* Pearson Education Limited: Essex, UK.

[13] Cattell, R.B. (1966). The Scree Test For The Number Of Factors. *Multivariate Behavioral Research,* 1(2), 245 – 276.

[14] Bentler, P.M. and Yuan, K.H. (1996). Test of linear trend in eigenvalues of a covariance matrix with application to data analysis. *British Journal of Mathematical and Statistical Psychology*, 49(2), 299-312.

Appendix

Table A1. Questionnaire items and related coding.

Question Order	Wording	Coding
1	I consult my textbook before lectures as preparation for the class	B4
2	I usually do not compare my performance with anybody's in my class (Reversed)	S4
3	I could not concentrate in class because I always think of my family all the time (Reversed)	F4
4	My teacher is good	T4
5	I think I need help to pass my mathematics course	HELP4
6	I have no difficulty in understanding lectures in English	E4
7	I do not enjoy the KFUPM campus environment (Reversed)	U4
8	I consult my friends about my class work	FR4
9	I can not see why I am studying mathematics (Reversed)	M4
10	My high school grade in mathematics is always good	HS4
11	I would not be happy if I got low grade	S6
12	The textbook is easy to read	B6
13	High school mathematics lessons are useless in understanding university mathematics (Reversed)	HS6
14	I follow my teacher's advice on how to prepare for the exams and quizzes	T6
15	I think there is too much mathematics required in engineering and sciences (Reversed)	U6
16	I always find mathematics easy	M6
17	All my friends are very serious about their studies	FR6
18	I do not need to study to get a good grade (Reversed)	S1
19	My family does not mind if I get any grade (Reversed)	F1
20	I like reading books	B1
21	I do not like KFUPM (Reversed)	UI
22	I have problems in understanding lectures in English (Reversed)	E1
23	The teacher does not explain the ideas in class (reversed)	T1
24	I am very serious about my studies and work very hard	S5
25	Most of my friends are doing poorly in mathematics (Reversed)	FR1
26	I did not understand mathematics lessons in high school (Reversed)	HS1r
27	I will be happy if I get a good friend that will help me get better grade	HELP1
28	I do not like mathematics	M1r
29	The University does not have a good counseling facility to help me in my courses (Reversed)	U8
30	I understand all lessons in class	T8
31	I rarely do my homework (Reversed)	S8
32	I do all my homework on time	S9
33	I am used to memorizing mathematics lessons (Reversed)	M8
34	It does not bother me whatever is my rank in class (Reversed)	S2
35	I have a poor background in English (Reversed)	E2
36	I like mathematics	M7
37	The textbook is very difficult to read (Reversed)	B2
38	My teacher contributes to my bad grade (Reversed)	T2
39	I will be happy if I get advice on how to get better grades	HELP2
40	My friends do not like studying (Reversed)	FR2
41	The textbook contains good ideas and material that helps me do well in my exams	B7
42	I will not make my family angry if I get a poor grade (Reversed)	F2
43	I have a very poor mathematics background from high school (Reversed)	HS2
44	I do not want Engineering as my major (Reversed)	U2
45	I always find mathematics difficult (Reversed)	M2
46	Doing mathematics homework is not necessary for getting a good grade (Reversed)	S7
47	Our teacher explains the material very well in the class	T7
48	There is no facility in the University to help me study efficiently for my courses (Reversed)	U7
49	I have good friends that help me and I also help them	FR7

50	I do not use to depend on a private tutor to explain my mathematics lessons in high school	HS7r
51	I do not mind getting any grade (Reversed)	S3
52	I do not understand the lesson in class (Reversed)	T3
53	My grade has nothing to do with my textbook	B3
54	KFUPM is too difficult for me (Reversed)	U3
55	I find it difficult to understand mathematics lessons taught in English (Reversed)	E3
56	Most of my friends are doing very well in mathematics	FR3
57	My poor grade is as a result of my family problems (Reversed)	F3
58	I was not paying attention to my mathematics lessons in high school (Reversed)	HS3
59	Mathematics is never my best subject (Reversed)	M3
60	I enjoy sharing my understanding of mathematics with my classmate	HELP3r
61	I do not enjoy mathematics (Reversed)	M5
62	I consult my text book after lectures to confirm my understanding of my daily lessons	B5
63	I think Engineering and sciences are difficult majors to enroll in (Reversed)	U5
64	My friends do not like talking about mathematics (Reversed)	FR5
65	I know all the mathematics material from high school	HS5
66	The teacher does not talk about how to prepare for the exams in the class (Reversed)	T5
67	My family forces me to go to university (Reversed)	F5
68	I like studying at KFUPM	U9
69	I understand mathematics, so I do not memorize it	M9
70	I want Engineering as my major	U10
71	Doing homework is a waste of time (Reversed)	S10

Table A2. Loadings for 10 Rotated (by Varimax) Principal Component Factors

Question Order	Coding	Component 1	2	3	4	5	6	7	8	9	10
20	B1	0.069	-0.099	0.039	0.221	0.081	0.045	-0.033	0.079	**0.343**	-0.128
37	B2	0.169	0.015	**0.562**	0.145	-0.030	0.098	-0.094	-0.035	**0.433**	-0.037
53	B3	-0.034	-0.110	-0.046	-0.026	-0.096	-0.018	-0.026	0.103	**-0.394**	-0.239
1	B4	-0.026	0.035	0.002	0.030	**0.494**	-0.002	0.047	0.091	**0.305**	0.023
62	B5	-0.035	-0.088	0.031	0.003	**0.629**	-0.018	0.027	0.043	0.234	0.010
12	B6	0.101	-0.040	**0.545**	0.091	-0.004	0.097	-0.041	0.023	**0.505**	-0.071
41	B7	0.146	0.141	0.101	0.013	0.071	-0.059	-0.048	0.049	**0.618**	0.238
22	E1	0.035	0.128	**0.828**	0.064	0.029	-0.073	-0.083	-0.035	-0.018	0.062
35	E2	-0.144	0.032	**0.742**	0.034	0.004	0.033	-0.004	0.093	-0.012	0.078
55	E3	0.123	0.131	**0.813**	0.137	0.090	-0.060	-0.082	-0.029	0.014	0.067
6	E4	0.011	0.108	**0.822**	0.068	0.044	-0.082	-0.044	0.007	0.042	0.025
19	F1	0.027	0.018	-0.023	-0.039	0.012	**0.428**	0.039	0.169	0.052	-0.025
42	F2	0.004	0.021	-0.059	0.006	-0.052	**0.497**	-0.085	0.136	-0.039	-0.079
57	F3	0.114	0.004	0.167	0.033	-0.019	0.216	0.043	0.018	-0.244	**0.371**
3	F4	0.099	0.075	0.284	0.292	-0.021	0.110	-0.110	-0.045	-0.293	0.262
67	F5	0.151	0.076	0.114	**0.357**	0.001	0.241	0.101	-0.042	-0.158	0.117
25	FR1	-0.006	0.129	-0.005	0.081	-0.078	-0.051	-0.149	**0.552**	-0.049	0.126
40	FR2	0.069	-0.046	0.086	0.167	0.103	0.223	-0.004	**0.586**	-0.127	0.015
56	FR3	-0.045	0.043	-0.090	0.047	0.007	-0.078	-0.071	**0.606**	0.066	-0.069
8	FR4	-0.156	-0.083	-0.092	-0.197	-0.015	0.072	0.257	0.265	0.150	0.133
64	FR5	0.105	0.009	-0.100	0.228	0.005	0.057	-0.071	**0.466**	0.089	0.058
17	FR6	0.038	0.004	0.089	-0.091	0.105	0.060	0.101	**0.588**	0.026	-0.112
49	FR7	0.114	0.029	0.105	-0.053	0.203	-0.034	0.116	**0.447**	0.016	0.181
27	HELP1	-0.021	0.052	-0.111	0.003	0.013	**0.375**	**0.391**	-0.068	0.122	-0.153
39	HELP2	-0.022	-0.060	-0.085	0.057	0.131	**0.421**	**0.354**	0.089	0.055	-0.196
60	HELP3r	0.103	-0.108	-0.014	-0.035	**0.306**	0.079	0.095	0.058	**0.314**	0.030

5	HELP4	**-0.415**	-0.224	-0.104	-0.123	-0.058	0.206	0.271	0.167	0.061	**-0.327**
26	HS1r	**0.681**	-0.028	0.086	-0.121	0.067	0.093	0.093	0.065	-0.105	0.129
43	HS2	**0.554**	-0.117	0.033	-0.212	0.118	0.162	0.174	0.090	-0.097	0.123
58	HS3	**0.556**	-0.105	0.072	-0.146	0.154	0.188	0.057	0.131	-0.038	0.199
10	HS4	**0.429**	-0.090	-0.047	-0.285	0.112	0.147	0.267	0.020	-0.107	0.013
65	HS5	**0.325**	0.096	0.173	-0.050	-0.016	-0.073	-0.001	0.061	0.098	0.118
13	HS6	0.290	0.054	0.040	0.025	-0.032	-0.025	-0.036	0.006	0.112	**0.415**
50	HS7r	0.217	0.066	0.037	0.134	0.179	-0.019	-0.118	-0.088	-0.034	-0.046
28	M1r	**0.770**	0.068	-0.079	**0.311**	0.051	0.066	0.002	0.042	0.210	-0.040
45	M2	**0.779**	0.102	0.056	0.107	0.027	0.016	0.008	0.008	0.023	0.008
59	M3	**0.766**	0.061	-0.118	0.147	0.022	0.000	-0.053	0.038	0.149	-0.057
9	M4	**0.429**	0.119	0.139	0.222	-0.009	0.122	-0.003	0.042	0.031	0.123
61	M5	**0.657**	0.135	-0.084	0.220	0.034	0.100	-0.048	0.045	0.298	-0.006
16	M6	**0.713**	0.076	0.075	0.077	0.014	-0.093	-0.034	0.008	0.036	-0.082
36	M7	**0.772**	0.080	-0.078	0.288	0.084	0.057	0.017	0.016	0.208	-0.109
33	M8	**0.464**	0.053	0.108	0.038	-0.110	-0.032	0.027	-0.065	-0.182	0.299
69	M9	**0.450**	0.060	0.081	0.060	-0.032	-0.051	0.249	-0.128	-0.075	0.204
18	S1	-0.188	-0.042	-0.154	0.018	0.201	**0.450**	0.061	-0.017	0.050	0.116
34	S2	0.200	0.027	-0.029	0.105	0.032	**0.395**	0.007	-0.085	-0.012	-0.050
51	S3	0.149	-0.006	0.096	0.167	0.100	**0.532**	0.144	-0.049	-0.125	0.073
2	S4	0.036	-0.030	-0.044	0.026	0.042	0.148	-0.108	-0.039	-0.029	-0.218
24	S5	0.099	-0.118	0.054	0.018	**0.637**	0.107	0.119	0.093	0.028	0.019
11	S6	-0.020	-0.043	0.107	-0.018	-0.004	**0.334**	0.199	-0.008	-0.067	-0.162
46	S7	0.024	0.133	0.007	0.051	0.219	**0.513**	-0.163	-0.061	0.108	0.254
31	S8	0.082	-0.007	0.020	0.109	**0.687**	0.183	-0.099	-0.029	-0.163	0.050
32	S9	0.068	0.031	0.097	0.097	**0.731**	0.145	-0.037	0.015	-0.120	0.009
71	S10	0.050	0.046	0.074	0.157	0.207	**0.528**	-0.075	-0.062	0.146	0.261
23	T1	0.081	**0.870**	0.108	-0.015	-0.058	0.005	0.002	0.047	0.020	0.063
38	T2	0.059	**0.799**	0.079	0.096	-0.041	0.056	-0.054	0.022	-0.007	0.106
52	T3	0.231	**0.524**	**0.459**	0.064	0.181	0.071	-0.020	-0.090	-0.061	0.059
4	T4	0.064	**0.859**	0.076	0.010	-0.007	0.007	0.064	0.027	0.013	-0.014
66	T5	0.036	**0.694**	0.041	0.013	-0.061	0.026	0.003	0.069	0.032	0.199
14	T6	0.073	0.229	0.016	-0.085	**0.517**	0.055	0.225	0.086	0.134	-0.117
47	T7	0.071	**0.926**	0.076	0.020	0.015	0.012	0.021	0.031	0.040	0.040
30	T8	0.231	**0.438**	**0.492**	0.037	0.190	-0.095	-0.007	-0.010	-0.021	-0.091
21	Ul	0.030	0.018	0.061	**0.741**	0.064	0.219	0.193	0.119	0.076	0.081
44	U2	0.273	0.021	-0.020	0.257	0.072	-0.009	**0.665**	-0.019	-0.047	0.065
54	U3	0.187	0.043	0.155	**0.491**	0.045	-0.146	-0.212	0.075	0.065	0.132
7	U4	0.057	0.008	0.061	**0.608**	0.013	0.106	-0.018	0.045	0.044	0.165
63	U5	0.266	0.013	0.193	**0.392**	0.027	-0.203	-0.045	0.032	-0.008	-0.015
15	U6	0.047	-0.044	0.087	0.126	-0.044	-0.052	**-0.329**	0.019	0.047	0.009
48	U7	0.088	0.063	0.004	0.158	0.065	0.036	-0.032	0.061	0.050	**0.580**
29	U8	0.016	0.136	-0.054	0.195	0.032	-0.004	-0.025	0.056	0.103	**0.530**
68	U9	0.054	0.021	0.041	**0.729**	0.088	0.188	0.217	0.100	0.105	0.063
70	U10	0.176	-0.025	-0.067	0.255	0.059	-0.056	**0.746**	-0.079	0.022	0.056

List of Contributers

Mario Barra	Manipulation of virtual objects for the development of connections between geometry and probability as well as between the various dimensions of space.
Margot Berger	Activities with CAS: Construction, transformation and interpretation.
Deonarain Brijlal and Aneshkumar Maharaj	An APOS analysis of students' constructions of the concept of continuity of a single-valued function.
Kathleen M Clark, Alex James & Clemency Montelle	"Expert" vs. "Advanced": Investigating Differences in Problem-Solving Practices.
Tracy Craig	"It's not my job to teach them this stuff!" Development of school mathematics skills within the tertiary environment.
Temple H Fay	Period Doubling and Not.
Temple H Fay and William F Sanford	On Generating Primitive Pythagorean Triples.
MR. Freislich and A. Bowen-James	Tertiary mathematics students' achievement and perception of their learning environment: some gender differences.
John Hannah	The Language Of Linear Algebra.
M.C. Harrison and MG Petrie	sigma: providing university-wide mathematics support.
Belinda Huntley, Johann Engelbrecht & Ansie Harding	An assessment component taxonomy for alternative mathematics assessment formats.
Stephan V. Joubert and Temple H. Fay	A note on linear independence.
Gulden Karakok and Barbara Edwards	Students' transfer of learning of eigenvalues and eigenvectors.
Harvey B. Keynes, Rebecca J. Schmitz & Brian J. Lindaman	Student Understanding and Misconceptions About Foundations of Sequences and Series.
Victor Martinez-Luaces	Modelling, applications and inverse modelling: innovations in differential equations courses.
Victor Martinez-Luaces	Modelling and inverse modelling with second order P.D.E. in engineering courses.
Clemency Montelle	Using history to enrich undergraduate mathematics:beyond the anecdote.
Judy Paterson	Teachers' learning during mathematics content based professional development: results and a model of processes and outcomes.
Juan Carlos Ponce-Campuzano & Antonio Rivera-Figueroa	Reflections on the method for computing primitives: $u = \tan \frac{x}{2}$
Karsten Schmidt	Teaching Matrix algebra in the PC Lab.
Moira Statham & Sheena Parnell	The role of technology in conversational support for learning: lecturer-student communication in a university setting .
Sepideh Stewart and Wayne S. Stewart	Promoting contingent teaching and active learning in lectures with classroom response systems.
Paul J. van Staden & M.T. (Theodor) Loots	Teaching the concept of kurtosis in introductory statistics courses using Mathematica: searching for platypuses and kangaroos beneath the cloth of Table Mountain.
Cristina Varsavsky & Marta Anaya	Undergraduate mathematics education: tales of two worlds.
B. Yushau, M. H. Omar, H. Al-Attas, & B. Al-Absi	Exploring the Sources of Student Motivation in Learning Mathematics at the Preparatory Year Level: The Case of Bilingual Arabs Students.

science & technology
Department: Science and Technology
REPUBLIC OF SOUTH AFRICA

National Research Foundation
NRF

CASIO

SAMF
SOUTH AFRICAN MATHEMATICS FOUNDATION

SLALEY

SLALEY

Sida

Classroom to Boardroom
Waltons
Your Solution!

JUTA
AND COMPANY LTD